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SUMMABILITY OF ORTHONORMAL POLYNOMIAL SERIES [1]

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Abstract

In this note we discuss the convergence of H_α-summability of the Fourier type expansions considered by H. Pollard in [11] *and M. Wing in* [15].

Introduction

This paper is based on the results communicated and commented on the *Rey Pastor Lecture* I gave on the occasion of the September 18-21, year 2000, meeting of the Unión Matemática Argentina, Rosario, Argentina. Throughout the coming sections I shall discuss the pointwise convergence, a.e. of the strong summability of the partial sums of certain orthonormal polynomial expansions of Fourier type under fairly general assumptions. I will use the assumptions introduced by H. Pollard in papers [10] and [11] and later simplified by G. M. Wing in [15]. Here, strong summability refers to the H-summability discussed in Zygmund [16], pp. 180, 181. This mode of summability was first introduced by Hardy and Littlewood in 1913 in the L^2 case (see [8]). These results were discussed and somewhat generalized by Borgen [3], Carleman [4] and Sutton [14] in the Fourier Series context. The most powerful known result on H-summability in the Fourier Series context is due to J. Marcynkiewicz and it is discussed in [16], p. 184. We shall use a local form of the Haussdorf-Young inequality to obtain the basic estimates leading to the summability result announced below as Theorem 1. The approach presented here in the global case goes back to Hardy and Littlewood [8] and a generalized version is presented in [16], p. 182.

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Let $\{p_n(x)\}$ be a complete system of orthonormal polynomials with respect to the weight $w(x)$ in $(-1,1)$. Let $\{q_n(x)\}$ be the associated system following Pollard-Bernstein (see [11], p. 356) and (see Wing [15], p. 802). As indicated in the references, the system:

$$q_n(1-x^2)^{-1}$$

is orthogonal with respect to the weight:

$$(1-x^2)w(x).$$

Pollard-Wing conditions are:

(H_1)

$$w(x) \geq 0$$

(H_2)

$$\int_{-1}^{1} |\log w(x)|(1-x^2)^{-1/2}dx < \infty$$

(H_3)

$$(1-x^2)^{1/4}|p_n(x)|w^{1/2}(x) \leq A$$

(H_4)

$$(1-x^2)^{-1/4}|q_n(x)|w^{1/2}(x) < B$$

Summability H_α

We define the Fourier type series of f, $f \in L^p(-1,1)$ as:

$$f \sim \sum c_n w^{1/2}(x)p_n(x). \tag{1.1}$$

Where the $p_n(x)$ and $w(x)$ satisfy the conditions H_i, $i = 1,2,3,4$ above. The coefficients are defined as:

$$c_n = \int_{-1}^{1} f(x)w^{1/2}(x)p_n(x)dx. \tag{1.2}$$

The above integral is taken in the Legesgue's sense. Call $S_n(x)$ the partial sums of the series (1.1). We say that the series (1.1) is H_α summable to sum s at the point x if

$$\frac{1}{n+1}\sum_{k=0}^{k=n}|S_k(x)-s|^\alpha \to 0 \tag{1.3}$$

as n tends to ∞.

The main result of this paper is:

Theorem 1 *Let $f \in L^p(-1,1)$, $p > 4/3$. Then $S_n(x)$ is H_α summable to sum f for* a.e. *x in $(-1,1)$, $\alpha > 0$.*

Throughout the next section we will discuss the auxiliary lemmas needed for the proof of Theorem 1.

2 Auxiliary Lemmas

Lemma 2.1 *Let $f \in L^p(-1,1)$, $1 < p \leq 2$ and suppose that $f = 0$ if $b < |x| < 1$, for some $b > 0$. Let us denote by c_n the Fourier coefficients introduced in* (1.2) *above. Then*

$$\left(\sum_0^\infty |c_n|^q\right)^{1/q} \leq C(b,q)\|f\|_p, \quad q = \frac{p}{p-1}.$$

$C(b,q)$ depends on b and q only.

Proof. Riesz's interpolation theorem [16], II p. 93, will give the local analog of Haussdorf-Young inequality [16], p. 101, that we need for this context is to interpolate the following norm inequalities.

For $p = 1$ we have

$$|c_n| \leq \frac{A}{(1-b^2)^{1/4}} \int_{-b}^{b} |f| dx, \quad n = 0,1,2\ldots \tag{2.1.1}$$

on the other hand, for $p = 2$ we have

$$\left(\sum_0^\infty |c_n|^2\right)^{1/2} \leq \left(\int_{-b}^{b} |f|^2 dx\right)^{1/2}. \tag{2.1.2}$$

Finally the thesis follows by taking $\frac{1}{p} = \frac{t}{2} + \frac{1-t}{1}$, $\frac{1}{q} = \frac{t}{2} + \frac{1-t}{\infty}$ or $p = \frac{q}{q-1}$. □

