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MULTIPLE SOLUTIONS OF A STATIONARY NONHOMOGENEOUS ONE-DIMENSIONAL NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT

In this work we study the multiplicity of solutions for a stationary nonhomogeneous problem associated to the nonlinear one-dimensional Schrödinger Equation. We prove the existence of a number $k_0 \in \mathbb{N}$ such that for every $j > k_0$ there exist at least two solutions of the Dirichlet problem with exactly j zeroes in (a, b). Moreover, if the forcing term f is constant, then for $j = 2n - 1 > k_0$ these solutions are T-periodic with $T = \frac{b-a}{n}$.

1. INTRODUCTION

This paper is devoted to the study of boundary value problems related to the nonlinear Schrödinger equation

$$\partial_t \phi = i\partial_x^2 \phi + i|\phi|^{2\sigma}\phi \tag{1}$$

with initial condition

$$\phi(x, 0) = \phi_0(x)$$

for $\sigma > 0$ and ϕ a function in an appropriate Sobolev Space. Problem (1) arises on the propagation of electromagnetic waves in a nonlinear medium, as a laser beam in an optical fiber [C], [Be], [L], [R], [S], [W1-3].

Under appropriate conditions existence results of local [B] and global [K] solutions can be proved using different conservations laws, such as mass and energy conservation given by the following functionals:

$$\mathcal{N}(\phi) = \frac{1}{2}\|\phi\|_2^2$$

$$\mathcal{H}(\phi) = \frac{1}{2}\|\partial_x \phi\|_2^2 - \frac{1}{2\sigma+2}\|\phi\|_{2\sigma+2}^{2\sigma+2}$$

This leads to the eigenvalue problem

$$\phi''(x) + |\phi(x)|^{2\sigma}\phi(x) = E\phi(x)$$

where the constant E is the energy of the system.

We consider the more general nonhomogeneous problem

$$\phi''(x) + |\phi(x)|^{2\sigma}\phi(x) = f(x) + E(x)\phi(x) \tag{2}$$

where E and f are continuous functions. For example, a nonconstant function $E(x) = E + V(x)$ is obtained when a term $iV(x)\phi(x)$ is added to the right hand side of equation (1).

Under a Dirichlet condition

$$\phi(a) = \phi(b) = 0 \tag{D}$$

we prove the existence of $k_0 \in \mathbb{N}$ such that for every $j > k_0$ there exist at least two solutions of (2-D) with exactly j zeroes in (a, b). Moreover, if E and the forcing term f are constant, these solutions are $\frac{b-a}{n}$- periodic for $j = 2n - 1 > k_0$.

We remark that the zero-order term $g(\phi) = -E\phi + |\phi|^{2\sigma}\phi$ is superlinear, namely

$$\lim_{|\phi|\to\infty} \frac{g(\phi)}{\phi} = +\infty$$

For g *sublinear*, topological and iterative arguments are appliable [AMS] and the solution of (2-D) is unique.

2. MULTIPLE SOLUTIONS OF PROBLEM (2-D)

In this section we prove the existence of infinitely many solutions of (2-D). Our main result is the following:

THEOREM 1

There exists $k_0 \in \mathbb{N}$ *such that for every* $j > k_0$ *problem (2-D) admits at least two solutions with exactly* j *zeroes in* (a, b).

Without loss of generality, we shall assume that $(a, b) = (0, 1)$ and consider for $\lambda \in \mathbb{R}$ ϕ_λ as the unique local solution of the initial value problem

$$\text{(IVP)} \begin{cases} \phi'' + |\phi|^{2\sigma}\phi = E\phi + f \\ \phi(0) = 0, \qquad \phi'(0) = \lambda \end{cases}.$$

Then we have:

LEMMA 2

Let us assume that ϕ_λ *is defined over* $[0, T]$ *for* $T \leq 1$. *Then there exists a positive constant* α *independent of* T *such that for any* R *large it holds:*

$$\text{(3)} \qquad \text{if } \|\phi_\lambda\|_{C^1([0,T])} \geq R, \quad \text{then } \phi_\lambda^2 + (\phi_\lambda')^2 \geq R^\alpha \quad \text{on } [0, T]$$

Proof

We may assume that $\|\phi_\lambda\|_{C^1([0,T])} = \max\{\|\phi_\lambda\|_\infty, \|\phi_\lambda'\|_\infty\} = R$. Integrating equation (2) we obtain:

$$\text{(4)} \qquad (\phi_\lambda')^2 + \frac{|\phi_\lambda|^{2\sigma+2}}{\sigma+1} = \lambda^2 + 2\int_0^x \left(E\phi_\lambda\phi_\lambda' + f\phi_\lambda'\right)$$

