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WEIGHTED BMO_ϕ SPACES AND THE HILBERT TRANSFORM

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Abstract

We obtain estimates for the distribution of values of functions in the weighted BMO_ϕ spaces, $BMO_\phi^w(\mathbf{R})$, that let us find equivalent norms. It is also obtained that a suitable redefinition of the Hilbert transform is a bounded operator from these spaces into themselves. This is achieved for a certain class of weights w.

1 Introduction.

A non negative function w defined on $\mathbf{R}$ is called a weight if it is locally integrable. We denote by $|I|$ the Lebesgue measure of I and $w(I) = \int_I w(x)dx$. The letter C denotes a constant, not necessarily the same at each occurrence. A weight is said to belong to the class A_p, $1 < p < \infty$, if there exists a constant C such that

$$\left(\frac{1}{|I|}\int_I w(x)dx\right)\left(\frac{1}{|I|}\int_I w(x)^{-\frac{1}{p-1}}dx\right)^{p-1} \leq C$$

for every interval $I \subset \mathbf{R}$. The class A_1 is defined replacing the above inequality by

$$\frac{1}{|I|}\int_I w(x)\,dx \leq C \operatorname*{ess\,inf}_I w.$$

On the other hand w is said to belong to A_∞ if there exist α and β such that $0 < \alpha, \beta < 1$ and for every interval I and every measurable subset E of I, $\int_E w\,dx < \beta \int_I w\,dx$ holds whenever $|E| < \alpha|I|$. The statement that $w \in A_\infty$ is equivalent to $w \in A_p$ for some p. A proof of these facts may be found at [1].

A weight is said to satisfy a doubling condition if there exists a constant C such that

$$w(2I) \leq Cw(I)$$

for every interval $I \subset \mathbf{R}$. As it is easy to check any weight in A_∞ satisfies a doubling condition.
Now let us introduce the main function spaces which concern us in this work. Let $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ be a non-decreasing function satisfying the Δ_2 Orlicz's condition $\phi(2r) \leq C\phi(r)$ for some positive constant C and every $r > 0$.

Definition 1.1 *Let f be a locally integrable function on $\mathbf{R}$. We say that f belongs to $BMO_\phi^w(\mathbf{R})$ if there exists a constant C such that*

$$\frac{1}{w(I)\phi(|I|)} \int_I |f(x) - f_I| dx \leq C \tag{1}$$

for every finite interval $I \subset \mathbf{R}$. Here f_I denotes the average of f over the interval I, i.e.,

$$f_I = \frac{1}{|I|} \int_I f(x)dx.$$

The smallest constant C satisfying (1) will be denoted by $||f||$ and it defines a norm in $BMO_\phi^w(\mathbf{R})$. In case that (1) only holds for those I lying in some fixed interval I_0, not necessarily of finite measure, we say that f belongs to $BMO_\phi^w(I_0)$. In that case, the smallest constant C satisfying the inequality will be called $||f||_{I_0}$. We note that if $\phi \equiv 1$ the space BMO_1^w coincides with the space $BMO(w)$ defined by Muckenhoupt and Wheeden in [5].
We now introduce a class of weights which appears in connection with the boundedness of the Hilbert transform on the BMO_ϕ^w spaces.

Definition 1.2 *Let w be a weight. We say that $w \in H(\phi, \infty)$ if there exists a constant C such that*

$$\frac{|I|}{\phi(|I|)} \int_{\mathbf{R}-I} \frac{w(y)\phi(|x_0 - y|)}{|x_0 - y|^2} dy \leq C\frac{w(I)}{|I|}$$

for every finite interval $I \subset \mathbf{R}$, where x_0 denotes the center of I.

In the previous definitions we have not imposed any constraints on the growth of ϕ. It is known that if $w \equiv 1$ and $\phi(t) = t^\beta$, with $\beta > 1$, the unique functions belonging to BMO_ϕ^w are the constant ones. In the weighted case, the spaces BMO_ϕ^w may be non-trivial for such functions ϕ: that depends on the weight w. In fact, there are examples showing this situation. However, it may be proved that if the function $\frac{\phi(t)}{t^2}$ is still increasing then BMO_ϕ^w is trivial for every weight w.

Similar considerations hold for the classes $H(\phi,\infty)$. In other words, if ϕ increases in such a way that $\frac{\phi(t)}{t^2}$ is non-decreasing, the only weight belonging to the class is $w\equiv 0$. We can also point out that if $\phi(t)$ grows faster than t (let us say $\frac{\phi(t)}{t}$ is non-decreasing) but slower than t^2, then the class $H(\phi,\infty)$ is non-trivial, although the weight $w\equiv 1$ does not belong to it.
Finally, though we will not impose additional restrictions on the functions ϕ in the statement of the theorems, it is clear that they become trivial when ϕ increases faster than t^2.
We present now a pair of lemmas that will be necessary later. See [5] for a proof of the following lemma.

