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# AN ALTERNATIVE EXISTENCE PROOF FOR GALE'S MODEL OF A DISCRETE EXCHANGE ECONOMY WITH MONEY

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**Abstract.** In this note we present an alternative existence proof for a discrete exchange equilibrium, as stated in Gale [2]. Our proof is obtained under a slightly different set of assumptions, but has the advantage of guaranteeing a non zero equilibrium price system.

## 1. Introduction.

Gale, in [2], proves an existence theorem for an exchange equilibrium in an economy with indivisible goods and only one perfectly divisible good, which can be thought of as money. In this note we present an alternative proof which is obtained under a slightly different set of assumptions. It allows us to guarantee that the equilibrium price system is always a non zero vector.
Gale's work is strongly related to Quinzii [3]. The proof given here used a technique developed in [1].

## 2. Preliminaries

We first present a brief description of the model. For further interpretation of its elements, we refer the reader to Gale [2]. Our presentation follows closely that of Gale.
The economy has $n$ *traders*, indexed by the set $\mathbf{N} = \{1, ..., n\}$. The traders will be identified with Greek letters. Each trader owns an *object.* The set of objects is also denoted by $\mathbf{N}$.
An *assignment* is a one to one mapping $\sigma$ from $\mathbf{N}$ into $\mathbf{N}$. A *price system* is a vector $p$ of $\mathbb{R}^n_+ = \{x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n, x_i \geq 0, i = 1, 2, ..., n\}$. Here $\mathbb{R}$ stands for the set of real numbers and $\mathbb{R}^n$ for the $n$-fold Cartesian product of $\mathbb{R}$.
To specify the demand, we assume that corresponding to each $\alpha \in \mathbf{N}$,there is a covering

$$\mathcal{C}^\alpha = \{C^\alpha_j\}_{j \in \mathbf{N}} \cup C^\alpha_0$$

of $\mathbb{R}^n$.The interpretation is that if $p \in C^\alpha_j, j > 0$, then trader $\alpha$ will demand object $j$ at prices $p$ while if $p \in C^\alpha_0$, then $\alpha$ will demand no object, but he his willing to exchange his object for some amount of money.

An *exchange equilibrium* is a pair $(p, \sigma)$ consisting of a non zero price system $p$ and an assignment $\sigma$ such that $p \in \mathcal{C}^{\alpha}_{\sigma(\alpha)}$ for all $\alpha \in \Gamma$.
Now, we give the definition of a $KKM$ covering. Let

$$\Delta^{n-1} = \{x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n, x_i \geq 0, i = 1, 2, ..., n, \sum_{i \in \mathbf{N}} x_i = 1\}$$

For any $\mathbf{S} \subseteq \mathbf{N}$,we denote by $\mathbb{F}_{\mathbf{S}}$ the face of $\Delta^{n-1}$spanned by the unit vectors $\mathbf{e}^i, i \in \mathbf{S}$. Here $\mathbf{e}^i = (0, ..., 1, ..., 0)$ ,with the one placed in the $i-th$ position. A family of closed sets $\mathcal{D} = \{\mathcal{D}_j\}_{j \in \mathbf{N}}$ is called a $KKM$ covering if

$$\mathbb{F}_{\mathbf{S}} \subseteq \bigcup_{i \in \mathbf{S}} \mathcal{D}_i$$

for any $\mathbf{S} \subseteq \mathbf{N}$.
A famous result due to Knaster, Kuratowsky and Mazurkiewicz, the $KKM$-lemma ([4]), states that if $\mathcal{D} = \{\mathcal{D}_j\}_{j \in \mathbf{N}}$ is a closed $KKM$ covering of $\Delta^{n-1}$,then

$$\bigcap_{j \in \mathbf{N}} \mathcal{D}_j \neq \Phi$$

In [2], Gale proves an extension of that result.. We state below an equivalent version which we are going to use next.
**Lemma 1.** Let $\mathcal{D}^{\alpha} = \{\mathcal{D}^{\alpha}_j\}_{j \in \mathbf{N}}, \alpha \in \mathbf{N}$ be $n$ $KKM$ coverings of $\Delta^{n-1}$.Then there is a permutation $\sigma$ on $\mathbf{N}$ and a point $x \in \Delta^{n-1}$ such that $x \in \mathcal{D}^{\sigma(j)}_j$ for all $j \in \mathbf{N}$.

