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ON A NON-LINEAR ELLIPTIC PROBLEM[1]

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Abstract

For the non-linear operator

$$A(u) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right), \qquad p > 2,$$

we consider the existence of solution, in $W_0^{1,p}(\Omega)$, $\underline{\Omega}$, open and bounded in $\mathbb{R}^n$ for the equation $A(u) + g(x,u) = f(x)$ with appropriate $\underline{f}$ and $\underline{g}$, using motonicity methods.

1. Introduction

Let

$$A(u) = -\sum \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right), \qquad p > 2.$$

The problem: solve

$$A(u) + g(x,u) = f(x) \qquad (*)$$

in $\Omega \subset \mathbb{R}^n$, open and bounded subset is considered for instance in [3], in the cases $g \equiv 0$, and $g(x,r) = |r|^{p-2} r$, using Galerkin method. We will consider the problem with more general g, using monotonicity methods. Here, the function $g(x,r)$ will be supposed to have the properties:

(a) $g(x,r)$ is measurable in $\underline{x} \in \Omega$, for fixed $r \in \mathbb{R}$. It is a continuous function in r, for each fixed x. For each $x \in \Omega$, $g(x,0) = 0$ and, for all $r \in \mathbb{R}$ and $x \in \Omega$, $g(x,r)r \geq 0$;

(b) there exists a non-decreasing continuous function $h\colon \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ such that for a given $C \in \mathbb{R}$: $|g(x,r)| \leq |h(r)|$ and $|h(r)| \leq C\{|g(x,r)| + |r|^{p-1} + 1\}$, for all $x \in \Omega$ and $r \in \mathbb{R}$.

We remark that $u \in W_0^{1,p}(\Omega) = V$ is a solution of (*) if equality, there, holds in the sense of distributions, that is, for each $\varphi \in \mathcal{D}(\Omega)$,

$$(A(u), \varphi) + (g(x,u), \varphi) = (f, \varphi)$$

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where $(\cdot, \cdot)$ is the duality between V' and V.

2. Existence of solution

Let us define the functions $g_n(x,r)$, by:

$$g_n(x,r) = \begin{cases} g(x,r) & \text{if} \quad |g(x,r)| \leq n \\ n & \text{if} \quad g(x,r) \geq n \\ -n & \text{if} \quad g(x,r) \leq -n \end{cases}$$

Lemma 1. *For each n, natural and $f \in V'$ there exists $u_n \in V$ such that*

$$A(u_n) + g_n(x, u_n) = f.$$

Proof. Let $u \in V$. Since $|g_n(x, u(x))| \leq n$, the function from Ω to $\mathbb{R}$, that to each $x \in \Omega$ associates $g_n(x, u(x))$ belongs to $L^{p'}(\Omega)$ and since $u \in L^p(\Omega)$, the functional from V to $\mathbb{R}$ associating to each $v \in V$,

$$\int g(x, u(x))v(x)\, dx$$

is well defined. It is bounded on V and therefore, is an element of V', denoted by $g_n(x, u)$. Hence the functional:

$$u \longmapsto A(u) + g_n(x, u)$$

is (non-linear), bounded on V. By the definition of $g_n(x,r)$ and the hypothesis (a) on $g(x,r)$ we will have that

$$A(u) + g_n(x, u) \tag{1}$$

is coercive since A is coercive. $A(u)$ is bounded, hemicontinuous and monotonic and therefore

$$A(u) + g_n(x, u)$$

is pseudomonotonic, since $g_n(x, u)$ is pseudomonotonic (see [3], p. 189). We also have

$$A(u) + g_n(x, u),$$

is surjective (proposition in [3], p. 247) and for each $\underline{n}$, natural and $f \in V'$ there exists $u_n \in V$ such that

$$A(u_n) + g_n(x, u_n) = f.$$

Since $u_n \in V$ and $g_n(x, u_n) \in V'$ is an element of $L^{p'}(\Omega)$ we may write

$$(A(u_n), u_n) + \int_\Omega g_n(x, u_n)u_n\, dx = (f, u_n). \tag{2}$$

Since $g_n(x, u_n)u_n \geq 0$, we have

$$\alpha||u_n||^p \leq (Au_n, u_n) \leq (f, u_n) \leq ||f||_{V'}\, ||u_n||,$$

for some $\alpha \in \mathbb{R}$ and there is $M > 0$ such that

$$||u_n|| \leq M, \quad \text{for all} \quad n \in \mathbb{N}.$$

V beeing reflexive there is a subsequence also denoted by $(u_n)_n$ such that u_n converges weakly in V to $u \in V$ and $A(u_n)$ converges weakly in V' for $\eta \in V'$ that is

$$u_n \rightharpoonup u \quad \text{and} \quad A(u_n) \rightharpoonup \eta, \quad n \to \infty. \tag{3}$$

Lemma 2. *In the conditions of Lemma* 1,

$$g_n(x, u_n) \to g(x, u), \quad \text{in} \quad L^1(\Omega).$$

Proof. We will verify that $g_n(x, u_n)$ are equiuniformly integrable, that is: for each $\varepsilon > 0$ there is $\delta > 0$ such that for each $B \subset \Omega$ measurable, with $\mu(B) < \delta$,

$$\int_B |g_n(x, u_n)|\, dx < \varepsilon, \quad \text{for all} \quad n \in \mathbb{N}.$$

Since $A(u) + g_n(x, u)$ is coercive, we obtain

$$\int_\Omega g_n(x, u_n) dx = (f, u_n) - (A(u_n), u_n) \leq ||f||_{V'}\, M + M_1\, M = M_2\,.$$

Let $R \in \mathbb{N}$, $\quad R > 0$, arbitrary.