Lemma 1.1 *Let $1<p<\infty$ and $\frac{1}{p}+\frac{1}{p'}=1$. If $w\in A_p$, there exists a constant C such that*

$$w(\{x\in I: w(x)<\beta\})\le C\left(\beta\frac{|I|}{w(I)}\right)^{p'}w(I)$$

for every interval I and every $\beta>0$.

Lemma 1.2 is quite similar to Lemma (4.7) in [4], which has been adapted to this context.

Lemma 1.2 *Let w be a weight satisfying a doubling condition. If $f\in BMO_\phi^w(\mathbf{R})$ then there exists a constant C such that*

$$\int_{\mathrm{R}-I}\frac{|f(y)-f_I|}{|x_0-y|^2}dy\le C||f||\int_{\mathrm{R}-I}\frac{w(y)\phi\left(|x_0-y|\right)}{|x_0-y|^2}dy$$

for every interval $I\subset\mathbf{R}$, where x_0 is the center of I.

Proof. Given an interval $I=I(x_0,R)$ let $I_j=I(x_0,2^jR)$. Using that $f\in BMO_\phi^w(\mathbf{R})$, we have

$$\begin{aligned}\int_{\mathrm{R}-I}\frac{|f(y)-f_I|}{|x_0-y|^2}dy &= \sum_{j=0}^{\infty}\int_{2^jR\le|x_0-y|<2^{j+1}R}\frac{|f(y)-f_I|}{|x_0-y|^2}dy\\ &\le\sum_{j=0}^{\infty}\frac{1}{2^{2j}R^2}\int_{I_{j+1}-I_j}|f(y)-f_I|dy\\ &\le C|I|^{-1}\sum_{j=0}^{\infty}\frac{2^{-j}}{|I_{j+1}|}\int_{I_{j+1}}|f(y)-f_I|dy\end{aligned}$$

$$
\begin{aligned}
&\le C|I|^{-1}\sum_{j=0}^{\infty}\frac{2^{-j}}{|I_{j+1}|}\sum_{k=0}^{j+1}\frac{1}{|I_k|}\int\limits_{I_k}|f(y)-f_{I_k}|dy\\
&\le C|I|^{-1}||f||\sum_{j=0}^{\infty}2^{-j}\sum_{k=0}^{j+1}\frac{w(I_k)\phi\left(|I_k|\right)}{|I_k|}\\
&= C|I|^{-1}||f||\sum_{k=0}^{\infty}2^{-k}\frac{w(I_k)\phi\left(|I_k|\right)}{|I_k|}\\
&\le C||f||\int\limits_{\mathrm{R}-I}\frac{w(y)\phi\left(|x_0-y|\right)}{|x_0-y|^2}dy.
\end{aligned}
$$

The last inequality is a result of using the properties of ϕ, the doubling condition for w and the following relations

$$
\begin{aligned}
\int\limits_{\mathrm{R}-I}\frac{w(y)\phi\left(|x_0-y|\right)}{|x_0-y|^2}dy &= \sum_{k=0}^{\infty}\int\limits_{2^kR\le|x_0-y|<2^{k+1}R}\frac{w(y)\phi\left(|x_0-y|\right)}{|x_0-y|^2}dy\\
&\ge C|I|^{-1}\sum_{k=0}^{\infty}2^{-k}\frac{\phi\left(|I_k|\right)}{|I_k|}w\left(I_{k+1}-I_k\right)\\
&\ge C|I|^{-1}\sum_{k=0}^{\infty}2^{-k}\frac{\phi\left(|I_k|\right)}{|I_k|}w(I_k),
\end{aligned}
$$

which completes the proof. □

2 Behavior of the distribution function and equivalent norms

Let $f \in BMO_\phi^w(I_0)$. The question is: how does the distribution function $w(\{x \in I : |f(x)-f_I|w^{-1}(x) > \alpha\})$, for $\alpha > 0$ and $I \subset I_0$, behave? An answer is given in [5] for the case $\phi \equiv 1$ and it may be used in our case. The result obtained by Muckenhoupt and Wheeden is the following theorem:

Theorem. *Let f be of bounded mean oscillation with weight w on I_0, that is*

$$
\int\limits_I |f-f_I|dx \le C\int\limits_I w\,dx; \qquad f_I=\frac{1}{|I|}\int\limits_I f\,dx, \qquad I\subset I_0.
$$

a) If $w \in A_1$, there are positive constants c_1 and c_2 such that

$$
w(\{x\in I : |f(x)-f_I|w^{-1}(x) > \alpha\}) \le c_1 e^{-c_2\alpha}w(I)
$$

for $\alpha > 0$ *and* $I \subset I_0$.
b) If $w \in A_p$, $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$, *there is a constant* c_3 *such that*

$$w(\{x \in I : |f(x) - f_I|w^{-1}(x) > \alpha\}) \leq c_3 \, (1+\alpha)^{-p'} w(I)$$

