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ON BEST LOCAL APPROXIMANTS IN $L_2(\mathbb{R}^n)$

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ABSTRACT. Given $f \in L_2(\mathbb{R}^n)$, $\epsilon > 0$ and $x \in \mathbb{R}^n$ we consider $P(t)$ to be the polynomial of best approximation to f in the L_2-norm by elements of $\Pi_m(t_1, ..., t_n)$ over the set $x + \epsilon C$, where C denote a suitable parallelepiped. Let $T^\epsilon_\alpha f(x)$ be the α^{th}-coefficient of P when it has been developed in the base $\{t^\alpha/\alpha!\}$. In this paper we show that the operator T^ϵ_α is a composition of a convolution operator with the differential operator D_α. As a consequence of this fact we can extend the operator T^ϵ_α to $L_1 + L_\infty$ and obtain an answer to questions on a.e. convergence and maximal inequalities in the setting of the L_2-norm.

INTRODUCTION AND NOTATIONS

In this paper we consider a problem related to best local approximation. The notion may be stated as follows.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function in a normed linear space X with norm $\|.\|$. Let V denote a subset of X. Let $C(\boldsymbol{x}, \epsilon)$ denote a net set containing $\boldsymbol{x}$ with diameters shrinking to 0 as $\epsilon \to 0$. For each $\epsilon > 0$ suppose that we have $f_\epsilon \in V$ which minimizes $\|(f - g)\chi_{C(x,\epsilon)}\|$ for $g \in V$, where $\chi_{C(x,\epsilon)}$ is the characteristic function of $C(\boldsymbol{x}, \epsilon)$. If $f_\epsilon \to f_x \in V$ then f_x is said to be the best local approximant of f at $\boldsymbol{x}$. In [2] Chui, Diamond and Raphael proved that if $f \in C^{m+1}(\mathbb{R}^n)$, $\|.\|$ is the p-norm and the subspace $V \subset C^{m+1}(\mathbb{R}^n)$ is uniquely interpolating at $\boldsymbol{x}$ of order m then the best local approximant of f at $\boldsymbol{x}$ from V is the unique $f_x \in V$ whose derivatives up to order m match those of f at $\boldsymbol{x}$.

In two papers, Macías and Zó([3]) and Zó([7]) extended, for V certain subspace of polynomials, the previous results to weighted L^p spaces. They worked with functions which satisfy a weaker condition than to be $m + 1$ times differentiable. In fact, they considered functions f with a derivative in the L^p sense([1], [5]).

In this paper we denote by $\alpha, \beta, ...$ n-tuples of nonnegative integers. For a vector $\boldsymbol{x} = (x_1, ..., x_n)$ in $\mathbb{R}^n$ and a n-tuple α as usual we write $\boldsymbol{x}^\alpha = x_1^{\alpha_1} ... x_n^{\alpha_n}$,

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$|\alpha| = \alpha_1 + ... + \alpha_n$ and $\alpha! = \alpha_1!...\alpha_n!$. Briefly we write 0 for the n-tuple with all the coordinates zero.

Also we consider the particular case when $V = \Pi_m(x_1, ..., x_n)$, is the space of all polynomials in n variables of degree at most m. Let $C(\boldsymbol{x}, \epsilon)$ be a family of subsets of $\mathbb{R}^n$ as above. Given $\epsilon > 0$, $\boldsymbol{x} \in \mathbb{R}^n$ and $1 < p < \infty$ let $P := P_{\epsilon,x}$ be the best p-approximant polynomial to f in $\Pi_m(x_1, ..., x_n)$ over the $C(\boldsymbol{x}, \epsilon)$, i.e.

$$\int_{C(x,\epsilon)} |f - P|^p \leq \int_{C(x,\epsilon)} |f - Q|^p \quad \text{for every } Q \in \Pi_m(x_1, ..., x_n).$$

Now we write the polynomial P in the following manner

$$P(\boldsymbol{t}) := \sum_{|\alpha| \leq m} \frac{T_\alpha^\epsilon f(\boldsymbol{x})}{\alpha!} (\boldsymbol{t} - \boldsymbol{x})^\alpha.$$

According to [7] and using properties of functions in a Sobolev space ([1],[5]) we have that if $f \in W^{m,2}(\mathbb{R}^n)$ then

$$T_\alpha^\epsilon f(\boldsymbol{x}) \to D_\alpha f(\boldsymbol{x}) \quad \text{a.e. when } \epsilon \to 0.$$

Henceforth we assume that the set $C(\boldsymbol{x}, \epsilon)$ is of the form $\boldsymbol{x} + \epsilon C$, where C is the parallelepiped $[a_1, b_1] \times \cdots \times [a_n, b_n]$. We observe that x might not belong to $C(\boldsymbol{x}, \epsilon)$.

