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Spline wavelets in periodic Sobolev spaces and application to high order collocation methods

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Abstract

In this paper, we present a particular family of spline wavelets constructed from the Chui-Wang Riesz basis of $L^2(\mathbb{R})$. The construction is explicit, allowing the study of specific functional properties and rather easy handling in numerical computations. This family constitutes a Riesz hierarchical basis in periodic Sobolev spaces. We also present a necessary and sufficient condition of strong ellipticity for pseudodifferential operators obtained with respect to these splines. It uses a new expression for the numerical symbol of the boundary integral operators. This expression allows us to use efficiently collocation methods with different meshes and splines.

Key words. Sobolev spaces, splines, wavelets, collocation methods.
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1 Introduction

Collocation methods using splines is a natural and widely used technique for solving strongly elliptic pseudodifferential equations on closed curves (see [2], [8], [20]). However, stability, asymptotic convergence, good condition numbers and efficient compression are not so easy to obtain. For smooth boundaries, the convergence of these methods has been proved by Arnold, Saranen and Wendland ([1], [2], [19]). Several recent papers use these methods in a more general setting (see for example [12], [14], [15]).

Wavelets can be used in this context because they provide Riesz bases, allow progressive computations and give good compression schemes. Moreover, one can adapt the construction of the basis so that the properties of the functions solve or reduce technical and numerical difficulties.

In this paper, we first present new explicit constructions of Riesz bases of spline wavelets in periodic Sobolev spaces based on the Chui-Wang wavelets and we study typical properties of these bases. We focus on spline functions especially because they are easy to handle in implementation. We also show how to obtain the dual bases explicitly.

Then we discuss the usefulness of these bases to obtain good convergence and asymptotic stability for collocation methods in the resolution of boundary integral equations. In this framework, we present a result concerning the numerical symbol of periodic pseudodifferential operators. We give a proof of the characterization of the coercivity condition which leads to relations on the meshes and order of the splines that is easy to handle (see Theorem 7).

As a typical example of application, we treat the simple and double layer potentials for the Dirichlet problem of the Laplace operator. The first one involves Sobolev spaces of half integer orders and the second one Sobolev space of integer orders. Some numerical computations of the condition number are presented and confirm the theoretical results.

2 Spline wavelets in periodic Sobolev spaces

2.1 Chui-Wang wavelets

For any strictly positive integer m and any integer j, denote by $V_j^{(m)}$ the set of functions on $\mathbb{R}$ which are smooth splines of degree $m-1$ with respect to the mesh $\{2^{-j}k : k \in \mathbb{Z}\}$ and belong to $L^2(\mathbb{R})$. If $\delta \in [0,1[$, we denote by $V_{j,\delta}^{(m)}$ the same set of splines but with respect to the mesh $\{2^{-j}(k+\delta) : k \in \mathbb{Z}\}$. The corresponding sets of 1-periodic splines are respectively denoted by $\mathcal{V}_j^{(m)}$ and $\mathcal{V}_{j,\delta}^{(m)}$.

Let

$$N_m = \underbrace{\chi_{[0,1]} * \ldots * \chi_{[0,1]}}_{m}$$

be the cardinal spline function. The classical Chui-Wang spline biwavelet $\psi_m \in V_1^{(m)}$ is defined by

$$\widehat{\psi}_m(2\xi) = p_m(\xi)\widehat{N}_m(\xi)$$

with

$$\begin{aligned} p_m(\xi) &= e^{-i(m-1)\xi}\left(\frac{1-e^{-i\xi}}{2}\right)^m \omega_m(\xi+\pi) \\ \omega_m(\xi) &= \sum_{k=-\infty}^{+\infty} |\widehat{N}_m(\xi+2k\pi)|^2 = \sum_{k=-m+1}^{m-1} e^{-ik\xi} N_{2m}(m+k). \end{aligned}$$

It is well known (see [6]), that the functions $\psi_m(x-k)$, $k \in \mathbb{Z}$, form a Riesz basis of the orthogonal complement $W_0^{(m)}$ of $V_0^{(m)}$ in $V_1^{(m)}$. Moreover $\mathrm{supp}(\psi_m) \subset [0, 2m-1]$. Since we have orthogonality between the levels, the functions

$$\psi_{m;j,k}(x) = 2^{j/2}\psi_m(2^j x - k), \quad j,k \in \mathbb{Z},$$

form a Riesz basis of $L^2(\mathbb{R})$.

