REVISTA DE LA
UNION MATEMATICA ARGENTINA
Volumen 42, Número 2, 2001, Páginas 103-111

A SIMPLE PROOF OF THE IRRATIONALITY OF THE TRILOG

PABLO PANZONE

ABSTRACT. We use orthogonal polynomials to give a simple proof of the irrationality of the trilog. An approximating formula for Riemann Zeta-function in the critical strip is derived.

0. Introduction.

In this note we give a simple proof of the irrationality of the trilog for certain values. More concretely we prove that $Li_3(1/d) = \sum_{n=1}^{\infty} \frac{1}{n^3 d^n}$ is irrational for $1173 \leqslant d \in N$. This result was first proved by M. Hata [6,7].

We want to point out that improved results had recently been obtained by Miladi in his thesis. Also the the techniques developed by Tanguy Rivoal in his thesis can be used effectively to get much better results.

This note is divided in two sections which are almost independent. In the first section we give an approximating formula for Riemann Zeta-function on the critical strip. In the second section we prove the mentioned result on the irrationality of the trilog.

In both cases we use orthogonal polynomials as in Borwein and Erdeli's book [1], appendix A2. Indeed the point to stress here is that orthogonal polynomials have an integral representation ([1] pg. 373) that permits to guess, at least in some cases, what kind of polynomials are needed to prove the irrationality results.

1. Let

$$f = f(d, \lambda_1, \lambda_2, \ell) := \int_0^1 \frac{x^{\lambda_1}}{(d \pm x^\ell)^{\lambda_2}} dx, \tag{1.0}$$

with parameters $\lambda_1, \lambda_2, d, \ell$ ranging in certain sets of values given below (here y^λ will stand for the positive λ-root of y if $y, \lambda > 0$).

We sometimes use the well-known notation $(a)_0 = 1$, $(a)_n = a(a+1)\dots(a+n-1)$. In what follows $A_n, B_n, F_n(x), \alpha_n, \beta_n, \gamma_n$ may depend on $d, \lambda_1, \lambda_2, \ell$ but to simplify the notation dependence on n is only written. For $1 \leqslant n$ we define

$$F_n(x) := \sum_{j=0}^{n} \frac{(-1)^{n+j}(j + \frac{\lambda_1+1}{\ell})_n}{(n-j)!j!} x^{j\ell}$$

$$A_n := \int_0^1 \frac{(F_n(x) - F_n((\mp d)^{\frac{1}{\ell}}))}{(d \pm x^\ell)^{\lambda_2}} x^{\lambda_1} dx,$$

$$B_n := F_n((\mp d)^{\frac{1}{\ell}}), \ \ (-d)^{1/\ell} := d^{1/\ell} e^{\frac{i\pi}{\ell}},$$

1991 *Mathematics Subject Classification.* 11Mxx,11A55,11J72.
Key words and phrases. Irrationalities, trilog, Riemann Zeta-function.

Proposition 1. *Let f be defined by (1.0) with $\lambda_1, \lambda_2, d \in R$, $0 \leqslant \lambda_1$, $0 \leqslant \lambda_2 \leqslant 1$, $1 \leqslant d$ and $\ell = 1, 2, 3, \dots$ (if $d = 1$ and the sign is - then we restrict $0 \leqslant \lambda_2 < 1$). Then the following holds*

i)

$$\int_0^1 \frac{F_n(x)x^{\lambda_1}}{(d \pm x^{\ell})^{\lambda_2}} dx = A_n + B_n f,$$

$$0 < \left|\int_0^1 \frac{F_n(x)x^{\lambda_1}}{(d \pm x^{\ell})^{\lambda_2}} dx\right| \leqslant h(d)^n \frac{1}{d^{\lambda_2}\ell 2\pi} \frac{(\lambda_2)_n}{\Gamma(n+1)} \int_0^1 \tau^{\frac{\lambda_1+1}{\ell}-1}(1 \pm \frac{\tau}{d})^{-\lambda_2} d\tau,$$

where

$$h(d) = \begin{cases} (\sqrt{d} - \sqrt{d-1})^2 & \text{if the sign is -} \\ (\sqrt{d+1} - \sqrt{d})^2 & \text{if the sign is +} \end{cases},$$

ii) We have the following recurrence relation for $3 \leqslant n$,

$$B_n + (\mp \alpha_n d + \beta_n) B_{n-1} + \gamma_n B_{n-2} = 0,$$

$$A_n + (\mp \alpha_n d + \beta_n) A_{n-1} + \gamma_n A_{n-2} = \mp \alpha_n \int_0^1 F_{n-1}(x)(d \pm x^{\ell})^{1-\lambda_2} x^{\lambda_1} dx,$$

