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LINEAR COMBINATION OF A NEW SEQUENCE OF LINEAR POSITIVE OPERATORS

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ABSTRACT. In the present paper, we study the approximation of unbounded continuous functions of exponential growth by the linear combination of a new sequence of linear positive operators. First, we discuss a Voronoskaja type asymptotic formula and then obtain an error estimate in terms of the higher order modulus of continuity of the function being approximated.

1. INTRODUCTION

In [1] we introduced a new sequence of linear positive operators M_n to approximate a class of unbounded continuous functions of exponential growth on the interval $[0,\infty)$ as follows:

Let $\alpha > 0$ and $f \in C_\alpha[0,\infty) = \{f \in C[0,\infty) : |f(t)| \le M\, e^{\alpha t} \text{ for some } M > 0\}$. Then,

$$M_n(f(t);x) = n\sum_{\nu=1}^{\infty} p_{n,\nu}(x) \int_0^{\infty} q_{n,\nu-1}(t) f(t)dt + (1+x)^{-n} f(0), \tag{1.1}$$

where $p_{n,\nu}(x) = \binom{n+\nu-1}{\nu} x^\nu (1+x)^{-n-\nu}, x \in [0,\infty)$, and $q_{n,\nu}(t) = \dfrac{e^{-nt}(nt)^\nu}{\nu!}$, $t \in [0,\infty)$.

The space $C_\alpha[0,\infty)$ is normed by $\|f\|_{C_\alpha} = \sup_{0\le t<\infty} |f(t)| e^{-\alpha t}, f \in C_\alpha[0,\infty)$. Alternatively, the operator (1.1) may be written as $M_n(f(t);x) = \int_0^{\infty} W_n(t,x) f(t)dt$, where the kernel

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$W_n(t,x) = n\sum_{\nu=1}^{\infty} p_{n,\nu}(x) q_{n,\nu-1}(t) + (1+x)^{-n}\delta(t)$, $\delta(t)$ being the Dirac-delta function.

The operator (1.1) was studied for degree of approximation in simultaneous approximation in [1]. It turned out that the order of approximation of the operator (1.1) is, at best, $O(n^{-1})$ howsoever smooth the function may be. Therefore, in order to improve the rate of convergence of the operators (1.1), we apply the technique of linear combination introduced by May [4] and Rathore [5] to these operators. The approximation process is defined as:

Following Agrawal and Thamer [2], the linear combination $M_n(f,k,x)$ of $M_{d_j n}(f;x)$, $j=0,1,\ldots,k$ is defined as:

(1.2) $$M_n(f,k,x) = \frac{1}{\Delta}\begin{vmatrix} M_{d_0 n}(f;x) & d_0^{-1} & d_0^{-2} & \ldots & d_0^{-k} \\ M_{d_1 n}(f;x) & d_1^{-1} & d_1^{-2} & \ldots & d_1^{-k} \\ \ldots\ldots & \ldots & \ldots & \ldots & \ldots \\ M_{d_k n}(f;x) & d_k^{-1} & d_k^{-2} & \ldots & d_k^{-k} \end{vmatrix},$$

where $d_0, d_1, \ldots, d_k$ are $k+1$ arbitrary but fixed distinct positive integers and Δ is the Vandermonde determinant obtained by replacing the operator column of the above determinant with the entries 1. On simplification, (1.2) is reduced to

(1.3) $$M_n(f,k,x) = \sum_{j=0}^{k} C(j,k) M_{d_j n}(f;x),$$

where $C(j,k) = \begin{cases} \prod\limits_{\substack{i=0\\ i\neq j}}^{k} \dfrac{d_j}{d_j - d_i} & ,k \neq 0 \\ 1 & ,k = 0 \end{cases}$.

The object of the present paper is to show that by taking $(k+1)^{th}$ linear combination of the operators (1.1), $O(n^{-(k+1)})$ rate of convergence can be achieved for $(2k+2)$ times continuously differentiable functions on $[0,\infty)$. Also, the determinant form (1.2) of the linear combination makes the determination of the polynomials $Q(2k+1,k,x)$ and $Q(2k+2,k,x)$ occurring in the following Theorem 1 of this paper quite easy.

2. DEGREE OF APPROXIMATION

Throughout our work, let N^0 denote the set of nonnegative integers, $0 < a_1 < a_2 < b_2 < b_1 < \infty$ and $\| \cdot \|_{C[a,b]}$, the sup-norm on $C[a,b]$. To make the paper self contained, we restate below two lemmas from our paper [1].

