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THE DISJUNCTIVE PROCEDURE AND THE MATCHING PROBLEM

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Abstract

In this paper we study the polytope generated by an application of the Disjunctive Operator on the linear relaxation of the matching polytope. We find a characterization of all the valid inequalities that allows us to give an alternative proof of the well-known result of the integrality of the matching polytope on bipartite graphs.

1 INTRODUCTION

A Combinatorial Optimization Problem can be formulated as finding the maximum (or minimum) of a linear function on a subset K_0 of $\{0,1\}^n$. Clearly, optimizing such a function over K_0 is equivalent to do so on its convex hull, $\text{conv}(K_0)$. Although in many problems K_0 can be described as

$$K_0 = \{x \in \{0,1\}^n : Ax \leq b\} \tag{1}$$

where A is an $m \times n$ real matrix and $b \in \mathbb{R}^m$, in general, it is not easy to find a description of $\text{conv}(K_0)$ in terms of linear inequalities.
A *linear relaxation* of K_0 is any polyhedron K such that $K \cap \{0,1\}^n = K_0$. If K_0 is defined as in (1) then

$$K = \{x \in \mathbb{R}^n : 0 \leq x \leq 1\ ;\ Ax \leq b\} = \{x \in \mathbb{R}^n : \tilde{A}x \leq \tilde{b}\}$$

is called the *original linear relaxation.*
Given a linear relaxation Q of K_0, there exists a wide variety of *tightening procedures* that find, at every step, a tighter relaxation of Q, in order to arrive to $\text{conv}(K_0)$ after a finite number of applications.
In this paper we particularly work on the *Disjunctive Procedure*, developed by Balas, Ceria and Cornuéjols in [2]. The Disjunctive Procedure can be described as follows:

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Let $K = \{x \in \mathbb{R}^n : \tilde{A}x \leq \tilde{b}\}$ be a linear relaxation of K_0. For fixed j, $1 \leq j \leq n$, every inequality in the system $\tilde{A}x \geq \tilde{b}$ is multiplied by x_j and $1 - x_j$, obtaining a system of, in general, nonlinear inequalities. Then, x_j^2 is replaced by x_j and the products $x_i x_j$, for $i \neq j$, by new variables y_i, obtaining a new system of linear inequalities in the variables x and y. The polytope defined by this system of linear inequalities is denoted by $M_j(K)$. Finally, the polyhedron $P_j(K)$ is obtained by projecting back $M_j(K)$ onto the x-space, by eliminating the y variables.
The following result, proved in [1], gives an alternative definition for P_j:

Theorem 1.1 *For any j, $P_j(K) = \text{conv}\{x \in K : x_j \in \{0,1\}\}$. In particular, $\text{conv}(K_0) \subset P_j(K) \subset K$.*

Defining, for $\{i_1, ... i_k\} \subset \{1, ..., n\}$,

$$P_{i_1 \ldots i_k}(K) = P_{i_1}(P_{i_2 \ldots i_k}(K)),$$

the following lemma characterizes the Disjunctive Procedure as a sequential tightening procedure.

Lemma 1.2 *For a polyhedron $K \subset [0,1]^n$ the following assertions are true:*

1. $P_{i_1 \ldots i_k}(K) = \text{conv}\{x \in K : x_j \in \{0,1\}$ *for any* $j \in \{i_1, ..., i_k\}\}$
2. $P_{1 \ldots n}(K) = \text{conv}(K_0)$.

Integer polyhedra can be described in terms of the Disjunctive Procedure by the following lemma

Lemma 1.3 $K = \text{conv}(K_0)$ *if and only if* $P_j(K) = K$ *for any* $j \in \{1, ..., n\}$.

Therefore, K is an integral polyhedron if and only if for all $j \in \{1, ..., n\}$ every valid inequality of $P_j(K)$ is also valid for K_0. In general, to obtain explicitly the whole set of valid inequalities for $P_j(K)$ is an exponential task. Nevertheless, in some cases, it is possible to characterize these inequalities in terms of the particular structure of the problem. We will study the matching polytope, following Ceria in his work upon the stable set polytope [2].