where

$$\alpha_n = -\frac{(2n-1+\frac{\lambda_1+1}{\ell})(2n-2+\frac{\lambda_1+1}{\ell})}{n(n-1+\frac{\lambda_1+1}{\ell})},$$

$$\beta_n = (2n-2+\frac{\lambda_1+1}{\ell}) - \frac{(2n-1+\frac{\lambda_1+1}{\ell})(2n-2+\frac{\lambda_1+1}{\ell})(n-2+\frac{\lambda_1+1}{\ell})(n-1)}{n(n-1+\frac{\lambda_1+1}{\ell})(2n-3+\frac{\lambda_1+1}{\ell})},$$

$$\gamma_n = \frac{(n-1)(2+2n^2-3\frac{\lambda_1+1}{\ell}+(\frac{\lambda_1+1}{\ell})^2+n(3\frac{\lambda_1+1}{\ell}-5))}{n(n-1+\frac{\lambda_1+1}{\ell})(2n+\frac{\lambda_1+1}{\ell}-4)(2n-3+\frac{\lambda_1+1}{\ell})^2},$$

if $3 \leqslant n$. Moreover, if $\lambda_2 = 1$ and $3 \leqslant n$ then $\int_0^1 F_{n-1}(x)(d \pm x^{\ell})^{1-\lambda_2} x^{\lambda_1} dx = 0$.

PROOF. We have for $0 \leqslant x < 1$

$$\frac{x^{\lambda_1}}{(d \pm x^{\ell})^{\lambda_2}} = \frac{1}{d^{\lambda_2}}(x^{\lambda_1} \mp \frac{\lambda_2}{d}\frac{x^{\ell+\lambda_1}}{1!} + \frac{\lambda_2(\lambda_2+1)}{d^2}\frac{x^{2\ell+\lambda_1}}{2!} \mp \dots). \tag{1.1}$$

The idea is to construct polynomials which are orthogonal to the powers of x that appear in (1.1). For this we *define* $F_n(x)$ as

$$\frac{1}{2\pi i}\int_{\gamma} \frac{(t+\lambda_1+1)(t+\lambda_1+\ell+1)(t+\lambda_1+\ell 2+1)\dots(t+\lambda_1+\ell(n-1)+1)}{t(t-\ell)\dots(t-n\ell)} x^t dt$$

$$= \frac{1}{2\pi i}\int_{\gamma} P(t) x^t dt, \tag{1.2}$$

where γ is a positively oriented closed curve enclosing $0, \ell, 2\ell \dots n\ell$ lying in the half plane $Re(t) > -1/2$. Using residues one easily sees that $F_n(x)$ is of the form stated in Proposition 1. Now $F_n(x)$ is orthogonal to $x^{\lambda_1}, x^{\ell+\lambda_1}, x^{2\ell+\lambda_1}, \ldots, x^{(n-1)\ell+\lambda_1}$ on $[0,1]$ i.e. $\int_0^1 F_n(x)x^{\lambda_1+j\ell}dx = 0$ if $j = 0, \ldots, n-1$. This is straightforward upon using the definition of $F_n(x)$, integrating first in x, then in $t = Re^{i\theta}$ and taking $R \to \infty$ (see [1]).

Now we compute the integral $\int_0^1 \frac{F_n(x)x^{\lambda_1}}{(d\pm x^\ell)^{\lambda_2}}dx$ in two ways:

i) Due to the orthogonality of $F_n(x)$ and (1.1) we have (here $\delta = (-1)^n$ if $d+x^\ell$; $\delta = 1$ if $d - x^\ell$)

$$\int_0^1 \frac{F_n(x)x^{\lambda_1}}{(d\pm x^\ell)^{\lambda_2}}dx =$$

$$= \delta \int_0^1 F_n(x)\frac{1}{d^{\lambda_2}}\left(\frac{\lambda_2 \ldots (\lambda_2+n-1)}{d^n}\frac{x^{n\ell+\lambda_1}}{n!} \mp \frac{\lambda_2 \ldots (\lambda_2+n)}{d^{n+1}}\frac{x^{(n+1)\ell+\lambda_1}}{n+1!} + \ldots\right)dx =$$

$$= \delta \int_0^1 F_n(x)w(x)dx = \frac{\delta}{2\pi i}\int_0^1 \int_\gamma P(t)w(x)x^t dt dx = \frac{\delta}{2\pi i}\int_\gamma \int_0^1 P(t)w(x)x^t dx dt =$$

$$= \frac{\delta}{d^{\lambda_2}2\pi i}\int_\gamma P(t)\left(\frac{\lambda_2\ldots(\lambda_2+n-1)}{d^n n!(t+n\ell+\lambda_1+1)} \mp \frac{\lambda_2\ldots(\lambda_2+n)}{d^{n+1}(n+1)!(t+(n+1)\ell+\lambda_1+1)} + \ldots\right)dt. \tag{1.3}$$