Lemma 1. Let the m^{th} order moment $(m\in N^0)$ for the operators (1.1) be defined by

$$T_{n,m}(x)=M_n((t-x)^m;x)=n\sum_{\nu=1}^{\infty}p_{n,\nu}(x)\int_0^{\infty}q_{n,\nu-1}(t)(t-x)^m\,dt+(-x)^m(1+x)^{-n}.$$

Then $T_{n,0}(x)=1$, $T_{n,1}(x)=0$ and

$$nT_{n,m+1}(x)=x(1+x)T'_{n,m}(x)+mT_{n,m}(x)+mx(x+2)T_{n,m-1}(x),\ \ m\geq 1.$$

Further, we have the following consequences of $T_{n,m}(x)$:

(i) $T_{n,m}(x)$ is a polynomial in x of degree m, $m\neq 1$;

(ii) for every $x\in[0,\infty)$, $T_{n,m}(x)=O(n^{-[(m+1)/2]})$;

(iii) the coefficients of n^{-k} in $T_{n,2k}(x)$ and $T_{n,2k-1}(x)$ are $(2k-1)!!\ \{x(x+2)\}^k$ and $Cx^k(x+2)^{k-1}(x^2+3x+3)$ respectively, where C is a constant depending only on k and !! denotes the semi-factorial function.

Lemma 2. Let δ and γ be any two positive real numbers and $[a,b]\subset(0,\infty)$. Then, for any $m>0$ we have,

$$\sup_{x\in[a,b]}\left|n\sum_{\nu=1}^{\infty}p_{n,\nu}(x)\int_{|t-x|\geq\delta}q_{n,\nu-1}(t)\,e^{\gamma t}\,dt\right|=O(n^{-m}).$$

First, we prove the Voronoskaja type asymptotic result for the operator $M_n(f,k,x)$.

THEOREM 1. Let $f\in C_\alpha[0,\infty)$ and $f^{(2k+2)}$ exists at a point $x\in[0,\infty)$. Then

(2.1) $$\lim_{n\to\infty}n^{k+1}[M_n(f,k,x)-f(x)]=\sum_{m=k+2}^{2k+2}\frac{f^{(m)}(x)}{m!}\,Q(m,k,x)$$

and

(2.2) $$\lim_{n\to\infty}n^{k+1}[M_n(f,k+1,x)-f(x)]=0,$$

where $Q(m,k,x)$ are certain polynomials in x of degree m. Moreover,

$$Q(2k+1,k,x)=\frac{(-1)^k}{\prod_{j=0}^{k}d_j}\,Cx^k\,(x+2)^{k-1}\,(x^2+3x+3)$$

and

$$Q(2k+2,k,x)=\frac{(-1)^k}{\prod_{j=0}^{k}d_j}\,(2k+1)!!\,\{x\,(x+2)\}^{k+1},$$

where C is a constant depending only on k.

Further, if $f^{(2k+1)}$ exists and is absolutely continuous over $[0,b]$ and $f^{(2k+2)}\in L_\infty[0,b]$, then for any $[c,d]\subset(0,b)$ there holds

(2.3) $$\| M_n(f,k,x) - f(x) \|_{C[c,d]} \leq M\, n^{-(k+1)} \left[\|f\|_{C_\alpha} + \left\| f^{(2k+2)} \right\|_{L_\infty[0,b]} \right],$$

where M is a constant independent of f and n.

Proof: Since $f^{(2k+2)}$ exists at $x \in [0,\infty)$, it follows that

$$f(t) = \sum_{m=0}^{2k+2} \frac{f^{(m)}(x)}{m!} (t-x)^m + \varepsilon(t,x)(t-x)^{2k+2},$$

where $\varepsilon(t,x) \to 0$ as $t \to x$.