for $\alpha > 0$ *and* $I \subset I_0$.
If we denote by $[f]_{I_0} = \sup\limits_{I \subset I_0} \frac{1}{w(I)} \int\limits_I |f(x) - f_I| dx$, a careful look at the proof shows that in the case $[f]_{I_0} \leq 1$ the constants in the above estimates depend only on the A_1 or A_p condition for the weight w and not on f neither on the interval I_0. From this, the general case follows, as usual, taking $g = \frac{f}{[f]_{I_0}}$. Therefore $a)$ and $b)$ can be written as
a') If $w \in A_1$, *there exist positive constants* A *and* B *such that*

$$w(\{x \in I : |f(x) - f_I|w^{-1}(x) > \alpha\}) \leq A e^{-\frac{B}{[f]_{I_0}}\alpha} w(I)$$

for $\alpha > 0$ *and* $I \subset I_0$.
b') If $w \in A_p$, $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$, *there is a positive constant* D *such that*

$$w(\{x \in I : |f(x) - f_I|w^{-1}(x) > \alpha\}) \leq D\left(1 + \frac{\alpha}{[f]_{I_0}}\right)^{-p'} w(I)$$

for $\alpha > 0$ *and* $I \subset I_0$.
To estimate the distribution function for $f \in BMO^w_\phi(I_0)$ let us take a finite interval $\bar{I} \subset I_0$, then $BMO^w_\phi(I_0) \subset BMO^w_\phi(\bar{I})$, and the inclusion is a continuous operator, in fact, $||f||_{\bar{I}} \leq ||f||_{I_0}$.
Since by definition we have

$$\frac{1}{w(I)\phi(|I|)} \int\limits_I |f(x) - f_I| dx \leq ||f||_{\bar{I}},$$

for all $I \subset \bar{I}$, the fact that ϕ is an increasing function implies

$$[f]_{\bar{I}} \leq \phi(|\bar{I}|)||f||_{\bar{I}}.$$

Therefore, combining inequalities we may write

$$[f]_{\bar{I}} \leq \phi(|\bar{I}|)||f||_{I_0},$$

for $f \in BMO^w_\phi(I_0)$ and for all $\bar{I} \subset I_0$. Finally noting that $h(x) = e^{-\frac{c}{x}}, c > 0$, is an increasing function of x we may apply $a')$ and use the previous inequality to obtain for $f \in BMO^w_\phi(I_0)$ and $w \in A_1$

$$w(\{x \in \bar{I} : |f(x) - f_{\bar{I}}|w^{-1}(x) > \alpha\}) \leq A e^{-\frac{B}{[f]_{I_0}}\alpha} w(\bar{I})$$

$$\leq A e^{-\frac{B}{\phi(|\bar{I}|)||f||_{I_0}}\alpha} w(\bar{I})$$

for all $\alpha > 0$ and $\bar{I} \subset I_0$. Also, having that the function $g(x) = \left(1 + \frac{c}{x}\right)^{-p'}$ with $c > 0$ and $p' > 0$ is also increasing we may use $b')$ for $w \in A_p, 1 < p < \infty$, to get

$$w(\{x \in \bar{I} : |f(x) - f_{\bar{I}}|w^{-1}(x) > \alpha\}) \leq D\left(1 + \frac{\alpha}{\phi(|\bar{I}|)||f||_{I_0}}\right)^{-p'} w(\bar{I})$$

for all $\alpha > 0$ and $\bar{I} \subset I_0$.
So we have achieved our goal of knowing the behavior of the distribution function for $f \in BMO_\phi^w(I_0)$. Next theorem clearly states this result and gives in addition a hint on an alternative way to prove it. Moreover, it shows that the condition on the distribution of values is not only necessary but also sufficient for an f to belong to $BMO_\phi^w(I_0)$.

Theorem 2.1 *Let $f \in BMO_\phi^w(I_0)$, i.e., there exists a constant C such that*

$$\frac{1}{w(I)\phi(|I|)} \int_I |f(x) - f_I|dx \leq C \tag{2}$$

for every interval $I \subset I_0$. Then
a) If $w \in A_1$, there exist positive constants A and B such that

$$w(\{x \in I : |f - f_I|w^{-1}(x) > \alpha\}) \leq Ae^{-\frac{B}{\phi(|I|)||f||_{I_0}}\alpha} w(I) \tag{3}$$

for $\alpha > 0$ and $I \subset I_0$.
b) If $w \in A_p$, $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, there is a positive constant D such that

$$w(\{x \in I : |f - f_I|w^{-1}(x) > \alpha\}) \leq D\left(1 + \frac{\alpha}{\phi(|I|)||f||_{I_0}}\right)^{-p'} w(I) \tag{4}$$

for $\alpha > 0$ and $I \subset I_0$.
Conversely if $w \in A_1$ and there exist positive constants A and C such that

$$w(\{x \in I : |f - f_I|w^{-1}(x) > \alpha\}) \leq Ae^{-\frac{C}{\phi(|I|)}\alpha} w(I) \tag{5}$$

for $\alpha > 0$ and $I \subset I_0$ then $f \in BMO_\phi^w(I_0)$. On the other hand, if $w \in A_p$, $1 < p < \infty$ and there is a positive constant C' such that

$$w(\{x \in I : |f - f_I|w^{-1}(x) > \alpha\}) \leq C'\left(1 + \frac{\alpha}{\phi(|I|)}\right)^{-p'} w(I) \tag{6}$$

for $\alpha > 0$ and $I \subset I_0$ then $f \in BMO_\phi^w(I_0)$.