If $j = q\ell$, $q = n, n+1, \ldots$ and γ', γ'' are positively oriented curves, the first around $-j-\lambda_1-1$ of radius $\epsilon < 1$ and the second a circle centered at zero of radius much larger than $n\ell$ or $j+\lambda_1+1$ then

$$\frac{1}{2\pi i}\int_\gamma P(t)\frac{1}{(t+j+\lambda_1+1)}dt + \frac{1}{2\pi i}\int_{\gamma'} P(t)\frac{1}{(t+j+\lambda_1+1)}dt =$$

$$\frac{1}{2\pi i}\int_{\gamma''} P(t)\frac{1}{(t+j+\lambda_1+1)}dt. \tag{1.4}$$

Also $\frac{1}{2\pi i}\int_{\gamma''} P(t)\frac{1}{(t+j+\lambda_1+1)}dt = 0$ upon taking the radius of γ'' tending to infinity. Using this fact, (1.4) yields

$$\frac{1}{2\pi i}\int_\gamma P(t)\frac{1}{(t+j+\lambda_1+1)}dt = -P(-j-\lambda_1-1) = \frac{(q-n+1)_n}{\ell(q+\frac{\lambda_1+1}{\ell})_{n+1}}. \tag{1.5}$$

Let us recall the hypergeometric function formula (valid for $Re(c) > Re(b) > 0$)

$${}_2F_1(a,b,c,z) := \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)}\int_0^1 \tau^{b-1}(1-\tau)^{c-b-1}(1-\tau z)^{-a}d\tau = \sum_{n=0}^\infty \frac{(a)_n(b)_n}{(c)_n n!}z^n.$$

Using this formula and (1.5) we have that the last formula of (1.3) is equal to

$$\frac{\delta}{d^{\lambda_2}\ell 2\pi i}\frac{(\lambda_2)_n}{d^n(n+\frac{\lambda_1+1}{\ell})_{n+1}}\left(1 \mp \frac{(\lambda_2+n)_1(n+\frac{\lambda_1+1}{\ell})_1}{(2n+1+\frac{\lambda_1+1}{\ell})_1 1!d} + \frac{(\lambda_2+n)_2(n+\frac{\lambda_1+1}{\ell})_2}{(2n+1+\frac{\lambda_1+1}{\ell})_2 2!d^2} \mp \cdots\right) =$$

$$= \frac{\delta}{d^{\lambda_2}\ell 2\pi i}\frac{(\lambda_2)_n}{d^n(n+\frac{\lambda_1+1}{\ell})_{n+1}}\frac{\Gamma(2n+1+\frac{\lambda_1+1}{\ell})}{\Gamma(n+1)\Gamma(n+\frac{\lambda_1+1}{\ell})}\int_0^1 \tau^{n+\frac{\lambda_1+1}{\ell}-1}(1-\tau)^n(1\pm\frac{\tau}{d})^{-n-\lambda_2}d\tau.$$

Therefore we get that our integral is different from zero and this yields the left inequality of i). Now $\frac{\tau(1-\tau)}{(1-\frac{\tau}{d})}$ has its maximum at $\tau_0 = d - \sqrt{d(d-1)}$ being this maximum equal to $d(\sqrt{d}-\sqrt{d-1})^2$. Also $\frac{\tau(1-\tau)}{(1+\frac{\tau}{d})}$ has its maximum at $\tau_0 = \sqrt{d(d+1)} - d$ being this maximum equal to $d(\sqrt{d+1}-\sqrt{d})^2$. Therefore

$$\int_0^1 \tau^{n+\frac{\lambda_1+1}{\ell}-1}(1-\tau)^n(1\pm\frac{\tau}{d})^{-n-\lambda_2}d\tau \leqslant (max_{\tau\in[0,1]}\frac{\tau(1-\tau)}{(1\pm\frac{\tau}{d})})^n \cdot \int_0^1 \tau^{\frac{\lambda_1+1}{\ell}-1}(1\pm\frac{\tau}{d})^{-\lambda_2}d\tau.$$