As $\lambda^2 = \phi_\lambda'(0)^2 \leq R^2$, we deduce for R large the existence of a constant c depending only on $\|f\|_\infty$ and $\|E\|_\infty$ such that

$$\frac{|\phi_\lambda(x)|^{2\sigma+2}}{\sigma+1} \leq cR^2$$

for any $x \in [0, T]$, and hence

$$\|\phi_\lambda\|_\infty \leq c_1 R^{\frac{1}{\sigma+1}} < R$$

Thus, for some x_0 it holds that

$$|\phi_\lambda'(x_0)| = R$$

Replacing into (4), we obtain:

$$R^2 \leq (\phi_\lambda')^2(x_0) + \frac{|\phi_\lambda(x_0)|^{2\sigma+2}}{\sigma+1} \leq \lambda^2 + c_2 R^{1+\frac{1}{\sigma+1}} + c_3 R \leq \lambda^2 + \frac{R^2}{2}$$

if R is large enough. Hence,

$$\lambda^2 \geq \frac{R^2}{2}$$

Let us fix a constant α with $\alpha < \frac{1}{\sigma+1}$. Then, if x is such that $\phi_\lambda(x)^2 < R^\alpha$, we have that

$$\phi'_\lambda(x)^2 = \lambda^2 + 2\int_0^x \left(E\phi_\lambda\phi'_\lambda + f\phi'_\lambda\right) - \frac{|\phi_\lambda(x)|^{2\sigma+2}}{\sigma+1}$$

$$\geq \frac{R^2}{2} - c_4 R^{1+\frac{1}{\sigma+1}} - \frac{R^{2\alpha(\sigma+1)}}{\sigma+1} \geq R^\alpha$$

for R large, and the proof is complete. □

As a simple consequence we have:

LEMMA 3

ϕ_λ is defined over $[0,1]$

Proof

In the situation of the previous lemma, it suffices to choose R large such that $R^\alpha > \lambda^2$. Thus, $\|\phi_\lambda\|_{C^1([0,T])} \leq R$ for any $T \leq 1$ and the proof follows from classical results in ODE's theory. □

LEMMA 4

Let λ be large and $\mathcal{A}$ be the set defined by

$$\mathcal{A} = \{x \in [0,1] : \phi_\lambda(x) = 0 \text{ or } \phi'_\lambda(x) = 0\}$$

Hence

$$\mathcal{A} = \{0 = x_0 < x_1 < \ldots < x_N\},$$

where zeroes and critical points alternate. More precisely,

$$\phi_\lambda(x_{2j}) = 0 \neq \phi'_\lambda(x_{2j})$$

$$\phi'_\lambda(x_{2j+1}) = 0 \neq \phi_\lambda(x_{2j+1})$$

Proof

From (3) it follows that ϕ_λ and ϕ'_λ cannot vanish at the same point, and it is clear that between two zeroes ϕ_λ has a critical point. On the other hand, if $\phi'_\lambda(x) = 0$ then $|\phi_\lambda(x)|$ is large, and from equation (2), it follows that

$$sgn[\phi''_\lambda(x)] = -sgn[\phi_\lambda(x)]$$

Hence, if $\phi_\lambda(x) > 0$ then x is a maximum, and if $\phi_\lambda(x) < 0$ then x is a minimum, proving that between two critical points ϕ_λ has a zero. □

Let us define

$$\varphi(\lambda) = \frac{1}{2\pi}\int_0^1 \frac{(\phi'_\lambda)^2 + |\phi_\lambda|^{2\sigma+2} - \phi_\lambda(f + E\phi_\lambda)}{\phi_\lambda^2 + (\phi'_\lambda)^2}$$

By Lemma 2, if $|\lambda|$ is large, then $\varphi(\lambda)$ is well defined and the integrand is positive. A simple computation shows that

$$\frac{(\phi'_\lambda)^2 + |\phi_\lambda|^{2\sigma+2} - \phi_\lambda(f + E\phi_\lambda)}{\phi_\lambda^2 + (\phi'_\lambda)^2} = \frac{(\phi'_\lambda)^2 - \phi_\lambda\phi''_\lambda}{\phi_\lambda^2 + (\phi'_\lambda)^2} = \left[arctg(\frac{\phi'_\lambda}{\phi_\lambda})\right]'$$

Hence, $\varphi(\lambda)$ measures the net fraction of turns that the curve $\Phi_\lambda(x) = (\phi'_\lambda(x), \phi_\lambda(x))$ performs around the origin when x moves between 0 and 1. From the previous considerations it follows that Φ_λ winds clockwise, without passing through $(0,0)$. Moreover, each complete turn -starting at the point $(\lambda, 0)$- corresponds to a pair of zeros of ϕ_λ, which are obtained when $\Phi_\lambda(x)$ intersects the x-axis.