1.1 THE MATCHING PROBLEM

Let $G = (V, E)$ a graph with n nodes. We denote by $[i, j]$ the edge in E with extreme points i and j for $i, j \in V$. Given a subset U of V, $E(U)$ will denote the set of edges of G with both extreme points in U. If $\bar{U} \subset V$ is such that $U \cap \bar{U} = \emptyset$ then $(U : \bar{U})$ will denote the set of all the edges with one node in U and the other one in $\bar{U}$. For each node $i \in V$ we define the following set of nodes

$$\Gamma(i) = \{j \in V : [i, j] \in E\}.$$

Definition 1.4 *Given the graph $G = (V, E)$ a matching is a subset of edges M such that in the induced subgraph $G(M) = (V, M)$ each node has at most degree 1.*

The set of all the matchings in G can be characterized by

$$K_0(G) = \{x \in \{0,1\}^{|E|} : \sum_{j\in\Gamma(i)} x_{ij} \leq 1, \text{ for all } i \in V\} = \{x \in \{0,1\}^{|E|} : Dx \leq 1\}.$$

where D is the node-edge incidence matrix of G. Let

$$K(G) = \{x \in \mathbb{R}_+^{|E|} : Dx \leq 1\}. \tag{2}$$

In general, $K(G)$ it is not an integral polyhedron. In particular, if G contains an odd cycle $C \subset E$, the point $r \in \mathbb{R}^{|E|}$ defined as $1/2$ over C and 0 in any other case, belongs to $K(G)$. It is easy to prove that if $x \in K_0(G)$,

$$\sum_{[i,j]\in C} x_{ij} \leq \left\lfloor \frac{|C|}{2} \right\rfloor. \tag{3}$$

therefore, $r \notin \text{conv}(K_0(G))$.

That means that if G is not bipartite, $K(G)$ is not an integral polyhedron. The converse is also true, it can be obtained as a corollary of the following strong result due to Edmonds [3] which gives a description by linear inequalities of $\text{conv}(K_0(G))$.

Theorem 1.5 *The convex hull of the matchings in G is given by*

$$\sum_{j\in\Gamma(i)} x_{ij} \leq 1 \quad \textit{for all } i \in V,$$

$$\sum_{[i,j]\in E(U)} x_{ij} \leq \left\lfloor \frac{|U|}{2} \right\rfloor \quad \textit{for all } U \subset V \textit{ such that } |U| \geq 3 \textit{ odd},$$

$$x \in \mathbb{R}_+^{|E|}.$$

Another way of proving the integrality of $K(G)$ in a bipartite graph, is a consequence of the total unimodularity of the matrix D. In this work, we present an alternative proof, independent form the ones already mentioned, that uses the characterization of the valid inequalities of $P_j(K(G))$.

2 THE DISJUNCTIVE PROCEDURE AND THE MATCHING POLYTOPE

Let $G = (V, E)$ and $K(G)$ be the linear relaxation of the matching polytope given by (2). Our purpose is to find a description of the whole set of the valid inequalities for $P_{[r.s]}(K(G))$, where $P_{[r,s]}$ denotes the Disjunctive Operator related to the edge $[r, s] \in E$.

After eliminating redundancies, $M_{[r.s]}(K(G))$ is given by the following system of linear inequalities

$$\begin{array}{rcll}
 & \sum_{j\in\Gamma(i)} x_{ij} & \leq 1 & i \in V, \\
\sum_{j\in\Gamma(i)} x_{ij} + x_{rs} - 1 \leq & \sum_{j\in\Gamma'(i)} y_{ij} & \leq x_{rs} & i \in V' = V \setminus \{r, s\}, \\
0 \leq & y_{ij} & \leq x_{ij} & [i,j] \in E' = E \setminus \{[r,s]\},
\end{array}$$

where $\Gamma'(i) = \Gamma(i) \setminus \{r, s\}$ for every $i \in V$.
Therefore, given $x \in K(G)$, $x \in P_{[r,s]}(K(G))$ if and only if there exists $y \in \mathbb{R}^{|E'|}$ such that

$$\begin{array}{rcll} \sum_{j\in\Gamma(i)} x_{ij} + x_{rs} - 1 \leq & \sum_{j\in\Gamma'(i)} y_{ij} & \leq x_{rs} & \text{for } i \in V', \\ 0 \leq & y_{ij} & \leq x_{ij} & \text{for } [i,j] \in E', \end{array}$$