Also $\frac{\Gamma(2n+1+\frac{\lambda_1+1}{\ell})}{(n+\frac{\lambda_1+1}{\ell})_{n+1}\Gamma(n+\frac{\lambda_1+1}{\ell})} = 1$. From this the right inequality of i) follows.

ii) We compute the integral in a second way showing that it satisfies a recurrence relation.

$$\int_0^1 \frac{F_n(x)x^{\lambda_1}}{(d\pm x^\ell)^{\lambda_2}}dx =$$

$$\int_0^1 \frac{(F_n(x)-F_n((\mp d)^{\frac{1}{\ell}}))x^{\lambda_1}}{(d\pm x^\ell)^{\lambda_2}}dx + F_n((\mp d)^{\frac{1}{\ell}})\int_0^1 \frac{x^{\lambda_1}}{(d\pm x^\ell)^{\lambda_2}}dx := A_n + B_n f, \qquad (1.6)$$

where $(-d)^{1/\ell} := d^{1/\ell}e^{\frac{i\pi}{\ell}}$. From the orthogonality relations for $F_n(x)$ one has, if $3 \leqslant n$

$$H_n(x) := F_n(x) + (\alpha_n x^\ell + \beta_n)F_{n-1}(x) + \gamma_n F_{n-2}(x) = 0, \qquad (1.7)$$

for some constants $\alpha_n, \beta_n, \gamma_n$. In fact, the coefficient of $x^{n\ell}$ in $F_n(x)$ is not zero and α_n, β_n can be adjusted to give $F_n(x) + (\alpha_n x^\ell + \beta_n)F_{n-1}(x) = a_0 + a_1x^\ell + \cdots + a_{n-2}x^{(n-2)\ell}$. This can be written as a linear combination of $F_{n-2}(x), \ldots, F_1(x), 1$. If this equality is multiplied by x^{λ_1} and orthogonality is used then only the coefficient of $F_{n-2}(x)$ survives.

So if we let $x = d^{1/\ell}$ or $x = (-d)^{1/\ell}$ in (1.7) then

$$B_n + (\pm\alpha_n d + \beta_n)B_{n-1} + \gamma_n B_{n-2} = 0,$$

respectively. Also

$$0 = \int_0^1 \frac{(H_n(x)-H_n((\mp d)^{1/\ell}))x^{\lambda_1}}{(d\pm x^\ell)^{\lambda_2}}dx = \int_0^1 \frac{(F_n(x)-F_n((\mp d)^{1/\ell}))}{(d\pm x^\ell)^{\lambda_2}}x^{\lambda_1}dx$$

$$\pm\alpha_n\int_0^1 F_{n-1}(x)(d\pm x^\ell)^{1-\lambda_2}x^{\lambda_1}dx + (\mp\alpha_n d+\beta_n)\int_0^1 \frac{(F_{n-1}(x)-F_{n-1}((\mp d)^{1/\ell}))}{(d\pm x^\ell)^{\lambda_2}}x^{\lambda_1}dx +$$

$$+\gamma_n\int_0^1 \frac{(F_{n-2}(x)-F_{n-2}((\mp d)^{1/\ell}))}{(d\pm x^\ell)^{\lambda_2}}x^{\lambda_1}dx.$$

Thus,

$$A_n \pm \alpha_n\int_0^1 F_{n-1}(x)(d\pm x^\ell)^{1-\lambda_2}x^{\lambda_1}dx + (\mp\alpha_n d+\beta_n)A_{n-1} + \gamma_n A_{n-2} = 0,$$

and one gets the recurrence relations of ii). Observe that in the last formula the integral is zero due to the orthogonality if $\lambda_2 = 1$ and $3 \leqslant n$.

Finally, $\alpha_n, \beta_n, \gamma_n$ are obtained from (1.7) making the coefficients of $x^{n\ell}, x^{(n-1)\ell}$, $x^{(n-2)\ell}$ equal to zero. This tedious calculation is omitted. ■

Next we give two applications of Proposition 1.

In the next two examples we take the + sign in (1.0). Example 1 is quite well known. See [10] for example.

Example 1. If one puts $\lambda_1 = 0, \ell = 1, \lambda_2 = 1$, then $F_n(x)$ are the Legendre polynomials. They can be written more simply as

$$F_n(x) = \sum_{j=0}^{n} (-1)^{n+j} \binom{n+j}{j} \binom{n}{j} x^j.$$

Thus if A_n, B_n are as defined in Proposition 1 then $A_n r_n, B_n \in N$ where $r_n = lcm\{1, 2, \ldots, n\}$ (recall $r_n = O(e^{(1+\epsilon)n})$ by the prime number theorem). Thus by i) of Proposition 1 if $1 \leqslant d \in N$

$$0 < |A_n r_n + Log\frac{d+1}{d} B_n r_n| = O((\sqrt{d+1} - \sqrt{d})^{2n} e^{(1+\epsilon)n}),$$

giving the irrationality of $Log\frac{d+1}{d}$ for $1 \leqslant d \in N$.