LEMMA 5

$$\varphi(\lambda) \to +\infty \qquad as \qquad |\lambda| \to \infty$$

Proof

Let $\mathcal{A}$ be as in lemma 4, and let $x_k, x_{k+1} \in \mathcal{A}$. We shall prove that $x_{k+1} - x_k \to 0$ as $|\lambda| \to \infty$. From the computations in Lemma 2 we know that for $\|\phi_\lambda\|_{C^1([0,1])} \geq R$ it holds that

$$(\phi'_\lambda)^2 \geq cR^2 - \frac{|\phi_\lambda|^{2\sigma+2}}{\sigma+1}$$

and

$$\|\phi_\lambda\|_\infty \leq \overline{c}R^{\frac{1}{\sigma+1}}$$

for some positive constants $c, \bar{c}$. Let

$$I_c = \{x \in [x_k, x_{k+1}] : |\phi_\lambda(x)| \leq \left(cR^2(\sigma+1)\right)^{\frac{1}{2\sigma+2}} := M_c\}$$

As $\phi'_\lambda \neq 0$ on (x_k, x_{k+1}) we may assume without loss of generality that $\phi'_\lambda(x_{k+1}) = 0$, which implies that $I_c = [x_k, x_c]$ for some x_c. For $x \in I_c$ we have:

$$|\phi'_\lambda(x)| \geq \sqrt{cR^2 - \frac{|\phi_\lambda(x)|^{2\sigma+2}}{\sigma+1}}$$

and hence

$$x_c - x_k = \int_{x_k} x_c dx \leq \int_0^{M_c} \frac{d\phi}{\sqrt{cR^2 - \frac{\phi^{2\sigma+2}}{\sigma+1}}}$$

Substituting by $z = \frac{\phi}{M_c}$ we obtain that

$$x_c - x_k \leq \frac{M_c}{\sqrt{c}R} \int_0^1 \frac{dz}{\sqrt{1 - z^{2\sigma+2}}} \to 0$$

as $R \to \infty$. On the other hand, on $[x_c, x_{k+1}]$ we have:

$$|\phi''_\lambda| = |f + E\phi_\lambda - |\phi_\lambda|^{2\sigma}\phi_\lambda| \geq \frac{M_c^{2\sigma+1}}{2}$$

for R large. Then, for any $x \in [x_c, x_{k+1}]$

$$|\phi'_\lambda(x)| = \int_x^{x_{k+1}} |\phi''_\lambda| \geq \frac{M_c^{2\sigma+1}}{2}(x_{k+1} - x)$$

Thus,

$$|\phi_\lambda(x_{k+1}) - \phi_\lambda(x_c)| = \int_{x_c}^{x_{k+1}} |\phi'_\lambda| \geq \frac{M_c^{2\sigma+1}(x_{k+1} - x_c)^2}{4}$$

As

$$|\phi_\lambda(x_{k+1}) - \phi_\lambda(x_c)| \leq 2|\phi_\lambda(x_{k+1})| \leq 2\bar{c}R^{\frac{1}{\sigma+1}}, \qquad M_c^{2\sigma+1} = \tilde{c}R^{\frac{2\sigma+1}{\sigma+1}}$$

we conclude that $x_{k+1} - x_c \to 0$, and the proof is complete. □

Proof of Theorem 1

From classical ODE's theory we know that $\Phi_{\overline{\lambda}} \to \Phi_\lambda$ uniformly on $[0, 1]$ as $\overline{\lambda} \to \lambda$. Hence, φ is continuous, and by Lemma 5 we deduce that taking k_0 large enough, for any $j > k_0$ there exist $\lambda_j^- < 0 < \lambda_j^+$ such that

$$\varphi(\lambda_j^\pm) = \frac{j}{2}$$

This completes the proof. □

3. MULTIPLE PERIODIC SOLUTIONS OF (2)

In this section we prove the existence of infinitely many periodic solutions of (2) for constant E and f. More precisely,

THEOREM 6

There exists $k_0 \in \mathbb{N}$ such that for every $j > k_0$ odd equation (2) admits at least two periodic solutions with exactly j zeroes in (a,b) and period $T = 2\frac{b-a}{j+1}$.

Proof

Let k_0 be as in Theorem 5, and $j > k_0$ odd. As f and E are constant, if ϕ is a solution of (2-D) with j zeroes on (a,b) then

$$(\phi')^2 + \frac{|\phi|^{2\sigma+2}}{\sigma+1} = \phi'(a)^2 + 2f\phi + E\phi^2$$

Hence, if

$$a = c_0 < c_1 < ... < c_{j+1} = b$$

are the zeroes of ϕ we obtain that

$$\phi'(c_2)^2 = \phi'(a)^2$$

Furthermore, since ϕ vanishes only once in (a, c_2) it is clear that $\phi'(c_2) = \phi'(a)$. If we define

$$\psi(x) = \phi(x + c_2 - a)$$

then ψ satisfies (2) with

$$\psi(a) = \phi(c_2) = 0, \qquad \psi'(a) = \phi'(c_2) = \phi'(a)$$

By uniqueness, $\psi = \phi$. It follows that ϕ is $(c_2 - a)$-periodic, and hence

$$c_{2k} = a + k(c_2 - a)$$

In particular,

$$b = a + \frac{j+1}{2}(c_2 - a)$$

which completes the proof. □

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