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# ON CONVERGENCE OF DERIVATIVES OF A NEW SEQUENCE OF LINEAR POSITIVE OPERATORS

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**ABSTRACT.** In the present paper, we introduce a new sequence of linear positive operators to approximate a class of unbounded continuous functions of exponential growth on the interval $[0,\infty)$. First, we study the basic pointwise convergence theorem in simultaneous approximation and then proceed to study the degree of this approximation.

## 1. INTRODUCTION

Let $C[0,\infty)$ denote the class of all continuous functions on the interval $[0,\infty)$. For $\alpha>0$ and $f\in C_\alpha[0,\infty)=:\{f\in C[0,\infty):|f(t)|\le M e^{\alpha t}$ for some $M>0\}$, we define a sequence of linear positive operators $M_n$ as

$$M_n(f(t);x)=n\sum_{\nu=1}^{\infty}p_{n,\nu}(x)\int_0^{\infty}q_{n,\nu-1}(t)f(t)dt+(1+x)^{-n}f(0), \tag{1.1}$$

where $p_{n,\nu}(x)=\binom{n+\nu-1}{\nu}x^\nu(1+x)^{-n-\nu}$, $x\in[0,\infty)$, and $q_{n,\nu}(t)=\dfrac{e^{-nt}(nt)^\nu}{\nu!}$, $t\in[0,\infty)$.

The space $C_\alpha[0,\infty)$ is normed by $\|f\|_{C_\alpha}=\sup\limits_{0\le t<\infty}|f(t)|e^{-\alpha t}$, $f\in C_\alpha[0,\infty)$. Alternatively, the operator (1.1) may be written as $M_n(f(t);x)=\int_0^\infty W_n(t,x)f(t)dt$, where $W_n(t,x)=n\sum_{\nu=1}^{\infty}p_{n,\nu}(x)q_{n,\nu-1}(t)+(1+x)^{-n}\delta(t)$, $\delta(t)$ being the Dirac-delta function.

The object of this paper is to study some direct results in simultaneous approximation of the operator (1.1). Some important references for the study in this area are [2-4].

The study in simultaneous approximation (the approximation of derivatives of function by the corresponding order derivatives of the operators) was initiated by Lorentz [6], who established the pointwise convergence theorem in simultaneous approximation for Bernstein polynomials on [0,1]. His method for the pointwise convergence in simultaneous approximation has been successively applied by several workers to other operators (cf. [1], [5], [7] etc.).

**KEY WORDS:** Linear positive operators, Simultaneous approximation, Degree of approximation, Modulus of continuity.

## 2. DEFINITIONS AND AUXILIARY RESULTS

Let $N^0$ denote the set of non-negative integers. For $m \in N^0$, let the $m-th$ order moment of the Lupas operators is defined by $\mu_{n,m}(x) = \sum_{\nu=0}^{\infty} p_{n,\nu}(x)\left(\frac{\nu}{n} - x\right)^m$.

**LEMMA 1** [7]. For the function $\mu_{n,m}(x)$, we have $\mu_{n,0}(x) = 1$, $\mu_{n,1}(x) = 0$ and there holds the recurrence relation

$$n\mu_{n,m+1}(x) = x(1+x)[\mu'_{n,m}(x) + m\mu_{n,m-1}(x)], \text{ for } m \geq 1.$$

Consequently, we have

(*i*) $\mu_{n,m}(x)$ is a polynomial in x of degree at most *m*;

(*ii*) for every $x \in [0,\infty)$, $\mu_{n,m}(x) = O(n^{-[(m+1)/2]})$ where $[\beta]$ denotes the integer part of $\beta$.

Let the $m-th$ order moment ($m \in N^0$) for the operators (1.1) be defined by

$$T_{n,m}(x) = M_n((t-x)^m; x) = n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int_0^{\infty} q_{n,\nu-1}(t)(t-x)^m\,dt + (-x)^m(1+x)^{-n}.$$

**LEMMA 2.** For the function $T_{n,m}(x)$, there follow $T_{n,0}(x) = 1$, $T_{n,1}(x) = 0$ and

(1.2) $$nT_{n,m+1}(x) = x(1+x)T'_{n,m}(x) + mT_{n,m}(x) + mx(x+2)T_{n,m-1}(x), m \geq 1.$$

Further, we have the following consequences of $T_{n,m}(x)$:

(*i*) $T_{n,m}(x)$ is a polynomial in x of degree *m*, $m \neq 1$;

(*ii*) for every $x \in [0,\infty)$, $T_{n,m}(x) = O(n^{-[(m+1)/2]})$;

(*iii*) the coefficients of $n^{-k}$ in $T_{n,2k}(x)$ and $T_{n,2k-1}(x)$ are $C_1\{x(x+2)\}^k$ and $C_2 x^{k-1}(x+2)^{k-2}(x^2+3x+3)$ respectively, where $C_1$ and $C_2$ are some constants dependent on $k$.