The following example seems to be new.

Example 2. Assume $\ell = 1, \lambda_2 = 1, d = 1$ in (1.0) and $\lambda_1 = \lambda$. Then

$$f(\lambda) = \int_0^1 \frac{x^\lambda}{(1+x)} dx = \frac{1}{1+\lambda} - \frac{1}{2+\lambda} + \frac{1}{3+\lambda} - \ldots$$

Our interest in this function comes from the fact that if $0 < \sigma < 1$, $s = \sigma + it$ then

$$\zeta(s) = \frac{sin(\pi s)}{\pi(1 - 2^{1-s})} \int_0^\infty f(\lambda) \lambda^{-s} d\lambda,$$

where $\zeta(s)$ is the zeta function of Riemann ([3] formula (1.3)).

Let $g(s) := \int_0^\infty f(\lambda)\lambda^{-s} d\lambda = \frac{\zeta(s)\pi(1-2^{1-s})}{sin(\pi s)}$. In this case in Proposition 1 one gets $A_n := A_n(\lambda) = \int_0^1 \frac{F_n(x) - F_n(-1)}{x+1} x^\lambda dx$ and $(-1)^n B_n := (-1)^n B_n(\lambda) = \sum_{j=0}^n \frac{(j+\lambda+1)_n}{n-j!j!}$.

Theorem 1. *The analytic function* $g_n(s) := \int_0^\infty \frac{A_n(\lambda)}{B_n(\lambda)} \lambda^{-s} d\lambda, 0 < \sigma < 1$, *converges uniformly to* $-g(s)$ *'inside' the critical strip, i.e.* , $|g_n(s) + g(s)| \leqslant \frac{(\sqrt{2}-1)^{2n}}{\binom{2n}{n}} \frac{1}{\pi\epsilon}$ *if* $\epsilon \leqslant \sigma \leqslant 1 - \epsilon$ *and* $0 < \epsilon < 1/2$.

Proof: Note that $A_n(\lambda)$ is continuous in $[0, +\infty)$ and $O(\lambda^{n-1})$. Also $(-1)^n B_n(\lambda)$ is a positive, increasing function on $[0, +\infty)$ and behaves asymptotically as $c\lambda^n$ as $\lambda \to +\infty, (c \neq 0)$. From this it is easily seen that $g_n(s)$ is analytic in the strip $0 < \sigma < 1$. From i) of Proposition 1 we get $|A_n(\lambda) + B_n(\lambda) f(\lambda)| \leqslant \frac{(\sqrt{2}-1)^{2n}}{2\pi(\lambda+1)}$. Thus,

$$|\int_0^\infty (\frac{A_n(\lambda)}{B_n(\lambda)} + f(\lambda))\lambda^{-s} d\lambda| \leqslant \int_0^\infty \frac{|A_n(\lambda) + B_n(\lambda) f(\lambda)|}{(-1)^n B_n(\lambda)} \lambda^{-\sigma} d\lambda$$

$$\leqslant (-1)^n \frac{(\sqrt{2}-1)^{2n}}{2\pi B_n(0)} \int_0^\infty \frac{\lambda^{-\sigma}}{\lambda+1} d\lambda.$$

Notice that $\binom{2n}{n} \leqslant (-1)^n B_n(0)$ and $\int_0^\infty \frac{\lambda^{-\sigma}}{\lambda+1} d\lambda \leqslant 2/\epsilon$. This proves the theorem.■

2. On the trilog. We prove in this section the following:

Theorem 2. *The number* $Li_3(1/d) = \sum_{n=1}^\infty \frac{1}{d^n n^3} = \frac{1}{2}\int_0^1 \frac{Log(x)^2}{d-x}dx$ *is irrational for* $d \in N, 1173 \leqslant d$.

Proof : Let $n \in N$ and define

$$F_n(x) := \frac{1}{2\pi i}\int_\gamma \frac{(t+1)^3\ldots(t+n)^3}{(t)(t-1)\ldots(t-3n)} x^t dt = \frac{1}{2\pi i}\int_\gamma P(t)x^t dt.$$

Here γ is a positively oriented curve enclosing $0, 1, \ldots, 3n$ in the half plane $-\frac{1}{2} < Re(t)$. Notice that $F_n(x)$ is orthogonal to $Log(x)^2$, $xLog(x)^2, \ldots, x^{n-1}Log(x)^2$.