For almost all $x \in \Omega$ such that $|u_n(x)| \leq R$,

$$|g_n(x, u_n)| \leq |g(x, u_n)| \leq |h(u_n)| \leq \{h(R) + |h(-R)|\},$$

and almost all $x \in \Omega$ such that $|u_n(x)| \geq R$,

$$R|g_n(x, u_n)| \leq |u_n(x)|\, |g_n(x, u_n)| = u_n(x) g_n(x, u_n),$$

by conditions (a) and (b).

Hence, for each $R > 0$, $\quad R \in \mathbb{N}$ and for almost all $x \in \Omega$,

$$R|g_n(x, u_n)| \leq u_n(x) g_n(x, u_n) + R\{h(R) + |h(-R)|\}$$

and it follows that

$$\int_B |g_n(x, u_n)| dx \leq R^{-1} \int_B u_n g_n(x, u_n) dx + \mu(B)\{h(R) + |h(-R)|\},$$

for each $R > 0$ and for all $B \subset \Omega$, measurable. Let $\varepsilon > 0$ and R be such that

$$R > \frac{2M_2}{\varepsilon} \quad \text{and} \quad \delta = \frac{\varepsilon}{2\{h(R) + |h(-R)|\}}\,.$$

To the first step we recall that the $L^1(\Omega)$-convergence of $g_n(x,u_n)$ to $g(x,u)$ given in Lemma 2, may be interpreted as a $\mathcal{D}'(\Omega)$-convergence as well as a weak-convergence in V' of $A(u_n)$ to η.

Hence, by (1), (3) and (4),

$$\eta + g(x,u) = f, \quad \text{in} \quad \mathcal{D}'(\Omega). \tag{5}$$

Let $H(r) = \int_0^r h(s)\,ds$. H is continuous, convex and $H(0) = 0$. Moreover,

$$|H(u)| = \left| \int_0^{u(x)} h(s)ds \right| \leq |h(u(x))u(x)| = |u|\,|h(u)| \leq C|u|\{|g(\,\cdot\,,u)| + |u|^{p-1} + 1\}.$$

Therefore, $H(u) \in L^1(\Omega)$ and there is a sequence $(v_j)_j$ in $C_0^\infty(\Omega)$ such that

$$\begin{array}{lll} v_j \to u & \text{strongly} & \text{in} \quad V \\ v_j \to u & \text{a.e.} & \text{in} \quad \Omega, \end{array}$$

and by Lemma 3, p. 11, [1], $H(v_j)$ is bounded for all $\underline{j}$ by a fixed function in $L^1(\Omega)$.

Let $v \in V$ and $x \in \Omega$. If $u(x) < v(x)$ then, for some $u < \xi < v$,

$$H(v) - H(u) = H'(\xi)(v-u) = h(\xi)(v-u) \geq h(u)(v-u).$$

If $v(x) < u(x)$ then,

$$H(u) - H(v) = H^{'}(\zeta)(u-v) = h(\zeta)(u-v) \leq h(u)(u-v).$$

Therefore $H(v) - H(u) \geq h(u)(v-u)$ or

$$h(u)v \leq H(v) - h(u) + h(u)u \leq H(v) + h(u)u$$

since $H(u) \geq 0$.

Now, if x is such that $u(x)$ and $v(x)$ have the same signal, then $\underline{v}$ has the same signal as $h(u)$; therefore,

$$|g(x,u)v| \leq |h(u)v| = h(u)v \leq H(v) + h(u)u.$$

And, if $\underline{u}$ and $\underline{v}$ have different signals then $g(x,u)$ and $\underline{v}$ have distinct signals. Since $H(v) + h(u)u \geq 0$ we have: $g(x,u)v \leq H(v) + h(u)u$.

By the above inequalities,

$$(g(x,u)v_j)^+ \leq H(v_j) + h(u)u.$$

The second member of the inequality is dominated by a $L^1(\Omega)$ function, for each $j \in \mathbb{N}$.

We have $(g(x,u)v_j)^+ \to (g(x,u)u)^+ = g(x,u)u$, a.e. in Ω.

Therefore, by Lebesgue theorem,

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Therefore, $H(u) \in L^1(\Omega)$ and there is a sequence $(v_j)_j$ in $C_0^\infty(\Omega)$ such that

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and by Lemma 3, p. 11, [1], $H(v_j)$ is bounded for all $\underline{j}$ by a fixed function in $L^1(\Omega)$.

