Revista de la
Unión Matemática Argentina
Volumen 42, Nro. 1, 2000, 109-112

DOUBLING PROPERTY FOR THE HAAR MEASURE ON QUASI-METRIC GROUPS [1]

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Abstract: *In* [L-S], [V-K1] *and* [V-K2], *doubling measures on doubling complete metric spaces are constructed, solving a problem posed by Dyn'kin* [D]. *In* [M] *a metric on self-similar fractals is defined in such a way that Hausdorff measure is doubling, in other words, for self-similar fractals the problem of Dyn'kin is solved by the natural measure for the underlying fractal structure. In this note we show that Haar measure solves Dyn'kin's conjecture on complete quasi-metric doubling groups.*

1.Introduction

In a recent paper by Luukkainen and Saksman [L-S], based on the previous work of Vol'berg and Konyagin,[V-K1] and [V-K2], a non trivial doubling measure is constructed on each complete metric space with the doubling or weak homogeneity property: there exists a constant N such that any subset A of a ball $B(x,r)$ with the property $d(y,z) \geq r/2$, $y \in A$, $z \in A$, $y \neq z$; has at most N points. This weak homogenity was first observed by Coifman and Weiss, [C-W], as a necessary condition in a space of homogeneous type: (X, d, μ) is a space of homogeneous type if d is a non-negative, symmetric and faithfull function defined on $X \times X$, satisfying for some $K \geq 1$ the quasi-triangle inequality $d(x,z) \leq K(d(x,y) + d(y,z))$, x, y, $z \in X$; μ is a doubling measure defined on a σ-algebra containing the d-balls, $0 < \mu(B(x,2r)) \leq A\ \mu(B(x,r)) < \infty$, for some A, every $x \in X$ and every $r > 0$.
In the euclidean space $\mathbb{R}^n$, the tipical space of homogeneous type with its standard metric and measure, Lebesgue measure is at once Hausdorff measure of dimension n and Haar measure for the underlying group structure. In a recent paper, U. Mosco [M] proves that any self-similar compact fractal K, in the sense of Hutchinson, with the Hausdorff measure of the apropriate dimension carries a quasi-distance such that K becomes a space of homogeneous type. In this note we consider the doubling property for Haar measures on groups. We show that for a complete abelian group X, whose topology is given by a translation invariant quasi-distance with the weak homogeneity property, (X, d, μ) is a space of homogeneous type, if μ is the Haar measure on X.

[1] 1991 Mathematics Subject Classification 28A12. Supported by Progr. CAI+D UNL-CONICET, Argentina.

2. Doubling Haar measures

As shown by Macías and Segovia [M-S], every quasi-metric space is metrizable in the sense that there exist a distance ρ and a positive number α such that ρ^α is equivalent to d. So that, without loosing generality, we shall assume that d-balls are open sets. We shall follow closely the developement and notation in [K-T], page 254 for the construction of the Haar measure. Let us write B_r for the d-ball with radius $r > 0$ centered at the identity 0 of X.

Let $\mathcal{C}_0$ denote the class of all bounded and open non-empty subsets of X. Given H_1 and H_2 in $\mathcal{C}_0$, the minimal number $(H_1 : H_2)$ of translates of H_2 that are needed to cover H_1 is finite. This follows from the weak homogeneity property, since H_1 is contained in a ball and H_2 contains a ball. Of course we also have $(H_1 : H_2) \geq 1$ and $(H_1 : H_2) = (x + H_1 : H_2)$ for every $x \in X$. Moreover, for H_1 , H_2 and H_3 in $\mathcal{C}_0$, we have

$$(2.1) \qquad (H_1 : H_3) \leq (H_1 : H_2)(H_2 : H_3).$$

For a given $n \in \mathbb{N}$, the expression: $\lambda_n(G) = (G : B_{1/n})(B_1 : B_{1/n})^{-1}$, defines a finite set function on the class $\mathcal{C}_0$, which from (2.1) and the weak homogeneity property satisfies a basic uniform doubling condition as shown in the next lemma.

Lemma 2.2. *The set functions λ_n satisfy uniform doubling conditions:*

$$\lambda_n(B(x, 2r)) \leq A\lambda_n(B(x, r)),$$

for every $x \in X$, every $r > 0$ and some constant A bounded by the weak homogeneity constant of (X, d).

Proof. Take $H_1 = B(x, 2r)$, $H_2 = B(x, r)$ and $H_3 = B_{1/n}$ in (2.1), then

$$(2.3) \qquad (B(x, 2r) : B_{1/n}) \leq (B(x, 2r) : B(x, r))(B(x, r) : B_{1/n}).$$

Notice now that $(B(x, 2r) : B(x, r)) = (B_{2r} : B_r)$, the number of translates of B_r needed to cover B_{2r} is bounded by the weak homogeneity constant N. Dividing both sides of (2.3) by $(B_1 : B_{1/n})$ we obtain the result. □

Next we apply a Hahn-Banach extension argument to produce a content λ, as a generalized limit of λ_n, satisfying all the relevant properties of each λ_n.