In order to motivate our approach to this subject we consider the following simple example. When $p = 2$ and $m = 0$ we have

$$T_0^\epsilon f(\boldsymbol{x}) = \frac{1}{|C(\boldsymbol{x}, \epsilon)|} \int_{C(x,\epsilon)} f(\boldsymbol{t}) d\boldsymbol{t},$$

where $|C|$ denotes the Lebesgue measure of the set C. This equation shows that we can extend the operator T_0^ϵ to the space $(L_1 + L_\infty)(\mathbb{R}^n)$. In this case there is a classical result of convergence, in fact it is well known by the Lebesgue Differentiation Theorem that $T_0^\epsilon f(\boldsymbol{x}) \to f(\boldsymbol{x})$ a.e. for any function $f \in L_1 + L_\infty$. These questions can be studied using the maximal function of Hardy-Littlewood defined by $T_0^* f(\boldsymbol{x}) := \sup_{\epsilon>0} |T_0^\epsilon f(\boldsymbol{x})|$. It satisfies two inequalities of weak-type $(1, 1)$ and strong-type (p, p) for $p > 1$.

The aim of this paper is to extend the Lebesgue Differentiation Theorem and the maximal inequalities from the point of view of the best local approximants. Following the previous example we have arrived to the next three conjectures:

Suppose $1 < p < \infty$, m a nonnegative integer, α a multi-index, with $|\alpha| \leq m$.

C1) We can extend the operator T_α^ϵ in a "natural" manner to the space $(L_{p-1} + L_\infty)(\mathbb{R}^n)$.

C2) We define the following maximal functions

$$T_\alpha^* f(x) = \sup_{\epsilon>0} |T_\alpha^\epsilon f(x)|.$$

Then the operator T_α^* satisfies the following weak inequality

$$|\{x : T_\alpha^* f(x) > \lambda\}| \leq \frac{A}{\lambda^{p-1}} \|f\|_{W^{|\alpha|,p-1}}^{p-1}$$

for every $f \in W^{|\alpha|,p-1}$, where the constant A depends only of n, m, p and α. We observe that when $p < 2$ the space $W^{|\alpha|,p-1}$ is not a normed space. However we also use this notation in this case.

C3) Let $f \in W^{k,p\prime}(\mathbb{R}^n), p\prime \geq p-1$. Then for $|\alpha| \leq k, T_\alpha^\epsilon f(x) \to D_\alpha f(x)$ a.e., as $\epsilon \to 0$, where $D_\alpha f$ now denotes the α^{th} weak derivative of f.

We observe that in the case $p\prime = p$, the conjecture C3) was obtained in [7]. In a previous paper ([4]) we have proved these conjectures when $m = 0$ and $p > 1$. In the present paper we prove the conjectures when $p = 2$ and m is an arbitrary nonnegative integer. It will be an immediate consequence of the fact that the operator T_α^ϵ is a composition of the differentiation operator D_α with a convolution.

Proof of the conjectures in the case $p = 2$

Suppose that the set of multi-index α, with $|\alpha| \leq m$, is ordered in some way. We define the following square matrix of order $\binom{m+n-1}{m}$

$$C_{\alpha,\beta} := \frac{1}{\alpha!\beta!}\int_C t^{\alpha+\beta}dt,$$

As the matrix $C_{\alpha,\beta}$ is the Gramm matrix of the system $\{t^\alpha/\alpha!\}_{|\alpha|\leq m}$, basis of $\Pi_m(x_1, ..., x_n)$, it is invertible. We call $A_{\alpha,\beta}$ the inverse matrix of $C_{\alpha,\beta}$.

Next for each multi-index γ, with $|\gamma| \leq m$, we define the functions

$$K[\gamma, t] := \sum_{|\alpha|\leq m} \frac{A_{\alpha,\gamma}}{\alpha!}t^\alpha. \tag{1}$$

Clearly the functions $K[\gamma, t]$ satisfy the equation

$$\int_C K[\gamma, t]t^\beta dt = \delta_{\gamma,\beta}\beta!. \tag{2}$$

Lemma 1. *Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a multi-index. Suppose $\gamma_j \neq 0$. Then*

$$\int_{a_j}^{b_j} t_j^i K[\gamma, t]dt_j \equiv 0 \text{ for } i = 0, \ldots, \gamma_j - 1 \text{ and for every } t_1, \ldots t_{j-1}, t_{j+1} \ldots t_n. \tag{3}$$