Let $v \in V$ and $x \in \Omega$. If $u(x) < v(x)$ then, for some $u < \xi < v$,

$$H(v) - H(u) = H'(\xi)(v-u) = h(\xi)(v-u) \geq h(u)(v-u).$$

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Now, if x is such that $u(x)$ and $v(x)$ have the same signal, then $\underline{v}$ has the same signal as $h(u)$; therefore,

$$|g(x,u)v| \leq |h(u)v| = h(u)v \leq H(v) + h(u)u.$$

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By the above inequalities,

$$(g(x,u)v_j)^+ \leq H(v_j) + h(u)u.$$

The second member of the inequality is dominated by a $L^1(\Omega)$ function, for each $j \in \mathbb{N}$.

We have $(g(x,u)v_j)^+ \longrightarrow (g(x,u)u)^+ = g(x,u)u, \quad \text{a.e. in} \quad \Omega.$

Therefore, by Lebesgue theorem,

$$\int_\Omega (g(x,u)v_j)^+\,dx \longrightarrow \int_\Omega g(x,u)u\,dx. \tag{6}$$

Using (5), we have $(\eta, \varphi) + (g(x,u), \varphi) = (f, \varphi)$, for all $\varphi \in \mathcal{D}(\Omega)$.
For the functions $(v_j) \in C_0^\infty(\Omega)$ in Lemma 3, page 11, of [1], we have

$$(\eta, v_j) + (g(x,u), v_j) = (f, v_j), \text{ for all } j \in \mathbb{N}, and (\eta, v_j) \longrightarrow (\eta, u). \tag{7}$$

Similarly $(f, v_j) \longrightarrow (f, u)$, and $(g(x,u), v_j) = \int_\Omega g(x,u) v_j \, dx$.

Therefore

$$(\eta, v_j) + \int_\Omega g(x,u) v_j \, dx = (f, v_j), \tag{8}$$

and

$$(\eta, v_j) = (f, v_j) - \int_\Omega g(x,u) v_j \, dx \geq (f, v_j) - \int_\Omega (g(x,u) v_j)^+ \, dx$$

and

$$(\eta, u) \geq (f, u) - \int_\Omega g(x,u) u \, dx, \text{ when } j \to \infty. \tag{9}$$

Let $Y_n = (Au_n - Av, u_n - v), \quad v \in V$, arbitrary.
By (2) we have

$$\begin{aligned} 0 \leq Y_n &= (Au_n, u_n) - (Au_n, v) - (Av, u_n) + (Av, v) = \\ &= (f, u_n) - \textstyle\int_\Omega g_n(x, u_n) u_n \, dx - (Au_n, v) - \\ &\quad -(Av, u_n) + (Av, v). \end{aligned}$$

Then, by (9),

$$\begin{aligned} 0 &\leq \limsup Y_n \leq (f, u) - \liminf \textstyle\int_\Omega g_n(x, u_n) u_n \, dx - (\eta, v) - \\ &\quad -(Av, u) + (Av, v) \leq (\eta - Av, u - v). \end{aligned}$$

That is, $\qquad (\eta - Av, u - v) \geq 0 \quad$ for each $v \in V$.

Let $v = u - \lambda w, \; w \in V, \; \lambda \in \mathbb{R}$ arbitrarily chosen.

Then, $(\eta - A(u - \lambda w), \lambda w) \geq 0$, and, for $\lambda > 0 : (\eta - A(u - \lambda w), w) \geq 0$.

As $\lambda^+ \longrightarrow 0$, we have, by the hemicontinuity of A: $(\eta - Au, w) \geq 0$.
If $\lambda < 0$, then $(\eta - A(u - \lambda w), w) \leq 0$ and as $\lambda \to 0$, we have $(\eta - Au, w) \leq 0$.
Hence, for all $w \in V$, $(\eta - Au, w) = 0$, and $\eta = A(u)$.
Using (5), we have

$$Au + g(x,u) = f, \quad in \;\; \mathcal{D}'(\Omega) \tag{10}$$

Finally we will prove that

$$(Au, u) + \int_\Omega g(x,u) u \, dx = (f, u).$$

Recall the equality (2)

$$A(u_n, u_n) + \int_\Omega g_n(x, u_n) u_n \, dx = (f, u_n).$$

From (9), we obtain

$$(f,u) \leq (Au,u) + \int_\Omega g(x,u)u\,dx. \tag{11}$$

On the other hand,

$$\begin{aligned}(f,u) &= \lim_{n\to\infty}(f,u_n) = \lim_{n\to\infty}\left[(Au_n,u_n) + \textstyle\int_\Omega g_n(x,u_n)u_n\,dx\right]\\ &\geq \liminf\left[(Au_n,u_n-u) + (Au_n,u) + \textstyle\int_\Omega g_n(x,u_n)u_n\,dx\right]\\ &\geq (Au,u) + \textstyle\int_\Omega g(x,u)u\,dx,\end{aligned} \tag{12}$$

since $\liminf\,(Au_n,u_n-u) \geq 0$.

From (11) and (12), we have our result.

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