**Proof:** It is easy to show that $T_{n,0}(x) = 1$ and $T_{n,1}(x) = 0$. Next, we prove (1.2). For $x = 0$, it clearly holds for all $m \geq 1$. For $x \in (0,\infty)$, we have

$$T'_{n,m}(x) = n\sum_{\nu=1}^{\infty}\left\{p'_{n,\nu}(x)\int_0^{\infty} q_{n,\nu-1}(t)(t-x)^m\,dt - mp_{n,\nu}(x)\int_0^{\infty} q_{n,\nu-1}(t)(t-x)^{m-1}\,dt\right\}$$
$$-m(-x)^{m-1}(1+x)^{-n} - n(-x)^m(1+x)^{-n-1}.$$

Since $x(1+x)p'_{n,\nu}(x) = (\nu - nx)p_{n,\nu}(x)$ and $tq'_{n,\nu}(t) = (\nu - nt)q_{n,\nu}(t)$, we have

$$x(1+x)T'_{n,m}(x)$$
$$= n\sum_{\nu=1}^{\infty}(\nu - nx)p_{n,\nu}(x)\int_0^{\infty} q_{n,\nu-1}(t)(t-x)^m\,dt - mx(1+x)T_{n,m-1}(x) + n(-x)^{m+1}(1+x)^{-n}$$
$$= n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int_0^{\infty}(\nu - nx)q_{n,\nu-1}(t)(t-x)^m\,dt - mx(1+x)T_{n,m-1}(x) + n(-x)^{m+1}(1+x)^{-n}$$

$$= n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int_0^{\infty}(\nu-1-nt+1)q_{n,\nu-1}(t)(t-x)^m dt + n^2\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int_0^{\infty} q_{n,\nu-1}(t)(t-x)^{m+1}dt$$
$$-mx(1+x)T_{n,m-1}(x)+n(-x)^{m+1}(1+x)^{-n}$$

$$= n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int_0^{\infty} tq'_{n,\nu-1}(t)(t-x)^m dt + n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int_0^{\infty} q_{n,\nu-1}(t)(t-x)^m dt + nT_{n,m+1}(x)$$
$$-mx(1+x)T_{n,m-1}(x)$$

$$= n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int_0^{\infty} q'_{n,\nu-1}(t)(t-x)^{m+1} dt + xn\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int_0^{\infty} q'_{n,\nu-1}(t)(t-x)^m dt + T_{n,m}(x)$$
$$-(-x)^m(1+x)^{-n}+nT_{n,m+1}(x)-mx(1+x)T_{n,m-1}(x).$$

Integrating by parts, we get

$$x(1+x)T'_{n,m}(x) = -(m+1)T_{n,m}(x)+(m+1)(-x)^m(1+x)^{-n} - mxT_{n,m-1}(x)$$
$$+mx(-x)^{m-1}(1+x)^{-n}+T_{n,m}(x)-(-x)^m(1+x)^{-n}+nT_{n,m+1}(x)-mx(1+x)T_{n,m-1}(x),$$

from which (1.2) is immediate.

From the values of $T_{n,0}(x)$ and $T_{n,1}(x)$, it is clear that the consequences (*i*) and (*ii*) hold for $m=0$ and $m=1$. The consequence (*i*) can be proved easily by using (1.2) and the induction on *m*. We sketch below the proof of the consequence (*ii*).

Suppose that the consequence (*ii*) be true for *m*, then by (1.2), we have

$$nT_{n,m+1}(x) = O(n^{-[(m+1)/2]}) + O(n^{-[(m+1)/2]}) + O(n^{-[m/2]})$$
$$= \begin{cases} O(n^{-[(m-1)/2]}), & \text{if } m \text{ is odd} \\ O(n^{-[m/2]}), & \text{if } m \text{ is even} \end{cases}$$

Then,

$$T_{n,m+1}(x) = \begin{cases} O(n^{-[(m+1)/2]}), & \text{if } m \text{ is odd} \\ O(n^{-[(m+2)/2]}), & \text{if } m \text{ is even} \end{cases}$$

Hence, for every $x\in[0,\infty)$ $T_{n,m+1}(x)=O(n^{-[(m+2)/2]})$. Thus, consequence (*ii*) holds for $m+1$.