Proof. Without loss of generality we may assume $j = 1$. We proceed by induction on i. Assume $i = 0$. From (2) we obtain for every multi-index β of the form $(0, \beta_2, ..., \beta_n)$, with $|\beta| \leq m$

$$\int_{[a_2,b_2]\times\cdots\times[a_n,b_n]} t_2^{\beta_2}...t_n^{\beta_n} \int_{a_1}^{b_1} K[\gamma, t]dt_1...dt_n = 0.$$

Therefore the following function of $\Pi_m(t_2, ..., t_n)$

$$\int_{a_1}^{b_1} K[\gamma, t]dt_1$$

is orthogonal to a basis of $\Pi_m(t_2, ..., t_n)$. Hence we get (3) for $i = 0$.
Suppose that for every $0 \leq j \leq i - 1$, with $i < \gamma_1$, we have

$$\int_{a_1}^{b_1} t_1^j K[\gamma, \boldsymbol{t}] dt_1 \equiv 0. \tag{4}$$

Write

$$K[\gamma, \boldsymbol{t}] = \sum_{\alpha_2 + ... + \alpha_n \leq m} \left[\sum_{0 \leq \alpha_1 \leq m - \alpha_2 ... - \alpha_n} \frac{A_{\alpha,\gamma}}{\alpha!} t_1^{\alpha_1} \right] t_2^{\alpha_2} ... t_n^{\alpha_n}.$$

From (4) we obtain for $0 \leq j \leq i - 1$

$$\int_{a_1}^{b_1} t_1^j \left[\sum_{0 \leq \alpha_1 \leq m - \alpha_2 ... - \alpha_n} \frac{A_{\alpha,\gamma}}{\alpha!} t_1^{\alpha_1} \right] dt_1 = 0. \tag{5}$$

In fact, the first member of (5) is a coefficient of the polynomial (4), which is zero. Thus if $\alpha_2 + ... + \alpha_n > m - i$ the polynomial of $\Pi_{i-1}(t_1)$

$$\sum_{0 \leq \alpha_1 \leq m - \alpha_2 ... - \alpha_n} \frac{A_{\alpha,\gamma}}{\alpha!} t_1^{\alpha_1}$$

is orthogonal by (5) to $\Pi_{i-1}(t_1)$, then we conclude that it is zero. As a consequence

$$\int_{a_1}^{b_1} t_1^i K[\gamma, \boldsymbol{t}] dt_1 \in \Pi_{m-i}(t_2, ..., t_n).$$

For every multi-index $(\beta_2, ..., \beta_n)$ with $\beta_2 + ... + \beta_n \leq m - i$, as $i < \gamma_1$ we have from (2)

$$\int_{[a_2,b_2] \times \cdots \times [a_n,b_n]} t_2^{\beta_2} ... t_n^{\beta_n} \int_{a_1}^{b_1} t_1^i K[\gamma, \boldsymbol{t}] dt_1 ... dt_n = 0.$$

As before the polynomial of $\Pi_{m-i}(t_2, ..., t_n)$

$$\int_{a_1}^{b_1} t_1^i K[\gamma, \boldsymbol{t}] dt_1$$

is orthogonal to $\Pi_{m-i}(t_2, ..., t_n)$, therefore it is zero $\square$.

Now for each multi-index γ we shall define a polynomial $P[\gamma, \boldsymbol{t}]$. If $\gamma = 0$ we set $P[0, \boldsymbol{t}] = K[0, \boldsymbol{t}]$. Assume $\gamma \neq 0$. For simplicity in the notations suppose that $\gamma = (\gamma_1, ..., \gamma_k, 0, ..., 0)$ with $\gamma_i \neq 0$ for $1 \leq i \leq k$, i.e., the only non-zero coordinates are the first ones, otherwise proceed analogously. Define

$$(6) \quad P[\gamma, \boldsymbol{t}] := \int_{a_1}^{t_1} \cdots \int_{a_k}^{t_k} G[\gamma, \boldsymbol{t}, s_1, ..., s_k] K[\gamma, s_1, ..., s_k, t_{k+1}, ..., t_n] ds_1 ... ds_k.$$

where

$$G[\gamma, \boldsymbol{t}, s_1, ..., s_k] = \frac{(t_1 - s_1)^{\gamma_1 - 1}}{(\gamma_1 - 1)!} \cdots \frac{(t_k - s_k)^{\gamma_k - 1}}{(\gamma_k - 1)!}.$$

The polynomial P has the following properties

$$(7) \qquad D_\gamma P[\gamma, \boldsymbol{t}] = K[\gamma, \boldsymbol{t}]$$

and for every multi-index γ' with $\gamma'_j \leq \gamma_j$, $1 \leq j \leq n$ and for each i such that $\gamma'_i \neq \gamma_i$ we have

$$(8) \qquad D_{\gamma'} P[\gamma, t_1, ..., t_{i-1}, a_i, t_{i+1}, ..., t_n] \equiv 0$$

and

$$(9) \qquad D_{\gamma'} P[\gamma, t_1, ..., t_{i-1}, b_i, t_{i+1}, ..., t_n] \equiv 0.$$

In fact (7) and (8) follow directly from the definition by the Leibniz's rule of derivation of the integral (6), while (9) is obtained expanding $(b_i - s_i)^{\gamma_i - \gamma'_i - 1}$ by the binomial formulae, using the mentioned rule and Lemma 1.