Lemma 2.2 *If $f \in L^p(-1,1)$, $p > 4/3$, then $c_n \to 0$ as $n \to \infty$. Here, as before, c_n stands for the Fourier coefficients defined in* (1.2) *above. This is a version of Riemann-Lebesgue's theorem.*

Proof. Select b, such that $0 < b < 1$ such that

$$\left(\int_{b<|x|<1} \left(\frac{A}{(1-x^2)^{1/4}}\right)^q dx\right)^{1/q} \cdot \left(\int_{b<|x|<1} |f|^p dx\right)^{1/p} < \epsilon. \tag{2.2.1}$$

Clearly

$$|c_n| \leq \left|\int_{-b}^{b} f(x)p_n(x)w^{1/2}(x)dx\right| + \epsilon. \tag{2.2.2}$$

Therefore

$$\overline{\lim_{n\to\infty}}|c_n| \leq \epsilon.$$

This proves the lemma when $4/3 < p \leq 2$. The case $p \geq 2$ is argued by observing $L^p(-1,1) \subset L^2(-1,1)$ for $p > 2$. □

2.3 Partial Sums

The kernel for the partial sums of order n is given by

$$K_n(x,y) = \sum_{k=0}^{k=n} \omega_k^{1/2}(x)\omega_k^{1/2}(y)p_k(x)p_k(y). \tag{2.3.1}$$

A crude estimate for $K_n(x,y)$ gives

$$|K_n(x,y)| \leq nA^2(1-x^2)^{-1/4} \cdot (1-y^2)^{-1/4}. \tag{2.3.2}$$

Lemma 2.4 *If $f \in L^p(-1,1)$; $1 \leq p < \infty$ and the support of f is contained in $[x-1/n, x+1/n]$, b has the same meaning as before, and $|x-b| > \frac{1}{n}$, $|x+b| > \frac{1}{n}$. Then under those assumptions we have*

(i) $|S_k(x)| \leq A^2(1-b^2)^{-1/2} n \int_{x-1/n}^{x+1/n} |f(x)|dt;\quad k \leq n$

or alternatively

(ii) $|S_k(x)| \leq A^2(1-b^2)^{-1/2}\left(n \int_{|x-y|<1/n} |f(y)|^p dy\right)^{1/p}, \quad k \leq n.$

Proof. It follows from the representation formula discussed in section 2.3 above and the corresponding estimates on $K_k(x,y)$. □

2.5 Darboux-Christoffel Formula

The kernel $K_n(x,y)$ is given by (see [16], p. 42):

$$K_n(x,y) = \lambda_n \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{(x-y)} \omega^{1/2}(x)\omega^{1/2}(y). \tag{2.5.1}$$

Under the assumptions H_1 through H_4 we have $|\lambda_n| \leq M$.

2.6 Bernstein-Pollard formula for the partial sums

For details see Pollard [11], pp. 357 and 358. See also Wing [15], p. 803.

$$\begin{aligned} K_n(x,y) &= \alpha_n\omega^{1/2}(x)\omega^{1/2}(y)\frac{p_{n+1}(x)q_n(y)}{x-y} + \beta_n\frac{\omega^{1/2}(x)\omega^{1/2}(y)p_{n+1}(y)q_n(x)}{x-y} \\ &\quad + \gamma_n\omega^{1/2}(x)\omega^{1/2}(y)p_{n+1}(x)q_{n+1}(y). \end{aligned} \tag{2.6.1}$$

Here, α_n, β_n γ_n being uniformly bounded.

Lemma 2.7 *Suppose that $f \in L^P(-1,1)$ and the support of f is contained in $(-b,b)$; where $0 < b < 1$ and $1 < p \leq 2$. Then if $q = \frac{p}{p-1}$ and $|x-b| > \frac{1}{n}$, $|x+b| > \frac{1}{n}$; we have*

$$\left(\frac{1}{n+1}\sum_{k=0}^{n}|S_k(x)|^q\right)^{1/q} \leq C_{p,b}\left(n^{1-p}\int\limits_{|x-y|>1/n}\frac{|f(y)-f(x)|^p}{|x-y|^p}dy\right)^{1/p}$$
$$+C'_{p,b}\left(n\int\limits_{|x-y|<1/n}|f(y)|^p dy\right)^{1/p}.$$

Proof. We write $f(y) = F_1(y) + F_2(y)$; where

$$\begin{aligned} F_1(y) &= f(y) \text{ if } |x-y| < \frac{1}{n} \text{ and zero otherwise} \\ F_2(y) &= f(y) \text{ if } |x-y| \geq \frac{1}{n} \text{ and zero otherwise.} \end{aligned} \tag{2.7.1}$$