Proof. That the condition $f \in BMO_\phi^w(I_0)$ implies (3) and (4) has been proved by the argument given above. This may also be shown following the steps of the proof in [5] but changing the definition of the function $\lambda(\alpha, I)$ there by

$$\lambda(\alpha, I) = w(\{x \in I : |f(x) - f_I|w^{-1}(x) > \alpha\phi(|I|)\}).$$

Then the same arguments can be carried out leading us to the required estimates. Conversely, if $w \in A_1$ and f satisfies (5) we have

$$\begin{aligned}
\int_I |f - f_I|dx &= \int_I |f - f_I|w^{-1}wdx \\
&= \int_0^\infty w(\{x \in I : |f - f_I|w^{-1} > \alpha\})d\alpha \\
&\leq Aw(I)\frac{\phi(|I|)}{C}\int_0^\infty e^{-\frac{C\alpha}{\phi(|I|)}}\frac{C\,d\alpha}{\phi(|I|)} \\
&\leq \bar{C}w(I)\phi(|I|),
\end{aligned}$$

where $\bar{C} = \frac{A}{C}$. Therefore $f \in BMO_\phi^w(I_0)$.
Similarly, if $w \in A_p$ and f satisfies (6)

$$\begin{aligned}
\int_I |f - f_I|dx &= \int_0^\infty w(\{x \in I : |f - f_I|w^{-1} > \alpha\})d\alpha \\
&\leq C'\phi(|I|)\,w(I)\int_0^\infty \left(1 + \frac{\alpha}{\phi(|I|)}\right)^{-p'}\frac{d\alpha}{\phi(|I|)} \\
&\leq C'w(I)\phi(|I|),
\end{aligned}$$

We conclude that $f \in BMO_\phi^w(I_0)$ which completes the proof of the theorem.□

The characterization of the distribution function of the elements in BMO_ϕ^w given in the previous theorem allows us to introduce some equivalent norms.

Theorem 2.2 *Let $1 \leq p < \infty$ and $w \in A_p$. Then $f \in BMO_\phi^w(I_0)$ if and only if there exists a constant C_r such that*

$$\left(\int_I |f - f_I|^r w^{1-r}dx\right)^{1/r} \leq C_r\phi(|I|)\,(w(I))^{1/r} \tag{7}$$

for every $I \subset I_0$ and every r such that $1 < r \leq p'$ and $r < \infty$. For every fixed r satisfying these conditions the infimum constant C_r defines an equivalent norm in $BMO^w_\phi(I_0)$.

Proof. Suppose that f satisfies (7) for some r, $1 < r < \infty$. Then, by Hölder's inequality

$$\begin{aligned}\int_I |f - f_I|dx &= \int_I |f - f_I| w^{\frac{1-r}{r}} w^{\frac{r-1}{r}} dx \\ &\leq (\int_I |f - f_I|^r w^{1-r} dx)^{1/r} (w(I))^{1/r'} \\ &\leq C_r\, \phi\,(|I|)\,(w(I))^{1/r+1/r'} \\ &= C_r\, \phi\,(|I|)\, w(I).\end{aligned}$$

Therefore, $f \in BMO^w_\phi(I_0)$. Conversely, let us suppose that $f \in BMO^w_\phi(I_0)$. Then, if $0 < r < \infty$

$$\begin{aligned}\int_I |f - f_I|^r w^{1-r} dx &= \int_I (|f - f_I| w^{-1})^r w dx \\ &= r \int_0^\infty \alpha^{r-1} w(\{x \in I : |f(x) - f_I| w^{-1}(x) > \alpha\}) d\alpha\end{aligned}$$

If $w \in A_1$, using Theorem 2.1 and denoting $C = ||f||_{I_0}$, we have

$$\begin{aligned}\int_I |f - f_I|^r w^{1-r} dx &\leq A\, r\, w(I) \int_0^\infty \alpha^{r-1} e^{-\frac{B}{\phi(|I|)C}\alpha} d\alpha \\ &= A\, r\, w(I) \int_0^\infty \alpha^r e^{-\frac{B}{\phi(|I|)C}\alpha} \frac{d\alpha}{\alpha} \\ &= A\, r\, w(I) \int_0^\infty (t\phi\,(|I|)\, C)^r e^{-Bt} \frac{dt}{t} \\ &= A\, r\, w(I) \phi^r\,(|I|)\, C^r \int_0^\infty t^{r-1} e^{-Bt} dt \\ &= A_r ||f||^r_{I_0} \phi^r\,(|I|)\, w(I).\end{aligned}$$