Consequently, by mathematical induction, it holds for all $m\in N^0$.

The proof of consequence (*iii*) follows easily from (1.2) using mathematical induction on k and hence the details are omitted ∎

Our next result is a Lorentz-type lemma for the derivatives of the kernel $W_n(t,x)$ of the operator $M_n$.

**LEMMA 3** [7]. There exist the polynomials $q_{i,j,r}(x)$ independent of *n* and $\nu$ such that

$$\frac{d^r}{dx^r}\left[x^\nu(1+x)^{-n-\nu}\right] = \sum_{\substack{2i+j\le r \\ i,j\ge 0}} n^i(\nu-nx)^j q_{i,j,r}(x)x^{\nu-r}(1+x)^{-n-\nu-r}.$$

**Corollary.** Let $\delta$ and $\gamma$ be any two positive real numbers and $[a,b] \subset (0,\infty)$. Then, for any $m>0$ we have,

$$\sup_{x\in[a,b]} \left| n\sum_{\nu=1}^{\infty} p_{n,\nu}(x) \int_{|t-x|\geq\delta} q_{n,\nu-1}(t)\, e^{\gamma t}\, dt \right| = O(n^{-m}).$$

Making use of Taylor's expansion, Schwarz inequality for integration and then for summation and Lemma 2, the proof of the Corollary easily follows, hence the details are omitted.

## 3. MAIN RESULTS

First, we prove that the derivatives of the operator (1.1) are approximation processes for corresponding order derivatives of the function, i.e., we prove that

$$M_n^{(r)}(f(t);x) \to f^{(r)}(x), \text{ as } n\to\infty,\ r=1,2,\ldots.$$

**THEOREM 1.** Suppose that $r\in N$, $f\in C_\alpha[0,\infty)$ for some $\alpha>0$ and $f^{(r)}$ exists at a point $x\in(0,\infty)$, then

$$\lim_{n\to\infty} M_n^{(r)}(f(t);x) = f^{(r)}(x). \tag{3.1}$$

Further, if $f^{(r)}$ exists and is continuous on $(a-\eta,b+\eta)\subset(0,\infty), \eta>0$, then (3.1) holds uniformly in $x\in[a,b]$.

**Proof:** By Taylor's expansion of $f$, we have

$$f(t) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^r,$$

where $\varepsilon(t,x)\to 0$ as $t\to x$. Hence

$$M_n^{(r)}(f(t);x) = \int_0^\infty W_n^{(r)}(t,x) f(t)\,dt$$

$$= \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \int_0^\infty W_n^{(r)}(t,x)(t-x)^i\,dt + \int_0^\infty W_n^{(r)}(t,x)\varepsilon(t,x)(t-x)^r\,dt := I_1 + I_2.$$

Using Lemma 2, we get that $M_n(t^m;x) = \int_0^\infty W_n(t,x)\, t^m\,dt$, is a polynomial in $x$ of degree exactly $m$, for all $m\in N^0$. Further, we can write it as

$$M_n(t^m;x) = \frac{(n+m-1)!}{n^m(n-1)!}x^m + m(m-1)\frac{(n+m-2)!}{n^m(n-1)!}x^{m-1} + O(n^{-2}), \tag{3.2}$$

and thus

$$I_1 = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} \frac{d^r}{dx^r}\left[\int_0^\infty W_n(t,x)\, t^j\,dt\right] = \frac{f^{(r)}(x)}{r!}\left[\frac{(n+r-1)!}{n^r(n-1)!} r!\right]$$

$$= f^{(r)}(x)\left[\frac{(n+r-1)!}{n^r(n-1)!}\right] \to f^{(r)}(x), \text{ as } n\to\infty.$$