An application of Lemma 2.4 above to $F_1(y)$ gives

$$\left(\frac{1}{n+1}\sum_{k=0}^{n}|S_k(x)|^q\right)^{1/q} \leq \max_{0\leq k\leq n}|S_k(x)|$$
$$\max_{0\leq k\leq n}|S_k(x)| \leq A^2(1-b^2)^{-1/2}\left(n\int\limits_{|x-y|<1/n}|F_1(y)|^p dy\right)^{1/p}. \tag{2.7.2}$$

On the other hand, an application of the Darboux-Christoffel formula to $F_2(y)$ gives for $S_k(x)$:

$$\begin{aligned} |S_k(x)| \leq\ & |\lambda_n|\ |p_{k+1}(x)|\omega^{1/2}(x)\Big|\int\limits_{|x-y|>1/n}\frac{F_2(y)}{x-y}\omega^{1/2}(y)p_k(y)dy\Big| \\ & +|\lambda_n|\ |p_k(x)|\omega^{1/2}(x)\Big|\int\limits_{|x-y|>1/n}\frac{F_2(y)}{(x-y)}\omega^{1/2}(y)p_{k+1}(y)dy\Big|. \end{aligned} \tag{2.7.3}$$

Therefore, we have for the $S_k(x)$ associated with $F_2(y)$ and using Lemma 2.1

$$\frac{1}{n+1}\sum_{k=0}^{n}|S_k(x)|^q \leq \frac{1}{n+1}2^{q-1}M^q(1-b^2)^{q/4}\sum_{k=0}^{\infty}|C_k(x)|^q \tag{2.7.4}$$

where

$$C_k(x) = \int\limits_{|x-y|>1/n}\frac{F_2(y)}{x-y}p_n(y)\omega^{1/2}(y)dy.$$

Consequently

$$(2.7.5)\quad \left(\frac{1}{n+1}\sum_{k=0}^{n}|S_k(x)|^q\right)^{1/q} \leq C(p,b)\left(n^{1-p}\int_{|x-y|>1/n}\frac{|F_2(y)|^p}{|x-y|^p}dy\right)^{1/p}.$$

This completes the proof of Lemma 2.7. □

2.8 Proof of Theorem 1

Let $f(y) \in L^p(-1,1)$ and $-1 < x < 1$. We will write $f(y) - f(x) = F_1(y) + F_2(y) + F_3(y)$, and choose b such that; $0 < b < 1$ and $-b < x < b$. Finally we shall consider n large enough so that

$$(2.8.1)\quad -1 < -b < x - \frac{1}{n} < x + \frac{1}{n} < b < 1.$$

Now we pass to define $F_1(y)$, $F_2(y)$ and $F_3(y)$:

$$(2.8.2)\quad \begin{aligned} F_1(y) &= f(y) - f(x) \text{ if } |x-y| < 1/n \text{ and zero otherwise} \\ F_2(y) &= f(y) - f(x) \text{ if } |x-y| \geq 1/n,\ -b < y < b, \text{ and zero otherwise} \\ F_3(y) &= f(y) - f(x) \text{ if } -1 < y < -b \text{ or } b < y < 1 \text{ and zero otherwise.} \end{aligned}$$

From Lemma 2.2 it follows that

$$(2.8.3)\quad \left(\frac{1}{n+1}\sum_{k=1}^{n}|S_k(F_3)(x)|^\alpha\right)^{1/\alpha} \to 0,\ \alpha > 0 \quad \text{if } |f(x)| < \infty.$$

On the other hand if $1 < p \leq 2$ and $q = \frac{p}{p-1}$ we have

$$(2.8.4)\quad \lim_{n\to\infty} n \int_{|x-y|<1/n} |f(x) - f(y)|^p dy = 0$$

for almost every x; and consequently

$$(2.8.5)\quad \lim_{n\to\infty}\frac{1}{n+1}\sum_{0}^{n}|S_k(F_i)|^q = 0 \quad i = 1,2 \text{ for almost every } x.$$

The above limit indicates that

$$(2.8.6)\quad |S(F_i)|^q \qquad (i = 1,2)$$

is an almost convergent sequence for a.e. x (see Zygmund [16], pp. 181, 182). Hence, it follows that

$$(2.8.7)\quad \frac{1}{n+1}\sum_{1}^{n}|S_k(f) - f(x)|^\alpha \to 0$$

for almost every x and $0 < \alpha \leq q$.

This concludes the proof since $L^p(-b,b) \subset L^{p_0}(-b,b), p > p_0 > 1$ and p_0 can be chosen arbitrarily close to 1 and α arbitrarily large. □

2.9 Additional Remarks to the above proof

First we notice that

(2.9.1)
$$n \int_{|x-y|<\frac{1}{n}} |f(y) - f(x)|^p dy \to 0$$

implies

(2.9.2)
$$n^{1-p} \int_{|x-y|><\frac{1}{n}} \frac{|f(y) - f(x)|^p}{|y-x|^p} dy \to 0.$$

Secondly, the burden of the need for $p > 4/3$ is carried only by $F_3(y)$.

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