Therefore

$$\left(\int_I |f - f_I|^r w^{1-r} dx \right)^{1/r} \leq A'_r ||f||_{I_0} \phi\,(|I|)\,(w(I))^{1/r}.$$

Now if $w \in A_p, 1 < p < \infty$, using part $b)$ of Theorem 2.1, we have

$$\int_I |f - f_I|^r w^{1-r} dx \le rDw(I) \int_0^\infty \left(1 + \frac{\alpha}{||f||_{I_0}\phi(|I|)}\right)^{-p'} \alpha^{r-1} d\alpha$$
$$= D'_r ||f||_{I_0}^r r\phi^r(|I|)\, w(I) \int_0^\infty \frac{t^{r-1}}{(1+t)^{p'}} dt$$
$$= D_r ||f||_{I_0}^r \phi^r(|I|)\, w(I),$$

since $p' > r$. Raising to the power $1/r$ both sides of the inequality we obtain (7).□

The result presented in the following corollary will be useful later.

Corollary 2.1 *Let $1 \le p < \infty$ and $w \in A_p$. If $f \in BMO_\phi^w(\mathbb{R})$ then there exists a number $q > 1$ such that $f \in L_{loc}^q(\mathbb{R})$. Moreover, for any finite interval I we have*

$$||f||_{L_q(I)} \le C||f||\phi(|I|)w(I)|I|^{\frac{1}{q}-1}.$$

Proof. Since $w \in A_p$, it satisfies a reverse Hölder inequality, i.e., there exists $\beta > 1$, depending only on p and on the A_p constant for w, such that, for every interval I

$$\left(\frac{1}{|I|}\int_I w^\beta dx\right)^{1/\beta} \le \frac{C}{|I|}\int_I w(x)dx,$$

with a constant C not depending on I (see [2]). Next let us choose $q > 1$ and $s > 1$ such that $1 < qs \le p'$ and $\frac{qs-1}{s-1} \le \beta$. So the reverse Hölder inequality also holds for $\frac{qs-1}{s-1}$. Using the previous theorem, we will have

$$\Big(\int_I |f|^q dx\Big)^{1/q} = \Big(\int_I |f|^q w^{\frac{1-qs}{s}} w^{\frac{qs-1}{s}} dx\Big)^{1/q}$$
$$\le \Big(\int_I |f|^{qs} w^{1-qs} dx\Big)^{1/qs} \Big(\int_{\tilde{I}} w^{\frac{(qs-1)}{s}s'} dx\Big)^{1/qs'}$$
$$= \Big(\int_I |f|^{qs} w^{1-qs} dx\Big)^{1/qs} \Big(\int_I w^{\frac{qs-1}{s-1}} dx\Big)^{\frac{s-1}{qs}}$$

$$\leq C\|f\|\phi(|I|)(w(I))^{1/qs}[(\frac{1}{|I|}\int_I w^{\frac{sq-1}{s-1}}dx)^{\frac{s-1}{sq-1}}]^{\frac{sq-1}{qs}}|I|^{\frac{s-1}{qs}}$$

$$\leq C\|f\|\phi(|I|)(w(I))^{1/qs}|I|^{\frac{1}{q}-\frac{1}{qs}}\frac{1}{|I|^{1-\frac{1}{qs}}}(w(I))^{1-\frac{1}{qs}}$$

$$= C\|f\|\phi(|I|)w(I)|I|^{\frac{1}{q}-1}.$$

for any fixed interval $I \subset \mathbb{R}$, and we obtain the desired result. $\square$

3 Hilbert Transform

Let f be a measurable function in $\mathbb{R}$. We define the operator

$$\mathcal{H}f(x) = \lim_{\epsilon \to 0+} \int_{|x-y|>\epsilon} \left[\frac{1}{x-y} + \frac{\mathcal{X}(y)}{y}\right] f(y)dy$$

where $\mathcal{X}(y)$ is the characteristic function of $|y| > 1$, provided the limit exists for almost every x. We denote by Hf the Hilbert transform, i.e.,

$$Hf(x) = \lim_{\epsilon \to 0+} \int_{|x-y|>\epsilon} \frac{1}{x-y} f(y)dy.$$

It is easy to see that if $\mathcal{H}f$ and Hf both exist for almost every x then they differ by a constant. This is the case if, for example, $f \in L^p$, $1 \leq p < \infty$. However $\mathcal{H}f$ may exist while Hf may not. As we will see later this happens, for example, when f is a constant function.
It is known that the Hilbert transform Hf is bounded in L^p_w if and only if $w \in A_p, 1 < p < \infty$ [3]. Moreover, for the case $p = \infty$ it is shown in [5] that the operator $\mathcal{H}$ previously defined is bounded from $L^{\infty}(w^{-1}) = \{f : \|fw^{-1}\|_{\infty} < c\}$ in $BMO(w)$ if and only if $w \in H(1,\infty) \cap A_{\infty}$. The next theorem shows that $\mathcal{H}f$ is well defined for $f \in BMO^w_{\phi}, w \in H(\phi,\infty) \cap A_{\infty}$ and that it is also a bounded operator. In particular, if $\phi \equiv 1$ we have an extension of Muckenhoupt and Wheeden's result, since $L^{\infty}(w^{-1}) \subseteq BMO(w)$.