Lemma 2.4. *We have a set function λ on $\mathcal{C}_0$ satisfying the following properties*

(2.5) $0 < \frac{1}{(B_1 : G)} \leq \lambda(G) \leq (G : B_1) < \infty$;

(2.6) $\lambda(G_1 \cup G_2) \leq \lambda(G_1) + \lambda(G_2)$;

(2.7) $\lambda(G_1 \cup G_2) = \lambda(G_1) + \lambda(G_2)$ *if* $d(G_1, G_2) > 0$;

(2.8) $\lambda(G_1) \leq \lambda(G_2)$ *if* $G_1 \subset G_2$;

(2.9) $\lambda(x + G) = \lambda(G)$;

(2.10) *for some constant A less than or equal to N, the inequalities*

$$0 < \lambda(B(x, 2r)) \leq A\lambda(B(x, r)) < \infty$$

hold for every $x \in X$ and every $r > 0$.

The last step is the use of the content λ to construct the metric outer measure

$$\mu^*(E) = \inf\{\sum_{i=1}^{\infty} \lambda(G_i) : G_i \in \mathcal{C}_0 \cup \{\emptyset, X\}; \bigcup_{i=1}^{\infty} G_i \supset E\},$$

with $\lambda(\phi) = 0$ and $\lambda(X) = +\infty$ if X is unbounded. The σ-algebra of μ^*-measurable subsets of X contains the Borel sets and, in particular, d-balls are μ^* -measurable sets. Let us check that μ^* satisfies the doubling property. Observe that, since $\mu^*(B) \leq \lambda(B) \leq (B : B_1)$, every ball has finite measure. Since bounded sets are totally bounded sets in quasi-metric spaces with the weak homogeneity property, if follows from the completeness that the closure of a ball is a compact set. Then if $B(x,s) \subset \bigcup_{j=1}^{\infty} G_j$, with $G_j \in \mathcal{C}_0$, $\overline{B(x,s/2)}$ and also $B(x,s/2)$ can be covered by only a finite number of the G_j's. Say $B(x,s/2) \subset \bigcup_{j=1}^{J} G_j$. From (2.10) and (2.6) we must have that

$$\begin{aligned} 0 < \lambda(B(x,s)) &\leq A\lambda(B(x,s/2)) \\ &\leq A\sum_{j=1}^{J} \lambda(G_j) \\ &\leq A\sum_{j=1}^{\infty} \lambda(G_j) \end{aligned}$$

In other words, $\lambda(B(x,s)) \leq A\sum_{j=1}^{\infty} \lambda(G_j)$ for every covering $\{G_j\}$ of $B(x,s)$. So that

$$0 < \lambda(B(x,s)) \leq A\mu^*(B(x,s)),$$

for every $x \in X$ and every $s > 0$. Thus

$$\begin{aligned} 0 < \mu^*(B(x,2r)) &\leq \lambda(B(x,2r)) \\ &\leq A\lambda(B(x,r)) \\ &\leq A^2\mu^*(B(x,r)) < \infty. \quad \square \end{aligned}$$

References

[C-W] Coifman, R. R. and Weiss, G., *"Analyse harmonique non-conmutative sur certains espaces homogènes"*, Lecture Notes in Math., Vol. 242, Springer-Verlag, Berlin, 1971.

[D] Dyn'kin, E. M., *"Problem S.11 in Linear and Complex Analysis Problem Book"*, Lecture Notes in Math., Vol. 43, Springer-Verlag, 1984.

[K-T] Kingman, J. F. C. and Taylor, S. J., *"Introduction to measure and probability"*, Cambridge University Press, Cambridge, 1977.

[L-S] Luukkainen, J. and Saksman, E., *"Every complete doubling metric space carries a doubling measure"*, Proceedings of the AMS, Vol. 126, Nro. 2, 1998, 531-534.

[M] Mosco, U., *"Lagrangian metrics on fractals"*, Proceedings of Symposia in Applied Mathematics, 54, 1998, 301-323.

[M-S] Macías, R. and Segovia, C., *"Lipschitz functions on spaces of homogeneous type"*, Advances in Math. 33, 1979, 257-270.

[V-K1] Vol'berg, A. L. and Konyagin, S. V., *"There is a homogeneous measure on any compact subset in* $\mathbb{R}^n$*"*, Soviet Math. Dokl. 30, 1984, 453-456.

[V-K2] ________________, *"On measures with the doubling condition"*, Math. USSR-Izv. 30, 1988, 629-638.

Recibido : 24 de Marzo de 2000
Aceptado : 29 de Marzo de 2001