or, equivalently, if and only if the system

$$\begin{array}{rcll} & \sum_{j\in\Gamma'(i)} y_{ij} + z_i & = x_{rs} & \text{for } i \in V', \\ 0 \leq & y_{ij} & \leq x_{ij} & \text{for } [i,j] \in E'' = E(V'), \\ 0 \leq & z_i & \leq 1 - \sum_{j\in\Gamma(i)} x_{ij} & \text{for } i \in V'. \end{array} \tag{4}$$

is feasible.
By Farkas' lemma, the system (4) has a solution if and only if the system

$$\begin{array}{rcl} -(u_i + u_j) + v_{ij} & \geq & 0, \text{ for } [i,j] \in E'', \\ -u_i + w_i & \geq & 0, \text{ for } i \in V', \\ -\sum_{i\in V'} u_i x_{rs} + \sum_{[i,j]\in E'} x_{ij} v_{ij} + \sum_{i\in V'} w_i (1 - \sum_{j\in\Gamma(i)} x_{ij}) & < & 0, \\ v, w & \geq & 0, \end{array} \tag{5}$$

is unfeasible.
Rewriting (5) as

$$\begin{array}{rcl} \max(0, u_i + u_j) & \leq & v_{ij} \text{ for } [i,j] \in E'', \\ \max(0, u_i) & \leq & w_i \text{ for } i \in V', \\ -\sum_{i\in V'} u_i x_{rs} + \sum_{[i,j]\in E'} x_{ij} v_{ij} + \sum_{i\in V'} w_i (1 - \sum_{j\in\Gamma(i)} x_{ij}) & < & 0, \end{array}$$

it is easy to prove that (5) is unfeasible if and only if there is no $u \in \mathbb{R}^{|V'|}$ such that

$$-\sum_{i\in V'} u_i x_{rs} + \sum_{[i,j]\in E''} x_{ij} \max(0, u_i + u_j) + \sum_{i\in V'} \max(0, u_i)(1 - \sum_{j\in\Gamma(i)} x_{ij}) < 0.$$

In other words, given $x \in K(G)$, $x \in P_{[r,s]}(K(G))$ if and only if, for every $u \in \mathbb{R}^{|V'|}$,

$$\sum_{i\in V'} u_i x_{rs} - \sum_{[i,j]\in E''} \max(0, u_i + u_j) x_{ij} + \sum_{i\in V'} \sum_{j\in\Gamma(i)} \max(0, u_i) x_{ij} \leq \sum_{i\in V'} \max(0, u_i)$$

or, equivalently, if for every $u \in \mathbb{R}^{|V'|}$,

$$\sum_{i\in V'} u_i x_{rs} - \sum_{[i,j]\in E'',\, u_i+u_j>0} (u_i + u_j) x_{ij} + \sum_{i\in V',\, u_i>0} \sum_{j\in\Gamma(i)} u_i x_{ij} \leq \sum_{i\in V',\, u_i>0} u_i. \tag{6}$$

Let us observe that for every $u \in \mathbb{R}^{|V'|}$, we have a natural partition of V' given by

$$P = \{i \in V' : u_i > 0\} \quad \text{and} \quad \bar{P} = V' \setminus P = \{i \in V' : u_i \leq 0\}.$$

For all the nodes $i \in \bar{P}$, redefine u_i as $-u_i$ in (6) to obtain

$$\left(\sum_{i \in P} u_i - \sum_{j \in \bar{P}} u_j\right) x_{rs} - \sum_{[i,j] \in (P:\bar{P}),\ u_i - u_j > 0} x_{ij}(u_i - u_j) - \sum_{[i,j] \in E(P)} x_{ij}(u_i + u_j)$$

$$+ \sum_{[i,j] \in (P:\bar{P})} u_i x_{ij} + \sum_{[i,j] \in E(P)} u_i x_{ij} + \sum_{i \in P \cap \Gamma(s)} u_i x_{is} + \sum_{i \in P \cap \Gamma(r)} u_i x_{ir} \leq \sum_{i \in P} u_i$$

or equivalently

$$\left(\sum_{i \in P} u_i - \sum_{j \in \bar{P}} u_j\right) x_{rs} + \sum_{i \in P \cap \Gamma(s)} u_i x_{is} + \sum_{i \in P \cap \Gamma(r)} u_i x_{ir}$$

$$+ \sum_{[i,j] \in (P:\bar{P})} \min(u_i, u_j) x_{ij} \leq \sum_{i \in P} u_i.$$