## 3. Existence proof.

Now we state the result of this note.
**Theorem.** Under the following assumptions

(1) The sets in $\{\mathcal{C}^{\alpha}_j\}_{j \in \mathbf{N}}$ are all closed for any $\alpha \in \Gamma$.
(2) There exists $M_1 > 0$ such that if $p_j \geq M_1$, then $p \notin \mathcal{C}^{\alpha}_j$ for any $\alpha \in \Gamma$.
(3) The family $\{\mathcal{C}^{\alpha}_j\}_{j \in \mathbf{N}}$ covers $M\Delta^{n-1}$.

the economy has an exchange equilibrium $(p, \sigma)$.
Here $M = (n-1) M_1$.
*Proof.* We first define the mapping $\varphi$ from $\Delta^{n-1}$ into $M\Delta^{n-1}$ by

$$(\varphi(x))_j = M \frac{(1 - x_j)}{n - 1} \tag{1}$$

The mapping $\varphi$ is continuous.
For each $\alpha \in \Gamma, j \in \mathbf{N}$,let

$$\mathcal{D}^{\alpha}_j = \varphi^{-1}\left(\mathcal{C}^{\alpha}_j \cap M\Delta^{n-1}\right) \tag{2}$$

Because of $(1), (3)$ and the continuity of $\varphi$,it follows that $\{\mathcal{D}^{\alpha}_j\}_{j \in \mathbf{N}}$ is a closed covering of $\Delta^{n-1}$for any $\alpha \in \Gamma$.

Now we are going to show that (2) is, in fact, a closed $KKM$ closed covering of $\Delta^{n-1}$. To do this, let us define

$$\mathcal{F}_{\mathbf{N}-\{j\}} = \{x \in \Delta^{n-1} : x_j = 0\}$$

From (1) it is easy to see that

$$\varphi\left(\mathcal{F}_{\mathbf{N}-\{j\}}\right) \subseteq \left\{p \in M\Delta^{n-1} : p_j = \frac{M}{n-1}\right\} \tag{3}$$

We now claim that

$$\mathcal{D}_j^\alpha \cap \mathcal{F}_{\mathbf{N}-\{j\}} = \Phi \tag{4}$$

for all $\alpha \in \Gamma$. In fact, if this was not true, let $x \in \mathcal{D}_j^\alpha \cap \mathcal{F}_{\mathbf{N}-\{j\}}$. Because of (2)

$$\varphi(x) \in \mathcal{C}_j^\alpha \cap M\Delta^{n-1} \subseteq \mathcal{C}_j^\alpha \tag{5}$$

and due to (3)

$$(\varphi(x))_j = \frac{M}{n-1} = M_1$$

On the other hand, this last relation and condition (2) above indicates that $\varphi(x) \notin \mathcal{C}_j^\alpha$,and this contradicts (5) .This contradiction proves the validity of our claim.
We claim that $\{\mathcal{D}_j^\alpha\}_{j\in\mathbf{N}}$ is a $KKM$ covering of $\Delta^{n-1}$ for each $\alpha \in \mathbf{N}$. To prove this, let $\Phi \neq \mathbf{Q} \subseteq \mathbf{N}$ and $x \in \Delta^{n-1}$ be such that $x_{\bar{\imath}} = 0$ for all $\bar{\imath} \notin \mathbf{Q}$. This last condition implies that $x \in \mathcal{F}_{\mathbf{N}-\bar{\imath}}$ for all $\bar{\imath} \notin \mathbf{Q}$. Therefore, because of (4) , $x \notin \mathcal{D}_j^\alpha$ for all $\bar{\imath} \notin \mathbf{Q}$. Taking into account that the family $\{\mathcal{D}_j^\alpha\}_{j\in\mathbf{N}}$ covers $\Delta^{n-1}$, there exists $j \in \mathbf{Q}$ such that

$$x \in \mathcal{D}_j^\alpha \subseteq \bigcup_{i\in\mathbf{Q}} \mathcal{D}_i^\alpha$$

The case $\mathbf{Q} = \Phi$,follows directly from the covering property of (2) . Thus, we have proved that this family forms a closed $KKM$ covering of $\Delta^{n-1}$.
Now, we are able to apply Lemma 1 to guarantee the existence of $x \in \Delta^{n-1}$ and a permutation $\sigma : \mathbf{N} \to \mathbf{N}$ such that

$$x \in \mathcal{D}_j^{\sigma(j)}$$

for all $j \in \mathbf{N}$.
Finally, by letting $p = \varphi(x)$ and $\tau = \sigma^{-1}$,we get that

$$p \in \mathcal{C}_{\tau(\alpha)}^\alpha$$

for all $\alpha \in \Gamma$.This completes our proof.

## 4. References.

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