Next, making use of Lemma 3, we have

$$I_2 = \int_0^\infty W_n^{(r)}(t,x)\varepsilon(t,x)(t-x)^r\,dt$$

$$= \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \frac{q_{i,j,r}(x)}{x^r(1+x)^r} n\sum_{\nu=1}^\infty p_{n,\nu}(x)(\nu-nx)^j \int_0^\infty q_{n,\nu-1}(t)\varepsilon(t,x)(t-x)^r\,dt$$

$$+(-1)^r\frac{(n+r-1)!}{(n-1)!}(1+x)^{-n-r}\varepsilon(0,x)(-x)^r.$$

Therefore,

$$|I_2| \le \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \frac{|q_{i,j,r}(x)|}{x^r(1+x)^r} n\sum_{\nu=1}^\infty p_{n,\nu}(x)|\nu-nx|^j \int_0^\infty q_{n,\nu-1}(t)|\varepsilon(t,x)||t-x|^r\,dt$$

$$+\frac{(n+r-1)!}{(n-1)!}(1+x)^{-n-r}|\varepsilon(0,x)|x^r := I_3 + I_4.$$

Since $\varepsilon(t,x)\to 0$ as $t\to x$, for a given $\varepsilon>0$, there exists a $\delta>0$ such that $|\varepsilon(t,x)|<\varepsilon$, whenever $0<|t-x|<\delta$. For $|t-x|\ge\delta$, there exist a constant $K>0$ such that $\left|\varepsilon(t,x)(t-x)^r\right|\le K e^{\alpha t}$. Hence,

$$I_3 \le C_1 \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^{i+1}\sum_{\nu=1}^\infty p_{n,\nu}(x)|\nu-nx|^j\left[\int_{|t-x|<\delta} q_{n,\nu-1}(t)\varepsilon|t-x|^r\,dt + \int_{|t-x|\ge\delta} q_{n,\nu-1}(t)K e^{\alpha t}\,dt\right] := I_5 + I_6,$$

where $C_1 = \sup\limits_{\substack{2i+j\le r\\ i,j\ge 0}} \dfrac{|q_{i,j,r}(x)|}{x^r(1+x)^r}$.

Now, applying Schwarz inequality for integration and then for summation we are led to

$$I_5 \le \varepsilon C_1 \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^{i+1}\sum_{\nu=1}^\infty p_{n,\nu}(x)|\nu-nx|^j\left[\int_0^\infty q_{n,\nu-1}(t)dt\right]^{1/2}\left[\int_0^\infty q_{n,\nu-1}(t)(t-x)^{2r}\,dt\right]^{1/2}$$

$$\le \varepsilon C_1 \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i\left[\sum_{\nu=1}^\infty p_{n,\nu}(x)(\nu-nx)^{2j}\right]^{1/2}\left[n\sum_{\nu=1}^\infty p_{n,\nu}(x)\int_0^\infty q_{n,\nu-1}(t)(t-x)^{2r}\,dt\right]^{1/2}.$$

From Lemma 1, we have

$$\sum_{\nu=1}^\infty p_{n,\nu}(x)(\nu-nx)^{2j} = n^{2j}\left[\sum_{\nu=0}^\infty p_{n,\nu}(x)\left(\frac{\nu}{n}-x\right)^{2j} - (1+x)^{-n}(-x)^{2j}\right]$$

$$= n^{2j}\left|O(n^{-j})+O(n^{-s})\right| = O(n^j)\ (\text{for any real } s>0). \tag{3.3}$$

Similarly, Lemma 2 yields us

$$n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int_0^{\infty} q_{n,\nu-1}(t)(t-x)^{2r}\,dt = T_{n,2r}(x)-(1+x)^{-n}(-x)^{2r}$$

(3.4) $$= O(n^{-r})+O(n^{-s}) = O(n^{-r}) \text{ (for any real } s>0\text{)}.$$

Therefore, $I_5 \le \varepsilon C_1 \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i\, O(n^{j/2})\,O(n^{-r/2}) = \varepsilon\, O(1)$.

Next, we can write $I_6 = C_2 \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^{i+1}\sum_{\nu=1}^{\infty} p_{n,\nu}(x)|\nu-nx|^j \int_{|t-x|\ge\delta} q_{n,\nu-1}(t)e^{\alpha t}\,dt$, where $C_2 = KC_1$.