Observe that

$$\int_0^1 \frac{F_n(x)}{d-x} Log(x)^2 dx = \int_0^1 \frac{F_n(x)-F_n(d)}{d-x} Log(x)^2 dx + F_n(d)\int_0^1 \frac{Log(x)^2}{d-x} dx =$$

$$:= A_n + B_n 2Li_3(1/d)$$

and also that

$$\int_0^1 \frac{F_n(x)}{d-x} Log(x)^2 dx = \frac{1}{d}\int_0^1 F_n(x)Log(x)^2 \left(\frac{x^n}{d^n} + \frac{x^{n+1}}{d^{n+1}} + \ldots\right)dx =$$

$$\frac{1}{2\pi i d}\int_0^1\int_\gamma P(t)x^t Log(x)^2 \left(\frac{x^n}{d^n} + \frac{x^{n+1}}{d^{n+1}} + \ldots\right)dtdx.$$

Changing the order of integration we obtain,

$$\frac{1}{\pi i d}\int_\gamma P(t)\left(\frac{1}{(t+n+1)^3 d^n} + \frac{1}{(t+n+2)^3 d^{n+1}} + \ldots\right)dt =$$

$$= -\frac{1}{d}\left(\frac{1}{d^n}\frac{d^2}{dt^2}P(-n-1) + \frac{1}{d^{n+1}}\frac{d^2}{dt^2}P(-n-2) + \ldots\right). \tag{2.1}$$

We have used the same idea of Proposition 1 to get the equality in the last formula (see (1.5) above). But $\frac{d}{dt}P(t) = P(t)(\frac{3}{t+1} + \cdots + \frac{3}{t+n} - \frac{1}{t-k} - \cdots - \frac{3}{t-k-3n}) := P(t)b(t)$. Therefore $\frac{d^2}{dt^2}P(t) = P(t)(b(t)^2 + b'(t))$. Notice that $b(-n-j)^2 + b'(-n-j) = h(n,j)$ with h defined by

$$h(n,j) := \left(\frac{3}{j} + \frac{3}{j+1} + \cdots + \frac{3}{n+j-1} - \frac{1}{n+j} - \frac{1}{n+1+j} - \cdots - \frac{1}{4n+j}\right)^2 -$$

$$-\left(\frac{3}{j^2} + \frac{3}{(j+1)^2} + \cdots + \frac{3}{(n+j-1)^2} - \frac{1}{(n+j)^2} - \frac{1}{(n+1+j)^2} - \cdots - \frac{1}{(4n+j)^2}\right)$$

$$= (\sum_1 - \sum_2)^2 - (\sum_3 - \sum_4), \tag{2.2}$$

Thus (2.1) is equal to

$$\frac{S(n,d)}{d^{n+1}(4n+1)\binom{4n}{3n}\binom{3n}{2n}\binom{2n}{n}}, \tag{2.3}$$

with $S(n,d)$ as defined in the following lemma which we prove later.

Lemma C. *Let*

$$S(n,d) := \sum_{j=0}^{\infty} \binom{n+j}{j}^3 \frac{(n+1)\dots(n+j)}{(4n+2)\dots(4n+j+1)} \frac{h(n,j+1)}{d^j}.$$

a) There exists n_0 such that $S(n,d)$ is non-zero if $4 \leqslant d$, $n_0 \leqslant n$

b) $S(n,d) = O(Log(n)(4n+1)\binom{4n}{3n}1.3718^{n+1} max_{t\in[01]}\{t^n(1-t)^{3n}\})$ if $1000 \leqslant d$ (the constant involved in this O-term is absolute).

We continue with the proof of Theorem 2. Up to this point we have a chain of equalities so using Lemma C a) we get that (2.3) is non-zero if n is large enough and $4 \leqslant d$. Using residues in the definition of $F_n(x)$ we obtain

$$F_n(x) = \frac{(-1)^n}{\binom{3n}{2n}\binom{2n}{n}} \sum_{j=0}^{3n} \binom{3n}{j}\binom{n+j}{j}^3 (-1)^j x^j,$$

and it is easily seen that $A_n r_n \in N$, $B_n\binom{3n}{2n}\binom{2n}{n} \in N$ if $r_n = \binom{3n}{2n}\binom{2n}{n}(lcm\{1,2,\dots,3n\})^3$.