In view of $M_n(1,k,x) = 1$, we can write

$$n^{k+1}[M_n(f,k,x) - f(x)] = n^{k+1} \sum_{m=1}^{2k+2} \frac{f^{(m)}(x)}{m!} M_n((t-x)^m, k, x)$$

$$+ n^{k+1} \sum_{j=0}^{k} C(j,k) M_{d_j n}(\varepsilon(t,x)(t-x)^{2k+2}; x)$$

$$= I_1 + I_2, \text{ say.}$$

Using Lemma 1, we have

$$T_{d_j n,m}(x) = \frac{P_1(x)}{(d_j n)^{[(m+1)/2]}} + \frac{P_2(x)}{(d_j n)^{[(m+1)/2]+1}} + \ldots + \frac{P_{[m/2]}(x)}{(d_j n)^{m-1}},$$

for certain polynomials $P_i, i = 1,2,\ldots,[m/2]$ in x of degree at most m.

Clearly,

$$\sum_{j=0}^{k} C(j,k) T_{d_j n,m}(x)$$

$$= \frac{1}{\Delta} \begin{vmatrix} \frac{P_1(x)}{(d_0 n)^{[(m+1)/2]}} + \frac{P_2(x)}{(d_0 n)^{[(m+1)/2]+1}} + \ldots + \frac{P_{[m/2]}(x)}{(d_0 n)^{m-1}} & d_0^{-1} & d_0^{-2} & \ldots & d_0^{-k} \\ \frac{P_1(x)}{(d_1 n)^{[(m+1)/2]}} + \frac{P_2(x)}{(d_1 n)^{[(m+1)/2]+1}} + \ldots + \frac{P_{[m/2]}(x)}{(d_1 n)^{m-1}} & d_1^{-1} & d_1^{-2} & \ldots & d_1^{-k} \\ \cdots\cdots\cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{P_1(x)}{(d_k n)^{[(m+1)/2]}} + \frac{P_2(x)}{(d_k n)^{[(m+1)/2]+1}} + \ldots + \frac{P_{[m/2]}(x)}{(d_k n)^{m-1}} & d_k^{-1} & d_k^{-2} & \ldots & d_k^{-k} \end{vmatrix}$$

(2.4)
$$= n^{-(k+1)}\{Q(m,k,x) + o(1)\},\ m = k+2, k+3, \ldots, 2k+2.$$

So, I_1 is determined by $\sum_{m=k+2}^{2k+2} \frac{f^{(m)}(x)}{m!} Q(m,k,x) + o(1)$.

The expression for $Q(2k+1,k,x)$ and $Q(2k+2,k,x)$ can be easily obtained from Lemma 1 in (2.4). Hence in order to prove (2.1) it suffices to show that

$I_2 \to 0$ as $n \to \infty$. For a given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|\varepsilon(t,x)| < \varepsilon$, whenever $|t-x| < \delta$, and for $|t-x| \geq \delta$, there exists a constant $K > 0$ such that $|\varepsilon(t,x)|(t-x)^{2k+2} \leq K e^{\alpha t}$.

Let $\Phi_\delta(t)$ be the characteristic function of the interval $(x-\delta, x+\delta)$, then

$$|I_2| \leq n^{k+1} \sum_{j=0}^{k} |C(j,k)| M_{d_j n}(|\varepsilon(t,x)|(t-x)^{2k+2} \Phi_\delta(t); x)$$

$$+ n^{k+1} \sum_{j=0}^{k} |C(j,k)| M_{d_j n}(|\varepsilon(t,x)|(t-x)^{2k+2} (1-\Phi_\delta(t)); x) := I_3 + I_4.$$

Again, using Lemma 1 we get $I_3 \leq \varepsilon n^{k+1} \left(\sum_{j=0}^{k} |C(j,k)| \right) \max_{0 \leq j \leq k} \left\{ T_{d_j n, 2k+2}(x) \right\} < K_1 \varepsilon$.

Now, applying Schwarz inequality for integration and then for summation and Lemma 2 we are led to

$$I_4 \leq K n^{k+1} \sum_{j=0}^{k} |C(j,k)| M_{d_j n}(e^{\alpha t} (1-\Phi_\delta(t)); x) = n^{k+1} O(n^{-m}), \text{ for any } m > 0.$$

$$= O(n^{k+1-m}) = o(1) \text{ for } m > k+1.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $I_3 \to 0$ for sufficiently large n. Combining the estimates of I_3 and I_4 we conclude that $I_2 \to 0$ as $n \to \infty$. The assertion (2.2) can be proved in a similar manner as $M_n((t-x)^m, k+1, x) = O(n^{-(k+2)})$, for all $m = k+3, k+4, \ldots, 2k+2$.