Therefore, we have proved the following theorem

Theorem 2.1 *Let D denote the node-edge incidence matrix of a graph $G = (V, E)$. Let $K(G) = \{x \in \mathbb{R}_+^{|E|} : Dx \leq 1\}$. If $[r, s] \in E$ and $V' = V \setminus \{r, s\}$ then $x \in P_{[r,s]}(K(G))$ if and only if $x \in K(G)$ and*

$$d x_{rs} + \sum_{i \in P \cap \Gamma(s)} u_i x_{is} + \sum_{i \in P \cap \Gamma(r)} u_i x_{ir} + \sum_{[i,j] \in (P:\bar{P})} \min(u_i, u_j) x_{ij} \leq \sum_{i \in P} u_i \tag{7}$$

for every $P \subset V'$ and $u \in \mathbb{R}_+^{|V'|}$, where $\bar{P} = V' \setminus P$ and $d = \sum_{i \in P} u_i - \sum_{j \in \bar{P}} u_j$.

Let us analyze the inequalities described in the previous theorem, with the purpose of eliminating redundancies.

Remark 2.2 *Given $P \subset V'$ and $u \in \mathbb{R}_+^{|V'|}$ such that $d \leq 0$, the inequality (7) associated to P and u is dominated by*

$$\sum_{i \in P \cap \Gamma(s)} u_i x_{is} + \sum_{i \in P \cap \Gamma(r)} u_i x_{ir} + \sum_{[i,j] \in (P:\bar{P})} u_i x_{ij} \leq \sum_{i \in P} u_i$$

which is a non-negative linear combination of the inequalities that define $K(G)$.

Remark 2.3 *Let $i \in V'$, $P = \{i\}$ and u given by*

$$u_k = \begin{cases} 1 & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases}$$

If $U_i = \{i, r, s\}$ then the inequality (7) is

$$\sum_{[k,l] \in E(U_i)} x_{kl} \leq 1. \tag{8}$$

Remark 2.4 *Let $P \subset V'$ and $u \in \mathbb{R}_+^{|V'|}$ such that $u_j = 0$ for every $j \in \bar{P}$, then the inequality (7) becomes*

$$\sum_{i \in P} u_i x_{rs} + \sum_{i \in P \cap \Gamma(s)} u_i x_{is} + \sum_{i \in P \cap \Gamma(r)} u_i x_{ir} \leq \sum_{i \in P} u_i$$

or

$$\sum_{i \in P} u_i \sum_{[k,l] \in E(U_i)} x_{kl} \leq \sum_{i \in P} u_i$$

which is a non-negative linear combination of the inequalities (8).

Taking into account Remarks 2.2, 2.3 and 2.4, in the description of $P_{[r,s]}(K(G))$ we need to consider the inequalities defining $K(G)$, the ones of the form (8), and the inequalities (7) such that $d > 0$ and $\sum_{i \in \bar{P}} u_j > 0$.

We close this section by proving that the odd cycle inequalities (3) containing the edge $[r, s]$, are valid for $P_{[r,s]}(K(G))$.

Lemma 2.5 *Let $G = (V, E)$, $[r, s] \in E$ and C an odd cycle in G such that $[r, s] \in C$ and $|C| \geq 5$. Then*

$$\sum_{[i,j] \in C} x_{ij} \leq \left\lfloor \frac{|C|}{2} \right\rfloor$$

is a valid inequality for $P_{[r,s]}(K(G))$.

Proof.

If $V(C)$ is the set of nodes in C, let us color the nodes in $V'(C) = V(C) \setminus \{r, s\}$. Suppose that we begin with the node next to r in C (different form s) in red, and we color alternatively, in blue and red until the one next to s (different form r). If P denotes the set of all the red nodes, then $|P| = \left\lfloor \frac{|C|}{2} \right\rfloor$.