Define $Q[\gamma, \boldsymbol{t}]$ by $Q[\gamma, \boldsymbol{t}] = (-1)^{|\gamma|} P[\gamma, -\boldsymbol{t}]$ if $\boldsymbol{t} \in C$ and $Q[\gamma, \boldsymbol{t}] = 0$ otherwise. Moreover write $Q_\epsilon[\gamma, \boldsymbol{t}] := \frac{1}{\epsilon^n} Q[\gamma, \frac{\boldsymbol{t}}{\epsilon}]$.

Theorem 2. *Suppose $f \in L_2(\mathbb{R}^n)$ and that there exist the weak derivatives $D_\alpha f$ for $|\alpha| \leq m$ which are locally integrable functions. Then for every γ, $|\gamma| \leq m$*

$$(10) \qquad T_\gamma^\epsilon f(\boldsymbol{x}) = \left(D_\gamma f * Q_\epsilon[\gamma, .]\right) = \frac{1}{\epsilon^n} \int D_\gamma f(\boldsymbol{t}) Q[\gamma, \frac{\boldsymbol{x} - \boldsymbol{t}}{\epsilon}] dt.$$

where the integral is taken on all $\mathbb{R}^n$.

Proof. It is well known that the best approximant polynomial is the orthogonal projection on $\Pi_m(t_1, ..., t_n)$. Then the coefficients $T_\alpha^\epsilon f(\boldsymbol{x})$ satisfy the linear system

$$(11) \qquad \sum_{|\alpha| \leq m} T_\alpha^\epsilon f(\boldsymbol{x}) \epsilon^{|\alpha| + |\beta| + n} C_{\alpha, \beta} = \int_{C(x, \epsilon)} f(\boldsymbol{t}) \frac{(\boldsymbol{t} - \boldsymbol{x})^\beta}{\beta!} dt, \qquad \text{for } |\beta| \leq m.$$

Now multiply (11) by $A_{\beta, \gamma} / \epsilon^{|\beta|}$ and adding on β for $|\beta| \leq m$ we get

$$T_\gamma^\epsilon f(\boldsymbol{x}) = \frac{1}{\epsilon^{|\gamma|+n}} \int_{C(\boldsymbol{x},\epsilon)} f(\boldsymbol{t}) K[\gamma, \frac{\boldsymbol{t}-\boldsymbol{x}}{\epsilon}] d\boldsymbol{t}. \tag{12}$$

Using integration by parts, (8) and (9) we obtain (10) □.

According to (12) we can extend the operator T_α^ϵ in a "natural" manner to the space $(L_1 + L_\infty)(\mathbb{R}^n)$. Therefore we have that C1) is true for the case $p = 2$.

In virtue of the representation (10) we can prove immediately the other conjectures C2) and C3) for the case $p = 2$. More precisely

Theorem 3. *Let $f \in W^{|\alpha|,1}$. Then*

1) *The operator T_α^* satisfies the following weak inequality*

$$|\{x : T_\alpha^* f(x) > \lambda\}| \leq \frac{A}{\lambda} \|D_\alpha f\|_1 \quad \text{for } \lambda > 0$$

where the constant A depends only of n, m, and α.

2) $T_\alpha^\epsilon f(\boldsymbol{x}) \to D_\alpha f(\boldsymbol{x})$ *a.e., when $\epsilon \to 0$.*

Proof. Property 1) follows from Zó's Theorem (see [6] Corollary 2.3, p. 284) and (10). From [6] (Corollary 2.4, p.284) and Theorem 2 we obtain

$$T_\alpha^\epsilon f(\boldsymbol{x}) \to D_\alpha f(x) \int Q[\alpha, \boldsymbol{t}] d\boldsymbol{t} \text{ a.e..}$$

Now taking $x = 0$, $\epsilon = 1$, $\alpha = \gamma$ and $f(\boldsymbol{t}) = \boldsymbol{t}^\alpha/\alpha!$ in (10) we get $\int Q[\alpha, \boldsymbol{t}] d\boldsymbol{t} = T_\alpha^1(\boldsymbol{t}^\alpha/\alpha!) = 1$ □.

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