Now, we shall prove (2.3). Let $\Psi(t)$ be the characteristic function of [0,b], then

$$M_n((f,k,x) = M_n(\Psi(t)(f(t)-f(x)), k, x) + M_n((1-\Psi(t))(f(t)-f(x)), k, x)$$

$$:= I_5 + I_6.$$

Proceeding as in the estimate of I_4, we have for all $x \in [c,d]$,

$$I_6 \leq \|f\|_{C_\alpha} O(n^{-m}), \text{ where } m > 0.$$

From the hypothesis on f, we can write, for all $t \in [0,b]$ and $x \in [c,d]$,

$$f(t) - f(x) = \sum_{i=1}^{2k+1} \frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{1}{(2k+1)!} \int_x^t (t-w)^{2k+1} f^{(2k+2)}(w)\, dw.$$

Therefore

$$I_5 = \sum_{i=1}^{2k+1} \frac{f^{(i)}(x)}{i!} M_n(\Psi(t)(t-x)^i, k, x)$$

$$+ \frac{1}{(2k+1)!} M_n(\Psi(t) \int_x^t (t-w)^{2k+1} f^{(2k+2)}(w)\, dw, k, x)$$

$$= \sum_{i=1}^{2k+1} \frac{f^{(i)}(x)}{i!} \left\{ M_n((t-x)^i, k, x) + M_n((\Psi(t)-1)(t-x)^i, k, x) \right\}$$

$$+ \frac{1}{(2k+1)!} M_n(\Psi(t) \int_x^t (t-w)^{2k+1} f^{(2k+2)}(w)\, dw, k, x)$$

$$:= \sum_{i=1}^{2k+1} \frac{f^{(i)}(x)}{i!} \{I_7 + I_8\} + I_9.$$

In view of (2.4), we have $I_7 = O(n^{-(k+1)})$, uniformly for all $x \in [c,d]$. Since $\Psi(t)$ is the characteristic function of $[0, \mathrm{b}]$ and $x \in [c,d]$, we can choose $\delta > 0$ such that $|t-x| \geq \delta$. Using Lemma 1, we have $I_8 = O(n^{-(k+1)})$. Again, applying Lemma 1, we get $\|I_9\|_{C[a,b]} \leq K_2\, n^{-(k+1)} \left\| f^{(2k+2)} \right\|_{L_\infty[0,b]}$. Combining the estimates of $I_7 - I_9$, we have

$$\|I_5\| \leq K_3\, n^{-(k+1)} \left(\sum_{i=1}^{2k+1} \left\| f^{(i)} \right\|_{C[a,b]} + \left\| f^{(2k+2)} \right\|_{L_\infty[0,b]} \right).$$

Now, applying Goldberg and Meir [3] property, the required result is immediate.

In our next theorem we estimate the degree of approximation of $M_n(f,k,x)$ to $f(x)$ in terms of the higher order modulus of continuity of f ■

Theorem 2. Let $f \in C_\alpha[0,\infty)$. Then, for sufficiently large n, there exists a constant M independent of n and f such that

(2.5) $$\|M_n(f,k,.) - f\|_{C[a_2,b_2]} \leq M \left\{ \omega_{2k+2}(f, n^{-1/2}, a_1, b_1) + n^{-(k+1)} \|f\|_{C_\alpha} \right\}.$$

Proof: For $f \in C_\alpha[0,\infty)$, the Steklov mean $f_{\eta,2k+2}(x) \in C^{2k+2}$ of $(2k+2)^{th}$ order is defined as

$$f_{\eta,2k+2}(x) = \frac{\eta^{-(2k+2)}}{\binom{2k+2}{k+1}} \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \left[(-1)^k \Delta^{-(2k+2)}_{\sum_{\nu=1}^{2k+2} u_\nu} f(x) + \binom{2k+2}{k+1} f(x) \right] \prod_{\nu=1}^{2k+2} du_\nu,$$

where $(k+1)^2 \eta < \min\{a_2 - a_1, b_1 - b_2\}$ and Δ_h^{-r} is the r^{th} symmetric difference operator defined by:

$$\Delta_h^{-(2k+2)} f(x) = \sum_{i=0}^{2k+2} (-1)^i \binom{2k+2}{i} f(x + (2k+2-i) \sum_{\nu=1}^{2k+2} u_\nu).$$

Then the function $f_{\eta,2k+2}(x)$ has the following properties:

(2.6) $$\left\| f^{(2k+2)}_{\eta,2k+2} \right\|_{C[a_2,b_2]} \leq M_1\, \eta^{-(2k+2)} \omega_{2k+2}(f, \eta, a_1, b_1);$$

(2.7) $$\left\| f - f_{\eta,2k+2} \right\|_{C[a_2,b_2]} \le M_2\, \omega_{2k+2}(f,\eta,a_1,b_1);$$

(2.8) $$\left\| f_{\eta,2k+2} \right\|_{C[a_2,b_2]} \le M_3 \left\| f \right\|_{C[a_1,b_1]} \le M_4 \left\| f \right\|_{C_\alpha},$$

where $M_4 = M_3 e^{b_1}$, M_i's are certain constants depending on k only and $\omega_{2k+2}(f,\eta,a_1,b_1)$ is the modulus of continuity of order $2k+2$ corresponding to f:

$$\omega_{2k+2}(f,\eta,a_1,b_1) = \sup_{\substack{|h|\le\eta \\ x,x+(2k+2)h\in[a_1,b_1]}} \left| \Delta_h^{2k+2} f(x) \right|.$$

Now, in order to prove (2.6), notice that

$$(-1)^k \binom{2k+2}{k+1} \eta^{2k+2} f_{n,2k+2}(x)$$

$$= \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \left[\sum_{i=0}^{2k+2} (-1)^i \binom{2k+2}{i} f(x + (k+1-i) \sum_{\nu=1}^{2k+2} u_\nu) + (-1)^k \binom{2k+2}{k+1} f(x) \right] \prod_{\nu=1}^{2k+2} du_\nu$$

$$= \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \left[\sum_{\substack{i=0 \\ i\ne k+1}}^{2k+2} (-1)^i \binom{2k+2}{i} f(x + (k+1-i) \sum_{\nu=1}^{2k+2} u_\nu) \right] \prod_{\nu=1}^{2k+2} du_\nu$$

$$= \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \left[\sum_{i=0}^{k} (-1)^i \binom{2k+2}{i} f(x + (k+1-i) \sum_{\nu=1}^{2k+2} u_\nu) \right.$$
$$\left. + \sum_{i=k+2}^{2k+2} (-1)^i \binom{2k+2}{i} f(x + (k+1-i) \sum_{\nu=1}^{2k+2} u_\nu) \right] \prod_{\nu=1}^{2k+2} du_\nu$$

$$= \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \sum_{i=0}^{k} (-1)^i \binom{2k+2}{i} \left\{ f(x + (k+1-i) \sum_{\nu=1}^{2k+2} u_\nu) \right.$$
$$\left. + f(x - (k+1-i) \sum_{\nu=1}^{2k+2} u_\nu) \right\} \prod_{\nu=1}^{2k+2} du_\nu .$$

Since

$$\frac{d^{2k+2}}{dx^{2k+2}} \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \left[f(x + \sum_{\nu=1}^{2k+2} u_\nu) + f(x - \sum_{\nu=1}^{2k+2} u_\nu) \right] \prod_{\nu=1}^{2+2} du_\nu = 2\Delta_\eta^{-(2k+2)} f(x),$$

and $\omega_{2k+2}(f;|k+1-i|\eta) \le |k+1-i|\, \omega_{2k+2}(f;\eta)$, we have,

$$\left\| f_{\eta,2k+2}^{(2k+2)} \right\|_{C[a_2,b_2]} = \frac{\eta^{-(2k+2)}}{\binom{2k+2}{k+1}} \left\| \sum_{i=0}^{k} (-1)^i \binom{2k+2}{i} 2\Delta_{(k+1-i)}^{-(2k+2)} f(x) \right\|_{C[a_1,b_1]}$$

$$\leq \frac{\eta^{-(2k+2)}}{\binom{2k+2}{k+1}} 2\sum_{i=0}^{k}\binom{2k+2}{i}(k+1-i)\,\omega_{2k+2}(f,\eta,a_1,b_1)$$

and thus (2.6) follows.
From the definition of $f_{\eta,2k+2}$, we have

$$\left|f - f_{\eta,2k+2}\right| \leq \frac{\eta}{\binom{2k+2}{k+1}} \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \left|\Delta^{-(2k+2)}_{\sum_{v=1}^{2k+2} u_v} f(x)\right| \prod_{v=1}^{2k+2} du_v$$