Hence, again using Schwarz inequality for integration and then for summation, (3.3) and Corollary, we have

$$I_6 \le C_2 \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^{i+1}\sum_{\nu=1}^{\infty} p_{n,\nu}(x)|\nu-nx|^j \left[\int_{|t-x|\ge\delta} q_{n,\nu-1}(t)\,dt\right]^{1/2}\left[\int_{|t-x|\ge\delta} q_{n,\nu-1}(t)e^{2\alpha t}\,dt\right]^{1/2}$$

$$\le C_2 \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^{i}\left[\sum_{\nu=1}^{\infty} p_{n,\nu}(x)(\nu-nx)^{2j}\right]^{1/2}\left[n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int_{|t-x|\ge\delta} q_{n,\nu-1}(t)e^{2\alpha t}\,dt\right]^{1/2}$$

$$= O(n^{(r/2)-s}) = o(1) \text{ (for } s>r/2\text{)}.$$

Now, since $\varepsilon>0$ is arbitrary, it follows that $I_3\to 0$ as $n\to\infty$. Also, $I_4\to 0$ as $n\to\infty$ and hence $I_2 = o(1)$. Combining the estimates of $I_1$ and $I_2$, (3.1) is immediate.

The uniformity assertion follows easily from the fact that in the above proof $\delta(\varepsilon)$ can be chosen to be independent of $x\in[a,b]$ and all the other estimates hold uniformly in $x\in[a,b]$ ∎

Our next result is a Voronoskaja type asymptotic formula for the operators $M_n^{(r)}(f;x)$.

**THEOREM 2.** Let $f\in C_\alpha[0,\infty)$ for some $\alpha>0$. If $f^{(r+2)}$ exists at a point $x\in(0,\infty)$, then

$$\lim_{n\to\infty} n\left(M_n^{(r)}(f(t);x)-f^{(r)}(x)\right)$$

(3.5) $$= \frac{r(r-1)}{2} f^{(r)}(x) + r(1+x)f^{(r+1)}(x) + \frac{x(x+2)}{2} f^{(r+2)}(x).$$

Further, if $f^{(r+2)}$ exists and is continuous on the interval $(a-\eta,b+\eta)\subset(0,\infty)$, $\eta>0$, then (3.5) holds uniformly on $[a,b]$.

**Proof:** By the Taylor's expansion of $f$, we get

$$M_n^{(r)}(f(t);x) = \sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}\int_0^{\infty} W_n^{(r)}(t,x)(t-x)^i\,dt + \int_0^{\infty} W_n^{(r)}(t,x)\varepsilon(t,x)(t-x)^{r+2}\,dt$$

$$:= I_1 + I_2, \text{ where } \varepsilon(t,x)\to 0 \text{ as } t\to x.$$

By Lemma 2 and (3.2), we have

$$I_1 = \sum_{i=r}^{r+2} \frac{f^{(i)}(x)}{i!} \sum_{j=r}^{i} \binom{i}{j} (-x)^{i-j} \left[ \int_0^\infty W_n^{(r)}(t,x) t^j \, dt \right]$$

$$= \frac{f^{(r)}(x)}{r!} \left[ \frac{(n+r-1)!}{n^r (n-1)!} r! \right] + \frac{f^{(r+1)}(x)}{(r+1)!} \left[ (r+1)(-x) \int_0^\infty W_n^{(r)}(t,x) t^r dt + \int_0^\infty W_n^{(r)}(t,x) t^{r+1} dt \right]$$

$$+ \frac{f^{(r+2)}(x)}{(r+2)!} \left[ \frac{(r+1)(r+2)}{2} (-x)^2 \int_0^\infty W_n^{(r)}(t,x) t^r dt + (r+2)(-x) \int_0^\infty W_n^{(r)}(t,x) t^{r+1} dt + \int_0^\infty W_n^{(r)}(t,x) t^{r+2} dt \right]$$

$$= f^{(r)}(x) \left[ \frac{(n+r-1)!}{n^r (n-1)!} \right] + \frac{f^{(r+1)}(x)}{(r+1)!} \left[ (r+1)(-x) \left( \frac{(n+r-1)!}{n^r (n-1)!} r! \right) + \left( \frac{(n+r)!}{n^{r+1} (n-1)!} (r+1)! x + (r+1)! r \frac{(n+r-1)!}{n^{r+1} (n-1)!} \right) \right] + \frac{f^{(r+2)}(x)}{(r+2)!} \left[ \frac{(r+1)(r+2)}{2} x^2 \left( \frac{(n+r-1)!}{n^r (n-1)!} r! \right) + (r+2)(-x) \left( \frac{(n+r)!}{n^{r+1} (n-1)!} (r+1)! x + (r+1)! \, r \frac{(n+r-1)!}{n^{r+1} (n-1)!} \right) + \left( \frac{(n+r+1)!}{n^{r+2} (n-1)!} \frac{(r+2)!}{2} x^2 + (r+2)(r+1) \frac{(n+r)!}{n^{r+2} (n-1)!} (r+1)! x \right) \right] + O(n^{-2}).$$