Theorem 3.1 *Let $f \in BMO^w_{\phi}(\mathbb{R})$. If $w \in A_{\infty} \cap H(\phi,\infty)$ then there exists a constant C such that*

$$\frac{1}{w(I)\phi(|I|)}\int_I |\mathcal{H}f(x) - (\mathcal{H}f)_I|dx \leq C\|f\| \quad \textit{for every } I \subset \mathbb{R},$$

i.e., $\mathcal{H}f$ is a bounded operator from $BMO^{\phi}_w(\mathbb{R})$ into itself.

Proof. To prove that the Hilbert transform is well defined over BMO_ϕ^w we will show first that if C is a constant then $\mathcal{H}C = 0$. We only need to prove this for $C = 1$. In that case we have

$$\mathcal{H}1\,(x) = \lim_{\epsilon\to 0} \int\limits_{|x-y|>\epsilon} \left(\frac{1}{x-y} + \frac{\mathcal{X}(y)}{y}\right) dy.$$

Note that

$$\lim_{\epsilon\to 0}\lim_{R\to\infty} \int\limits_{|x-y|>\epsilon} \mathcal{X}_{(-R,R)}(x-y)\left(\frac{1}{x-y} + \frac{\mathcal{X}(y)}{y}\right) dy = 0. \tag{8}$$

In fact, if ϵ and R are fixed numbers such that $0 < \epsilon < R$, then

$$\int\limits_{R>|x-y|>\epsilon} \frac{1}{x-y}dy = 0.$$

Also, if $|x| > 1$, considering $R > 2\,|x|$ and ϵ sufficiently small, we have

$$\begin{aligned}\int\limits_{\substack{R>|x-y|>\epsilon\\ |y|>1}} \frac{1}{y}\,dy &= \operatorname{sg}(x)\int\limits_{R-|x|}^{|x|+R} \frac{1}{y}\,dy - \int\limits_{|x-y|<\epsilon} \frac{1}{y}\,dy\\ &= \pm\left(\ln\left(|x|+R\right) - \ln\left(R-|x|\right)\right) - \int\limits_{|x-y|<\epsilon} \frac{1}{y}\,dy,\end{aligned}$$

therefore, the first term tends to zero if $R \to \infty$ and the second tends to zero if $\epsilon \to 0$.
When $|x| < 1$ and $R > 2$ we only have the first term and taking $\lim\limits_{R\to\infty}$ we obtain the result.
On the other hand, it is easy to see that

$$\int\limits_{|x-y|>\epsilon} \left|\frac{1}{x-y} + \frac{\mathcal{X}(y)}{y}\right| dy < \infty.$$

In fact

$$\int\limits_{|x-y|>\epsilon} \left|\frac{1}{x-y} + \frac{\mathcal{X}(y)}{y}\right| dy = \int\limits_{\substack{|x-y|>\epsilon\\ |y|<1}} \left|\frac{1}{x-y}\right| dy + \int\limits_{\substack{|x-y|>\epsilon\\ |y|>1}} \left|\frac{1}{x-y} + \frac{1}{y}\right| dy.$$

It is easy to check that the first term is finite. Also

$$\int\limits_{\substack{|x-y|>\epsilon \\ |y|>1}} \left|\frac{1}{x-y}+\frac{1}{y}\right| dy = \int\limits_{\substack{|x-y|>\epsilon \\ |y|>2|x| \\ |y|>1}} \frac{|x|}{|x-y|\,|y|} dy + \int\limits_{\substack{|x-y|>\epsilon \\ 2|x|>|y|>1}} \left|\frac{1}{x-y}+\frac{1}{y}\right| dy$$

$$\leq c\,|x| \int\limits_{|y|>2|x|} \frac{1}{|y|^2} dy + \int\limits_{\substack{3|x|>|x-y|>\epsilon \\ 2|x|>|y|>1}} \left(\left|\frac{1}{x-y}\right| + \left|\frac{1}{y}\right|\right) dy,$$

and this shows that both terms are finite. Using Lebesgue's dominated convergence theorem, from (8) we obtain $\mathcal{H}1 = 0$, as stated.