Let

$$\Psi_{m;j}(x) = 2^{j/2} \sum_{k=-\infty}^{+\infty} \psi_m(2^j(x-k)), \quad j \geq 0$$

and

$$\Psi_{m;j,k}(x) = \Psi_{m;j}(x - k2^{-j}), \; j \geq 0, 0 \leq k < 2^j.$$

It is readily seen that for every j, the functions $\Psi_{m;j,k}, 0 \leq k < 2^j$, form a Riesz basis of the orthogonal complement of $\mathcal{V}_j^{(m)}$ in $\mathcal{V}_{j+1}^{(m)}$ and that we have orthogonality between the levels. The Riesz bounds are independent of j. The spline functions 1 and $\Psi_{m;j,k}$, $j \geq 0$, $0 \leq k < 2^j$, form a Riesz basis of $L^2(]0,1[)$. Moreover, after normalization of the constant function 1, the Riesz bounds in $L^2(]0,1[)$ are the same as the ones obtained for the functions $\psi_{m;j,k}$, $j,k \in \mathbb{Z}$, in $L^2(\mathbb{R})$.

Let us consider the Sobolev spaces $H^s_{\text{per}}(]0,1[)$, $s \in \mathbb{R}$, of 1-periodic distributions endowed with the norm

$$\|u\|^2_{H^s_{\text{per}}} = |u_0|^2 + |u|^2_{H^s_{\text{per}}}$$

where

$$|u|^2_{H^s_{\text{per}}} = \sum_{k \neq 0} |k|^{2s} |u_k|^2$$

and u_k is the k-th Fourier coefficient of u. We also use the notation

$$\|u\|_0 = \|u\|_{L^2(]0,1[)}.$$

We recall also that in case $s > 1/2$, the norm $\|.\|_{H^s_{\text{per}}}$ is equivalent to the following one

$$\|u\|_s^2 = |u(0)|^2 + |u|^2_{H^s_{\text{per}}} = |u(0)|^2 + \sum_{k \neq 0} |k|^{2s} |u_k|^2.$$

The stability of Chui-Wang wavelets in Sobolev spaces has been studied in [16] and [10]. This result is covered in greater generality in various studies. However, for the sake of completeness, we give here a simple and very direct proof in the periodic setting with optimal indices.

Proposition 1 *If $|s| < m - \frac{1}{2}$ then there are $C, c > 0$ such that*

$$c \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j - 1} |c_{j,k}|^2 \leq \| \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j - 1} c_{j,k} 2^{-js} \Psi_{m;j,k} \|^2_{H^s_{\text{per}}} \leq C \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j - 1} |c_{j,k}|^2.$$

This result is optimal since $\Psi_{m;j} \in H^s_{\text{per}}(]0,1[)$ if and only if $s < m - \frac{1}{2}$.

Proof. Let

$$Q_j f = P_{j+1} f - P_j f = \sum_{k=0}^{2^j - 1} c_{jk} \Psi_{m;j,k}.$$

Here P_j is the L^2-orthogonal projection onto $\mathcal{V}_j^{(m)}$. Let s be such that $|s| < m - \frac{1}{2}$. We have

$$\|Q_j f\|_{H^s_{\text{per}}} \leq C 2^{js} \|Q_j f\|_0 \quad \forall j \in \mathbb{N}.$$