Let us define $u \in \mathbb{R}_+^{|V'|}$ such that

$$u_i = \begin{cases} 1 & \text{if } i \in V'(C), \\ 0 & \text{in any other case.} \end{cases} \tag{9}$$

The valid inequality (7) for $P_{[r,s]}(K(G))$ related to P and u becomes

$$x_{rs} + \sum_{i \in P \cap \Gamma(s)} x_{is} + \sum_{i \in P \cap \Gamma(r)} x_{ir} + \sum_{[i,j] \in (P:\bar{P})} x_{ij} \leq \left\lfloor \frac{|C|}{2} \right\rfloor$$

which is clearly stronger than

$$\sum_{[i,j] \in E(C)} x_{ij} \leq \left\lfloor \frac{|C|}{2} \right\rfloor.$$

■

3 MATCHING ON BIPARTITE GRAPHS AND THE DISJUNCTIVE PROCEDURE

In Section 1 we have mentioned some well-known proofs of the integrality of $K(G)$ when G is a bipartite graph. In this section we give an alternative proof of this result using Theorem 2.1.

Theorem 3.1 *If G is a bipartite graph then $K(G) = \text{conv}\,(K_0(G))$.*

Proof.
Let $G = (V, E)$ be a bipartite graph. If $|V| \leq 3$ then the proof is obvious. Therefore we prove the general result by induction on $|V|$. Let us suppose that for every bipartite graph G with $|V| \leq k$, $K(G) = \text{conv}\,(K_0(G))$.
Let G be a bipartite graph such that $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$, $E \subset V_1 \times V_2$ and $|V| = k + 1 \geq 4$. We will prove that every $[r, s] \in E$, any valid inequality for $P_{[r,s]}(K(G))$ is also valid for $K(G)$.
From now on we make use of the notation introduced in the previous section. Let $[r, s] \in E$ with $r \in V_1$ and $s \in V_2$.
It is clear that if G is bipartite, the inequalities (8) can be obtained as a linear non-negative combination of the inequalities defining $K(G)$. Therefore we only need to prove that for every $P \subset V'$ and $u \in \mathbb{R}_+^{|V'|}$ such that $d > 0$ and $\sum_{j \in \bar{P}} u_j > 0$,

$$d x_{rs} + \sum_{i \in P \cap \Gamma(s)} u_i x_{is} + \sum_{i \in P \cap \Gamma(r)} u_i x_{ir} + \sum_{[i,j] \in (P:\bar{P})} \min(u_i, u_j) x_{ij} \leq \sum_{i \in P} u_i, \tag{10}$$

is a valid inequality for $K(G)$.
Let us consider first $P = V_1 \setminus \{r\}$ and $\bar{P} = V_2 \setminus \{s\}$, where (10) takes the form

$$d x_{rs} + \sum_{i \in P} u_i x_{is} + \sum_{[i,j] \in (P:\bar{P})} \min(u_i, u_j) x_{ij} \leq \sum_{i \in P} u_i. \tag{11}$$

Let us observe that if $x \in K(G)$ and

$$\sum_{i \in P} (u_i - d) x_{is} + \sum_{[i,j] \in (P:\bar{P})} \min(u_i, u_j) x_{ij} \leq \sum_{j \in \bar{P}} u_j$$

then x satisfies (11) because

$$\begin{aligned}
& dx_{rs} + \sum_{i\in P} u_i x_{is} + \sum_{[i,j]\in(P:\bar{P})} \min(u_i,u_j)x_{ij} \\
& \leq d(1-\sum_{i\in P} x_{is}) + \sum_{i\in P} u_i x_{is} + \sum_{[i,j]\in(P:\bar{P})} \min(u_i,u_j)x_{ij} \\
& = d + \sum_{i\in P}(u_i - d)x_{is} + \sum_{[i,j]\in(P:\bar{P})} \min(u_i,u_j)x_{ij} \\
& \leq d + \sum_{j\in\bar{P}} u_j = \sum_{i\in P} u_i.
\end{aligned}$$

Therefore, it is enough to prove that for every $x \in K(G)$,

$$\sum_{i\in P}(u_i - d)x_{is} + \sum_{[i,j]\in(P:\bar{P})} \min(u_i,u_j)x_{ij} \leq \sum_{j\in\bar{P}} u_j. \tag{12}$$

Let us consider $\tilde{G} = (\tilde{V},\tilde{E})$ the subgraph of G induced by $\tilde{V} = V \setminus \{r\}$. Clearly, if $x \in K(\tilde{G})$, defining $\tilde{x} \in \mathbb{R}^{|\tilde{E}|}$ such that $\tilde{x}_{ij} = x_{ij}$ for all $[i,j] \in \tilde{E}$, we have that $\tilde{x} \in K(\tilde{G})$. Since the variables with positive coefficients in (12) are variables associated to arcs in $\tilde{E}$, and $\tilde{G}$ is a bipartite graph with $\left|\tilde{V}\right| = k$, by inductive hypothesis it is enough to prove (12) for every $x \in K(\tilde{G})$ such that $x \in \{0,1\}^{|\tilde{E}|}$.