$$\leq M'\omega_{2k+2}(f;\eta(k+1),a_1,b_1) \leq (k+1)M'\omega_{2k+2}(f;\eta,a_1,b_1)$$

$$= M_2\,\omega_{2k+2}(f;\eta,a_1,b_1) \text{ for all } x\in[a_2,b_2],$$

which proves (2.7). The proof of the inequality (2.8) is trivial and therefore we omit it.
Now, we shall prove (2.5). we can write

$$M_n(f,k,x) - f(x) = M_n(f - f_{\eta,2k+2},k,x) + (f_{\eta,2k+2}(x) - f(x))$$
$$+ (M_n(f_{\eta,2k+2},k,x) - f_{\eta,2k+2}(x)) = I_1(x) + I_2(x) + I_3(x), \text{ say.}$$

From (2.7) we have

$$\|I_2\|_{C[a_2,b_2]} \leq M_2\omega_{2k+2}(f;\eta,a_1,b_1) = M_2\omega_{2k+2}(f;n^{-1/2},a_1,b_1).$$

Next, proceeding as in the estimate of I_4 in the previous theorem, we have

$$|I_1(x)| \leq \sum_{j=0}^{k}|C(j,k)| \int_0^\infty W_{d_j n}(t,x)\left|f(t) - f_{\eta,2k+2}(t)\right| dt$$

and

$$\int_0^\infty W_{d_j n}(t,x)\left|f(t) - f_{\eta,2k+2}(t)\right| dt = \int_{|t-x|\leq\delta} + \int_{|t-x|>\delta}$$

$$\leq \left\|f - f_{\eta,2k+2}\right\|_{C[a_2-\delta,b_2-\delta]} + K_m n^{-m}\|f\|_{C_\alpha}, \text{ for all } m>0,$$

where, $\delta < \min\{a_2 - a_1, b_1 - b_2\}$. Hence, again in view of (2.7)

$$\|I_1\|_{C[a_2,b_2]} \leq M_2\,\omega_{2k+2}(f;n^{-1/2},a_1,b_1) + K_m n^{-m}\|f\|_{C_\alpha}.$$

Finally, in order to estimate $I_3(x)$, we observe that by Taylor expansion

$$f_{\eta,2k+2}(t) = \sum_{i=0}^{2k+2}\frac{f^{(i)}_{\eta,2k+2}(x)}{i!}(t-x)^i + \frac{1}{(2k+2)!}f^{(2k+2)}_{\eta,2k+2}(\xi)(t-x)^{2k+2}, \tag{2.9}$$

where ξ lies between t and x. Operating $M(\,.\,,k,x)$ on (2.9) and separating the integral into two parts as in the estimation of $I_1(x)$, from Lemma 1 and (2.4) we are led to

$$\left\| M_n(f_{\eta,2k+2},k,.) - f_{\eta,2k+2} \right\|_{C[a_2,b_2]}$$
$$\leq M_5\, n^{-(k+1)} \sum_{i=1}^{2k+2} \left\| f_{\eta,2k+2}^{(i)} \right\|_{C[a_2,b_2]} + K_m\, n^{-m} \left\| f_{\eta,2k+2} \right\|_{C_\alpha}.$$

Using [3], we get

$$\left\| f_{\eta,2k+2}^{(i)} \right\|_{C[a_2,b_2]} \leq M_6 \left(\left\| f_{\eta,2k+2} \right\|_{C[a_2,b_2]} + \left\| f_{\eta,2k+2}^{(2k+2)} \right\|_{C[a_2,b_2]} \right),$$

and choosing $m \geq k+1$, we have further that

$$\left\| M_n(f_{\eta,2k+2},k,.) - f_{\eta,2k+2} \right\|_{C[a_2,b_2]} \leq M_7\, n^{-(k+1)} \left(\left\| f_{\eta,2k+2} \right\|_{C_\alpha} + \left\| f_{\eta,2k+2}^{(2k+2)} \right\|_{C[a_2,b_2]} \right).$$

Now, applying (2.6), (2.8) and the definition of $f_{\eta,2k+2}$ we get:

$$\left\| I_3 \right\|_{C[a_2,b_2]} \leq M_8 \left(\omega_{2k+2}(f;n^{-1/2},a_1,b_1) + n^{-(k+1)} \left\| f \right\|_{C_\alpha} \right).$$

Combining the estimates of $I_1(x) - I_3(x)$ we obtain (2.5).

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