Indeed, if $s \geq 0$, this property follows from the inverse property of periodic splines in Sobolev spaces (see for example, Theorem 2.11 of [17]). If $s < 0$, we write

$$\begin{aligned}
\|Q_j f\|_{H^s_{\text{per}}} &\leq \sup_{\|g\|_{H^{-s}_{\text{per}}} \leq 1} |\langle Q_j f, g \rangle_0| \leq \|Q_j f\|_0 \sup_{\|g\|_{H^{-s}_{\text{per}}} \leq 1} \|Q_j g\|_0 \\
&\leq \|Q_j f\|_0 \sup_{\|g\|_{H^{-s}_{\text{per}}} \leq 1} \|g - P_j g\|_0 \\
&\leq C 2^{js} \|Q_j f\|_0
\end{aligned}$$

using the approximation property of splines (see Theorem 2.6 of [17]).

Now, if $\epsilon > 0$ satisfies $|s \pm \epsilon| < m - \frac{1}{2}$, we get

$$\begin{aligned}
\|\sum_{j=0}^{+\infty} 2^{-js} Q_j f\|^2_{H^s_{\rm per}} &\leq \sum_{j=0}^{+\infty} 2^{-2js} \|Q_j f\|^2_{H^s_{\rm per}} + 2\Re \sum_{j<k} 2^{-(j+k)s} \langle Q_j f, Q_k f\rangle_{H^s_{\rm per}} \\
&\leq C \sum_{j=0}^{+\infty} \|Q_j f\|_0^2 + 2\Re \sum_{j<k} 2^{-(j+k)s} \langle Q_j f, Q_k f\rangle_{H^s_{\rm per}} \\
&\leq C \sum_{j=0}^{+\infty} \|Q_j f\|_0^2 + 2 \sum_{j<k} 2^{-(j+k)s} \|Q_j f\|_{H^{s+\epsilon}_{\rm per}} \|Q_k f\|_{H^{s-\epsilon}_{\rm per}} \\
&\leq C \sum_{j,k=0}^{+\infty} 2^{-|k-j|\epsilon} \|Q_j f\|_0 \|Q_k f\|_0 \\
&\leq C_1 \sum_{j=0}^{+\infty} \|Q_j f\|_0^2.
\end{aligned}$$

Finally

$$\begin{aligned}
\sum_{j=0}^{+\infty} \|Q_j f\|_0^2 &= \left\langle \sum_{j=0}^{+\infty} 2^{js} Q_j f, \sum_{j=0}^{+\infty} 2^{-js} Q_j f \right\rangle_{L^2(]0,1[)} \\
&\leq \|\sum_{j=0}^{+\infty} 2^{js} Q_j f\|_{H^{-s}_{\rm per}} \|\sum_{j=0}^{+\infty} 2^{-js} Q_j f\|_{H^s_{\rm per}} \\
&\leq C \sqrt{\sum_{j=0}^{+\infty} \|Q_j f\|_0^2} \quad \|\sum_{j=0}^{+\infty} 2^{-js} Q_j f\|_{H^s_{\rm per}}.
\end{aligned}$$

This proves the proposition. □

2.2 Modified spline wavelets

Since, by construction, the function ψ_m has m vanishing moments, there is a unique spline function $\theta_m \in V_1^{(2m)}$ on $\mathbb{R}$ such that $D^m \theta_m = \psi_m$ and $\mathrm{supp}(\theta_m) \subset [0, 2m-1]$. Explicitly, we have

$$\theta_m(x) = \frac{1}{(m-1)!} \int_0^x (x-t)^{m-1} \psi_m(t)$$

and also

$$\widehat{\theta}_m(\xi) = \frac{\widehat{\psi}_m(\xi)}{(i\xi)^m}.$$

Let

$$\Theta_{m;j}(x) = 2^{-mj+j/2} \sum_{k=-\infty}^{+\infty} \theta_m(2^j(x-k)), \ j \geq 0.$$

It follows that

$$\operatorname{supp}(\Theta_{m;j}) \subset \bigcup_{k=-\infty}^{+\infty} [k, k+2^{-j}(2m-1)].$$

In $[0,1]$, this set is reduced to an interval with length $2^{-j}(2m-1)$ if j is large. With the previous notation, we get $D^m\Theta_{m;j} = \Psi_{m;j}$. We also consider the functions

$$\Theta_{m;j,k}(x) = \Theta_{m;j}(x - k2^{-j}), \quad j \geq 0,\ 0 \leq k < 2^j.$$

These functions are not orthogonal to constants since θ_m has no vanishing moment. But this property is "replaced" by the fact that they all vanish at 0, which is very useful in the sequel. We give now the proof of this property.