$$= f^{(r)}(x) \left[ \frac{(n+r-1)!}{n^r (n-1)!} \right] + f^{(r+1)}(x) \left[ \frac{r(1+x)}{n} \frac{(n+r-1)!}{n^r (n-1)!} \right] + f^{(r+2)}(x) \left[ \frac{nx^2 + 2nx + rx^2 + 2r^2 x + 2rx}{2n^2} \frac{(n+r-1)!}{n^r (n-1)!} \right] + O(n^{-2}).$$

Hence in order to prove (3.5) it suffices to show that $nI_2 \to 0$ as $n \to \infty$, which follows on proceeding along the lines of proof of $I_2 \to 0$ as $n \to \infty$ in Theorem 1. The uniformity assertion follows as in the proof of Theorem 1 ∎

Now, we present a theorem, which gives an estimate of the degree of approximation by $M_n^{(r)}(.;x)$ for smooth functions.

**THEOREM 3.** Let $f \in C_\alpha[0,\infty)$ for some $\alpha > 0$ and $r \le q \le r+2$. If $f^{(q)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,\infty)$, $\eta > 0$, then for sufficiently large *n*,

$$\left\| M_n^{(r)}(f(t);x) - f^{(r)}(x) \right\| \le C_1 n^{-1} \sum_{i=r}^{q} \left\| f^{(i)} \right\| + C_2 n^{-1/2} \omega_{f^{(r+1)}}(n^{-1/2}) + O(n^{-2}),$$

where $C_1, C_2$ are both independent of $f$ and $n$, $\omega_f(\delta)$ is the modulus of continuity of $f$ on $(a-\eta, b+\eta)$, and $\|.\|$ means the sup-norm on $[a,b]$.

**Proof:** By our hypothesis

$$f(t)=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{f^{(q)}(\xi)-f^{(q)}(x)}{q!}(t-x)^q\chi(t)+h(t,x)(1-\chi(t)),$$

where $\xi$ lies between $t$, $x$, and $\chi(t)$ is the characteristic function of the interval $(a-\eta,b+\eta)$. For $t\in(a-\eta,b+\eta)$ and $x\in[a,b]$, we get

$$f(t)=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{f^{(q)}(\xi)-f^{(q)}(x)}{q!}(t-x)^q .$$

For $t\in[0,\infty)\setminus(a-\eta,b+\eta)$ and $x\in[a,b]$, we define

$$h(t,x)=f(t)-\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i .$$

Now,

$$M_n^{(r)}(f(t);x)-f^{(r)}(x)=\left[\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}\int_0^\infty W_n^{(r)}(t,x)(t-x)^i\,dt-f^{(r)}(x)\right]$$

$$+\int_0^\infty W_n^{(r)}(t,x)\left\{\frac{f^{(q)}(\xi)-f^{(q)}(x)}{q!}(t-x)^q\chi(t)\right\}dt+\int_0^\infty W_n^{(r)}(t,x)h(t,x)(1-\chi(t))dt$$

$$:=I_1+I_2+I_3 .$$

By using Lemma 2 and (3.2), we get

$$I_1=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}\frac{d^r}{dx^r}\left[\int_0^\infty W_n(t,x)t^j\,dt\right]-f^{(r)}(x)$$

$$=\sum_{i=r}^{q}\frac{f^{(i)}(x)}{i!}\sum_{j=r}^{i}\binom{i}{j}(-x)^{i-j}\frac{d^r}{dx^r}\left[\frac{(n+j-1)!}{n^j(n-1)!}x^j+j(j-1)\frac{(n+j-2)!}{n^j(n-1)!}x^{j-1}+O(n^{-2})\right]-f^{(r)}(x).$$

Consequently, $\|I_1\|\le C_1 n^{-1}\left(\sum_{i=r}^{q}\|f^{(i)}\|\right)+O(n^{-2})$, uniformly in $x\in[a,b]$.