Let $x \in K(G) \cap \{0,1\}^{|\tilde{E}|}$.

- If $x_{is} = 0$ for all $i \in P$ then

$$\begin{aligned}
& \sum_{i\in P}(u_i - d)x_{is} + \sum_{[i,j]\in(P:\bar{P})} \min(u_i,u_j)x_{ij} \\
= & \sum_{[i,j]\in(P:\bar{P})} \min(u_i,u_j)x_{ij} \\
\leq & \sum_{j\in\bar{P}} u_j \sum_{i\in P} x_{ij} \leq \sum_{j\in\bar{P}} u_j.
\end{aligned}$$

- If $x_{ks} = 1$ for some $k \in P$, then $x_{is} = 0$ for every $i \in P \setminus \{k\}$ and $x_{kj} = 0$ for every $j \in \bar{P}$. Therefore

$$\begin{aligned}
& \sum_{i\in P}(u_i - d)x_{is} + \sum_{[i,j]\in(P:\bar{P})} \min(u_i,u_j)x_{ij} \\
& = u_k - d + \sum_{i\in P\setminus\{k\}} \sum_{j\in\bar{P}} \min(u_i,u_j)x_{ij} \\
& \leq u_k - d + \sum_{i\in P\setminus\{k\}} u_i \sum_{j\in\bar{P}} x_{ij} \\
& \leq \sum_{i\in P} u_i - d = \sum_{j\in\bar{P}} u_j.
\end{aligned}$$

In this way we have proved that for $P = V_1 \setminus \{r\}$, (7) is a valid inequality for $K(G)$, for any $u \in \mathbb{R}_+^{|V'|}$.
Now, suppose that P is any subset of V', $\bar{P} = V' \setminus P$ and $u \in \mathbb{R}_+^{|V'|}$.
Let us define

$$P_1 = P \cap V_1, \quad P_2 = P \cap V_2$$

$$\bar{P}_1 = \bar{P} \cap V_1, \quad \bar{P}_2 = \bar{P} \cap V_2.$$

Moreover, let us consider u' and u'' given by

$$u_i' = \begin{cases} u_i & \text{if } i \in P_1 \cup \bar{P}_2 \\ 0 & \text{in any other case} \end{cases}, \quad u_i'' = \begin{cases} u_i & \text{if } i \in \bar{P}_1 \cup P_2 \\ 0 & \text{in any other case} \end{cases}.$$

¿From the previous case, if $P' = V_1 \setminus \{r\}$ the inequalities (10) associated to P', u' and u'', are valid for $K(G)$ and they can be written as

$$\left(\sum_{i\in P_1} u_i - \sum_{j\in \bar{P}_2} u_j\right) x_{rs} + \sum_{i\in P_1} u_i x_{is} + \sum_{[i,j]\in(P_1:\bar{P}_2)} \min(u_i,u_j)x_{ij} \le \sum_{i\in P_1} u_i. \tag{13}$$

and

$$\left(\sum_{i\in P_2} u_i - \sum_{j\in \bar{P}_1} u_j\right) x_{rs} + \sum_{i\in P_2} u_i x_{ir} + \sum_{[i,j]\in(P_2:\bar{P}_1)} \min(u_i,u_j)x_{ij} \le \sum_{i\in P_2} u_i. \tag{14}$$

Clearly, the sum of (13) and (14) gives

$$dx_{rs} + \sum_{i\in P_1} u_i x_{is} + \sum_{i\in P_2} u_i x_{ir} + \sum_{[i,j]\in(P:\bar{P})} \min(u_i,u_j)x_{ij} \le \sum_{i\in P} u_i,$$

a valid inequality for $K(G)$. ■

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