Recall that by the prime number theorem $(lcm\{1,2,\dots,3n\})^3 \leqslant e^{(9+\epsilon)n}$ for large n. Therefore we have proved that

$$0 < |A_n r_n + B_n r_n 2Li_3(1/d)| \leqslant \frac{e^{(9+\epsilon)n}|S(n,d)|}{d^{n+1}(4n+1)\binom{4n}{3n}},$$

if n is large enough and $4 \leqslant d$. Using Lemma C b) and Stirling's formula one gets that $Li_3(1/d)$ is irrational whenever $\frac{1.3718e^9 3^3}{d4^4} < 1$ or what is the same when $1173 \leqslant d \in N$.■

The proof of Lemma C depends on Lemmas A and B which we give below.

Lemma A. *If $0 \leqslant w \leqslant 1/1000$ then*

$$\sum_{j=0}^{\infty}\binom{n+j}{j}^3 w^j \leqslant (\frac{10}{9})^{3(n+1)} \leqslant 1.3718^{n+1}.$$

Proof: Recall that $\sum_{j=0}^{\infty}\binom{n+j}{j}u^j = \frac{1}{(1-u)^{n+1}}$. Therefore, if $0 \leqslant w$

$$\sum_{j=0}^{\infty}\binom{n+j}{j}^3 w^j \leqslant (\sum_{j=0}^{\infty}\binom{n+j}{j}w^{\frac{j}{3}})^3 = (\frac{1}{(1-w^{\frac{1}{3}})^{n+1}})^3.$$

The lemma follows if we take $0 \leqslant w \leqslant 1/1000$. ■

Lemma B. *a) There exists a natural number n_0 and absolute constants $0 < c_1, c_2$ such that*

$$c_1 \leqslant h(n,j),$$

if $1 \leqslant j \leqslant 3n$, $n_0 \leqslant n$ and

$$h(n,j) \leqslant c_2,$$

if $n_0 \leqslant n$, $3n \leqslant j$.

b) We have $|h(n,j)| \leqslant c_2 Log(n)$ for all $1 \leqslant n, j$.

Proof: a) The proof is divided in four cases. Recall formula (2.2):

Case 1. $1 \leqslant j \leqslant Log(n)$. It is easily seen that in this case

$$3Log(\frac{n}{Log(n)}) + O(1) \leqslant \sum_1; \ \sum_2, \sum_3 \text{ and} \sum_4 \text{ are } O(1).$$

So the function $h(n,j)$ is greater than $2Log(\frac{n}{Log(n)})$ if n is large enough.

Case 2. $Log(n) \leqslant j \leqslant n$. In this case

$$3Log(2) + o(1) \leqslant \sum_1, \ \sum_2 \leqslant Log4 + o(1), \ \sum_3 \text{ and} \sum_4 \text{ are } o(1).$$

Notice that $(3Log2 - Log4) = .69\ldots.$ So the function $h(n,j)$ is greater than $(3Log2 - Log4)^2 + o(1)$ if n is large enough.

Case 3. $n \leqslant j \leqslant 2n$. The argument is the same as in case 2, i.e.

$$3Log(3/2) + o(1) \leqslant \sum_1, \ \sum_2 \leqslant Log(5/2) + o(1), \ \sum_3 \text{ and} \sum_4 \text{ are } o(1).$$

But $3Log(3/2) - Log(5/2) = .3\ldots.$ This yields that the function $h(n,j)$ is greater than $(3Log(3/2) - Log(5/2))^2 + o(1)$ if n is large enough.

Case 4. $2n \leqslant j \leqslant 3n$. Similarly,

$$3Log(4/3) + o(1) \leqslant \sum_1, \sum_2 \leqslant Log(2) + o(1), \sum_3 \text{ and} \sum_4 \text{ are } o(1).$$

Notice that $3Log(4/3) - Log(2) = .169\ldots$ So the function $h(n,j)$ is greater than $(3Log(4/3) - Log(2))^2 + o(1)$ if n is large enough. This proves the first inequality.