To estimate $I_2$ we proceed as follows:

$$|I_2|\le\int_0^\infty\left|W_n^{(r)}(t,x)\right|\left\{\frac{\left|f^{(q)}(\xi)-f^{(q)}(x)\right|}{q!}|t-x|^q\chi(t)\right\}dt$$

$$\le\frac{\omega_{f^{(q)}}(\delta)}{q!}\int_0^\infty\left|W_n^{(r)}(t,x)\right|\left(1+\frac{|t-x|}{\delta}\right)|t-x|^q\,dt$$

$$\le\frac{\omega_{f^{(q)}}(\delta)}{q!}\left[n\sum_{\nu=1}^{\infty}\left|p_{n,\nu}^{(r)}(x)\right|\int_0^\infty q_{n,\nu-1}(t)\left(|t-x|^q+\delta^{-1}|t-x|^{q+1}\right)dt\right.$$

$$\left.+\frac{(n+r-1)!}{(n-1)!}(1+x)^{-n-r}\left(|x|^q+\delta^{-1}|x|^{q+1}\right)\right],\ \delta>0.$$

Now, for $s=0,1,2,\ldots$, we have

$$n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\,|\nu-nx|^j \int_0^\infty q_{n,\nu-1}(t)|t-x|^s\,dt$$

$$\leq n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\,|\nu-nx|^j \left[\left(\int_0^\infty q_{n,\nu-1}(t)dt\right)^{1/2}\left(\int_0^\infty q_{n,\nu-1}(t)(t-x)^{2s}\,dt\right)^{1/2}\right]$$

$$\leq \left[\sum_{\nu=1}^{\infty} p_{n,\nu}(x)(\nu-nx)^{2j}\right]^{1/2}\left[n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int_0^\infty q_{n,\nu-1}(t)(t-x)^{2s}\,dt\right]^{1/2}$$

$$= O(n^{j/2})O(n^{-s/2}) = O(n^{(j-s)/2}),$$

uniformly in $x\in[a,b]$, in view of (3.3) and (3.4). Therefore by Lemma 3, we get

$$n\sum_{\nu=1}^{\infty}\left|p_{n,\nu}^{(r)}(x)\right|\int_0^\infty q_{n,\nu-1}(t)|t-x|^s\,dt$$

$$\leq n\sum_{\nu=1}^{\infty}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i\,|\nu-nx|^j\,\frac{|q_{i,j,r}(x)|}{x^r(1+x)^r}\,p_{n,\nu}(x)\int_0^\infty q_{n,\nu-1}(t)|t-x|^s\,dt$$

$$\leq K\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i\left[n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)|\nu-nx|^j\int_0^\infty q_{n,\nu-1}(t)|t-x|^s\,dt\right] = K\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i\,O(n^{(j-s)/2})$$

$$= O(n^{(r-s)/2}), \tag{3.6}$$

uniformly in $x\in[a,b]$, where $K=\sup\limits_{\substack{2i+j\leq r\\ i,j\geq 0}}\ \sup\limits_{x\in[a,b]}\dfrac{|q_{i,j,r}(x)|}{x^r(1+x)^r}$. Choosing $\delta=n^{-1/2}$ and applying (3.6), we are led to

$$\|I_2\|\leq\frac{\omega_{f^{(q)}}(n^{-1/2})}{q!}\left[O(n^{(r-q)/2})+n^{1/2}O(n^{(r-q-1)/2})+O(n^{-m})\right],\ \text{for any } m>0$$

$$\leq C_2\; n^{-(r-q)/2}\;\omega_{f^{(q)}}(n^{-1/2}).$$

Since $t\in[0,\infty)\setminus(a-\eta,b+\eta)$, we can choose $\delta>0$ in such a way that $|t-x|\geq\delta$ for all $x\in[a,b]$.

Thus, by Lemma 3, we obtain

$$|I_3|\leq n\sum_{\nu=1}^{\infty}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i\,|\nu-nx|^j\,\frac{|q_{i,j,r}(x)|}{x^r(1+x)^r}\,p_{n,\nu}(x)\int_{|t-x|\geq\delta} q_{n,\nu-1}(t)\,|h(t,x)|dt$$

$$+\frac{(n+r-1)!}{(n-1)!}(1+x)^{-n-r}|h(0,x)|.$$

For $|t-x|\geq\delta$, we can find a constant $M>0$ such that $|h(t,x)|\leq M\,e^{\alpha t}$. Finally using Schwarz inequality for integration and then for summation, (3.3), and Corollary, it easily follows that

$I_3 = O(n^{-s})$ for any $s > 0$, uniformly on $[a,b]$. Combining the estimates of $I_1, I_2, I_3$, the required result is immediate ∎

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