In order to see that $\mathcal{H}f(x)$ is finite almost everywhere we may write $\mathcal{H}f(x) = \mathcal{H}(f - f_I)(x)$. Let $x \in I = (-R, R)$ and $R > 1/2$. Then

$$\mathcal{H}\,(f - f_I)\,(x) = \lim_{\epsilon\to 0} \int\limits_{|x-y|>\epsilon} \left(\frac{1}{x-y} + \frac{\chi(y)}{y}\right) (f\,(y) - f_I)\, dy$$

$$= \lim_{\epsilon\to 0} \int\limits_{\substack{|x-y|>\epsilon \\ |y|>2R}} \left(\frac{1}{x-y} + \frac{\chi(y)}{y}\right) (f\,(y) - f_I)\, dy +$$

$$+ \lim_{\epsilon\to 0} \int\limits_{\substack{|x-y|>\epsilon \\ |y|<2R}} \left(\frac{1}{x-y} + \frac{\chi(y)}{y}\right) (f(y) - f_I)\, dy$$

$$= T_1(x) + T_2(x).$$

For the first term note that $|x-y| > |y| - |x| > |y| - R > |y|/2$, so using Lemma 1.2 and the fact that $w \in H(\phi, \infty)$, we have for $x \in I$

$$|T_1(x)| \leq \lim_{\epsilon\to 0} \int\limits_{\substack{|x-y|>\epsilon \\ |y|>2R}} \left|\frac{x}{(x-y)\,y}\right| |f(y) - f_I|\, dy$$

$$\leq 2R \int\limits_{|y|>2R} \frac{|f(y) - f_I|}{|y|^2} dy$$

$$\leq 2R \int\limits_{|y|>R} \frac{|f(y) - f_I|}{|y|^2} dy$$

$$\leq 2Rc\,\|f\| \int\limits_{|y|>R} \frac{w(y)\phi\,(|y|)}{|y|^2} dy$$

$$\leq c\,\|f\|\, w(I)\phi\,(|I|)\, |I|^{-1}.$$

On the other hand

$$T_2(x) = \lim_{\epsilon\to 0} \int\limits_{\substack{|x-y|>\epsilon \\ |y|<2R}} \frac{1}{x-y}(f(y)-f_I)\,dy + \lim_{\epsilon\to 0} \int\limits_{\substack{|x-y|>\epsilon \\ 2R>|y|>1}} \frac{f(y)-f_I}{y}dy.$$

From Corollary 2.1 we know that there exists a number $q>1$ such that $f(y)-f_I \in L^q_{loc}$. Since $Hg(x)$ is finite for almost every x when $g\in L^q$, considering $g(y) = \chi_{(-2R,2R)}(y)\,(f(y)-f_I)$ we conclude that the first term is finite a.e. The absolute value of the second term is bounded by

$$\int\limits_{2R>|y|>1} \frac{|f(y)-f_I|}{|y|}dy \le \int\limits_{|y|<2R} |f(y)-f_I|\,dy < \infty,$$

because $f(y)-f_I \in L^1_{loc}$. We have shown then that $\mathcal{H}f(x)$ is finite for almost every $x\in I$. Letting $R\to\infty$ our claim is completely proved.
To get the norm estimate let I be any finite interval. Since $\mathcal{H}C=0$ when C is constant, we have that

$$\mathcal{H}f(x) = \mathcal{H}(f-f_I)(x)$$

Let $g(x) = (f-f_I)(x)$. Then $\mathcal{H}f(x) = \mathcal{H}g(x)$.
If $\widetilde{I} = 2I$, we put $g = g_1 + g_2$, with $g_1 = g\mathcal{X}_{\widetilde{I}}$ and $g_2 = g\mathcal{X}_{(\widetilde{I})^C}$ where $(\widetilde{I})^C$ denotes the complement of $\widetilde{I}$. Then

$$\begin{aligned}\int\limits_I |\mathcal{H}f(x) - (\mathcal{H}f)_I|dx &= \int\limits_I |\mathcal{H}g(x) - (\mathcal{H}g)_I|dx \\ &\le \int\limits_I |\mathcal{H}g_1(x) - (\mathcal{H}g_1)_I|dx + \int\limits_I |\mathcal{H}g_2(x) - (\mathcal{H}g_2)_I|dx \\ &= I_1 + I_2\end{aligned}$$

Since $w\in A_\infty$ and $BMO^\phi_w(\mathbb{R})$, using Corollary 2.1 we obtain that there exists $q>1$ such that $f\in L^q_{loc}(\mathbb{R})$ and therefore $g_1\in L^q(\mathbb{R})$. Then we have that $\mathcal{H}g_1 = Hg_1 + C$ and so

$$\begin{aligned}\int\limits_I |\mathcal{H}g_1(x) - (\mathcal{H}g_1)_I|dx &= \int\limits_I |Hg_1(x) - (Hg_1)_I|dx \\ &\le 2\int\limits_I |Hg_1(x)|dx.\end{aligned}$$

Now, using Hölder's inequality and the fact that the operator H is of strong type (q,q) for $q>1$, we have

$$\begin{aligned}\int_I |Hg_1(x)|dx &\leq (\int_I |Hg_1(x)|^q dx)^{1/q}|I|^{1/q'}\\ &\leq (\int_{\mathbb{R}} |Hg_1(x)|^q dx)^{1/q}|I|^{1/q'}\\ &\leq C(\int_{\mathbb{R}} |g_1(x)|^q dx)^{1/q}|I|^{1/q'}.\end{aligned}$$