It is easy to see that the second inequality and b) hold. They are left to the reader.■

Proof of Lemma C: a) Write $S(n,d)$ as $\sum_{j=0}^{3n-1} + \sum_{j=3n}^{\infty} = \sum_5 + \sum_6$. Thus, using a) of Lemma B we get that if $n_0 \leqslant n$ then the sum $\sum_5$ is greater than one of its summands, namely that with $j = n$. Thus

$$c_1 \binom{2n}{n}^3 \frac{(n+1)\ldots(2n)}{(4n+2)\ldots(5n+1)} \frac{1}{d^n} \leqslant \sum_5. \tag{2.4}$$

Also, using a) of Lemma B one gets if $4 \leqslant d$, (here we use $\binom{n+j}{j} \leqslant \binom{4n}{3n}(\frac{4}{3})^{j-3n}$ if $3n \leqslant j$ which is proved using that $\binom{n+j+1}{j+1} = \frac{n+j+1}{j+1}\binom{n+j}{j})$)

$$|\sum_6| = O(\sum_{j=3n}^{\infty} \binom{n+j}{j}^3 \frac{(n+1)\ldots(n+j)}{(4n+2)\ldots(4n+j+1)} \frac{1}{d^j}) =$$

$$= O(\binom{4n}{3n}^3 \sum_{j=3n}^{\infty} (\frac{4}{3})^{3(j-3n)} \frac{(n+1)\dots(n+j)}{(4n+2)\dots(4n+j+1)} \frac{1}{d^j}) =$$

$$= O(\binom{4n}{3n}^3 \frac{(n+1)\dots(4n)}{(4n+2)\dots(7n+1)} (\frac{3}{4})^{9n} \sum_{j=3n}^{\infty} (\frac{(\frac{4}{3})^3}{d})^j =$$

$$= O((\frac{4^4}{3^3 d})^{3n} \frac{(n+1)\dots(4n)}{(4n+2)\dots(7n+1)}).$$

From this and (2.4) it is easily seen that $|\sum_6| < \sum_5$ if n is large enough. This proves a).

b) Noticing that $(4n+1)\binom{4n}{3n}\int_0^1 t^{n+j}(1-t)^{3n}dt = \frac{(n+1)\dots(n+j)}{(4n+2)\dots(4n+j+1)}$ and using b) of Lemma B one gets that

$$S(n,d) = O(\sum_{j=0}^{\infty} \binom{n+j}{j}^3 \frac{(n+1)\dots(n+j)}{(4n+2)\dots(4n+j+1)} \frac{Log(n)}{d^j}) =$$

$$= O(Log(n)\binom{4n}{3n}(4n+1)\int_0^1 t^n(1-t)^{3n}\{\sum_{j=0}^{\infty}\binom{n+j}{j}^3 \frac{t^j}{d^j}\}dt).$$

Now recall that $1000 \leqslant d$ and thus $0 \leqslant w = t/d \leqslant 1/1000$. This inequality together with Lemma A yields the desired result. ■

I want to express my gratitude to the referee for pointing out several references and for remarks that improved the final form of this note. Also I want to express my thanks to Tanguy Rivoal for his kindness in sending me his beautiful results and for answering my questions.

References

1. Borwein P. and Erdeli T., Polynomials and polynomial inequalities (Springer) (1995).
2. Hata Masatoshi, *Rational approximations to the dilogarithm*, Transaction of the AMS Vol 336 (1993).
3. Panzone P., *A note on certain integral formulas for $\zeta(s)$ in the critical strip*, submitted to The Ramanujan Journal (2000).
4. Berndt Bruce, Ramanujan's Notebooks II (Springer) (1989).
5. Alladi K., Robinson M.L., *Legendre polynomials and irrationality*, J. Reine Angew. Math. 318, 137-155, (1980).
6. Hata Masayoshi, *On the linear independence of the values of polylogarithmic functions*, J. Math. Pures Appl. IX Ser. 69, No. 2, 133-173 (1990).
7. Hata Masayoshi, *Irrationality measures of the values of hypergeometric functions*, Acta Arithmetica 60 No. 4, 335-347 (1992).
8. Huttner, M., *Irrationalite de certaines integrales hypergeometriques*, Journal of Number Theory 26, 166-178 (1987).
9. T. Rivoal, *Proprietes diophantiennes de la fonction zeta de Riemann aux entiers impairs*, These. Caen 2001. http://these-EN-ligne.in2p3.fr/documents/archives0/00/00/12/67/index_fr.html.
10. Baker, George A., Graves-Morris, Peter, *Pade approximants*, 2nd ed. Encyclopedia of Mathematics and Its Applications, 59. Cambridge: Cambridge Univ. Press xiv, 746p..
11. Habsieger, Laurent, *Linear recurrent sequences and irrationality measures*, Journal of Number Theory 37, No. 2, 133-145 (1991).

DEPTO. E INSTITUTO DE MATEMATICA, UNIVERSIDAD NACIONAL DEL SUR, AV. ALEM 1253, (8000) BAHIA BLANCA, ARGENTINA

E-mail address: papanzone@infovia.com.ar