Let us estimate $\|g_1\|_{L^q(dx)} = (\int_{\tilde{I}} |f-f_I|^q dx)^{1/q}$.

$$\begin{aligned}(\int_{\tilde{I}} |f-f_I|^q dx)^{1/q} &\leq (\int_{\tilde{I}} |f-f_{\tilde{I}}|^q dx)^{1/q} + (\int_{\tilde{I}} |f_I-f_{\tilde{I}}|^q dx)^{1/q}\\ &= 2\|f\|_{L^q(\tilde{I})} + A_2\\ &= A_1 + A_2.\end{aligned}$$

Corollary 2.1 gives us a bound for A_1. Now, let us estimate A_2.

$$\begin{aligned}A_2 &= \left(\int_{\tilde{I}} |f_I - f_{\tilde{I}}|^q dx\right)^{1/q}\\ &= C|f_I - f_{\tilde{I}}||\tilde{I}|^{1/q}\\ &\leq C\frac{1}{|I|}\int_I |f-f_{\tilde{I}}|dx\,|\tilde{I}|^{1/q}\\ &\leq C\frac{1}{|I|}\int_{\tilde{I}} |f-f_{\tilde{I}}|dx\,|I|^{1/q}\\ &\leq Cw(\tilde{I})\phi(|\tilde{I}|)|I|^{1/q-1}\,\|f\|.\end{aligned}$$

We have used the fact that $f \in BMO_\phi^w(\mathbb{R})$ in the last inequality. Therefore

$$A_1 + A_2 \leq C\,\|f\|\,\phi(|\tilde{I}|)w(\tilde{I})|\tilde{I}|^{\frac{1}{q}-1} \leq C\,\|f\|\,\phi\,(|I|)\,w(I)|I|^{\frac{1}{q}-1}$$

where we used the doubling condition for w. Finally,

$$\int_I |\mathcal{H}g_1(x) - (\mathcal{H}g_1)_I|dx \leq C\,\|f\|\,\phi\,(|I|)\,w(I).$$

Now, we will estimate I_2 using Lemma 1.2 and the fact that $w \in H(\phi, \infty)$. Let x_0 denote the center of I and put $R = \frac{|I|}{2}$, then we have

$$\begin{aligned}
|\mathcal{H}g_2(x) - \mathcal{H}g_2(z)| &= \left| \int\limits_{\tilde{I}^C} \left(\frac{1}{x-y} - \frac{1}{z-y} \right) g_2(y)\,dy \right| \\
&\leq \int\limits_{\tilde{I}^C} \frac{|z-x|}{|x-y||z-y|} |g_2(y)|\,dy \\
&\leq |z-x| \int\limits_{|x_0-y|\geq 2R} \frac{|f-f_I|}{|x-y||z-y|} dy \\
&\leq C|I| \int\limits_{|x_0-y|\geq R} \frac{|f-f_I|}{|x_0-y|^2} dy \\
&\leq C|I|||f|| \int\limits_{|x_0-y|\geq R} \frac{w(y)\phi\left(|x_0-y|\right)}{|x_0-y|^2} dy \\
&= C|I|^{-1}\phi\left(|I|\right) w(I) ||f||
\end{aligned}$$

Therefore

$$|\mathcal{H}g_2(x) - \mathcal{H}g_2(z)| \leq C|I|^{-1}\phi\left(|I|\right) w(I)||f|| \tag{9}$$

when $x, z \in I$. But since

$$\begin{aligned}
\int\limits_I |\mathcal{H}g_2(x) - (\mathcal{H}g_2)_I| dx &= \int\limits_I |\mathcal{H}g_2(x) - \frac{1}{|I|}\int\limits_I \mathcal{H}g_2(z)dz|dx \\
&= \int\limits_I |\frac{1}{|I|}\int\limits_I \mathcal{H}g_2(x)dz - \frac{1}{|I|}\int\limits_I \mathcal{H}g_2(z)dz|dx \\
&= \int\limits_I \frac{1}{|I|} |\int\limits_I [\mathcal{H}g_2(x) - \mathcal{H}g_2(z)]dz|dx \\
&\leq \frac{1}{|I|}\int\limits_I\int\limits_I |\mathcal{H}g_2(x) - \mathcal{H}g_2(z)|dzdx,
\end{aligned}$$

using (9) we have

$$\int\limits_I |\mathcal{H}g_2(x) - (\mathcal{H}g_2)_I| dx \leq \frac{1}{|I|}|I|^2 C|I|^{-1}\phi\left(|I|\right) w(I)||f||$$

that is to say

$$\int_I |\mathcal{H}g_2(x) - (\mathcal{H}g_2)_I|dx \leq C\,\phi\,(|I|)\,w(I)||f||.$$

Putting together the estimates for $\mathcal{H}g_1$ and $\mathcal{H}g_2$ we obtain the desired result. □

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