Lemma 2 *For every $m \in \mathbb{N}$ and every integer p, we have*

$$\theta_m(p) = 0.$$

It follows that for every $m \in \mathbb{N}, j \geq 0, k \in \{0, \ldots, 2^j - 1\}$, and $q \in \mathbb{Z}$, we have $\Theta_{m;j,k}(2^{-j}q) = 0$.

Proof. Since

$$\Theta_{m;j,k}(2^{-j}q) = 2^{-mj+j/2} \sum_{l=-\infty}^{+\infty} \theta_m(q - k - 2^j l)$$

it suffices to show that

$$\theta_m(p) = 0 \quad \forall p \in \mathbb{Z}.$$

From the relations

$$\begin{aligned}
\widehat{\psi}_m(2\xi) &= p_m(\xi)\widehat{N}_m(\xi) = 2^{-m}e^{-i(m-1)\xi}\omega_m(\xi+\pi)(i\xi)^m\widehat{N}_{2m}(\xi) \\
\widehat{\theta}_m(\xi) &= \frac{\widehat{\psi}_m(\xi)}{(i\xi)^m} \\
\omega_m(\xi) &= \sum_{k=-m+1}^{m-1} e^{-ik\xi}N_{2m}(m+k) = \sum_{k=-\infty}^{+\infty} e^{-ik\xi}N_{2m}(m+k)
\end{aligned}$$

we get

$$\begin{aligned}
\widehat{\theta}_m(2\xi) &= 2^{-2m}e^{-i(m-1)\xi}\ \omega_m(\xi+\pi)\widehat{N}_{2m}(\xi) \\
&= 2^{-2m}e^{-i(m-1)\xi} \sum_{k=-\infty}^{+\infty} (-1)^k e^{-ik\xi}N_{2m}(m+k)\widehat{N}_{2m}(\xi) \\
\theta_m(x) &= 2^{-2m+1} \sum_{k=-\infty}^{+\infty} (-1)^k N_{2m}(m+k)N_{2m}(2x-(k+m)+1).
\end{aligned}$$

It follows that for any integer p, if one changes the sum over k to a sum over k' with $k+m=2p-(m+k')+1$, one gets

$$\begin{aligned}\theta_m(p) &= 2^{-2m+1}\sum_{k=-\infty}^{+\infty}(-1)^k N_{2m}(m+k)\; N_{2m}(2p-(m+k)+1)\\ &= -\,2^{-2m+1}\sum_{k'=-\infty}^{+\infty}(-1)^{k'} N_{2m}(2p-(m+k')+1)\; N_{2m}(m+k')\\ &= -\theta_m(p).\end{aligned}$$

□

Proposition 3 *For any integer $m\geq 1$ and any real number s such that $\frac{1}{2}<s<2m-\frac{1}{2}$, the functions 1 and $2^{j(m-s)}\Theta_{m;j,k}$, $j\geq 0$, $0\leq k<2^j$, form a Riesz basis of $H^s_{\mathrm{per}}(]0,1[)$.*

Proof. By construction, we have $|(\Theta_{m;j,k})_l| = |(2\pi l)^{-m}(\Psi_{m;j,k})_l|$ for $l\in\mathbb{Z}, l\neq 0$ hence

$$\begin{aligned}|\sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1}c_{j,k}2^{j(m-s)}\Theta_{m;j,k}|_{H^s_{\mathrm{per}}} &= (2\pi)^{-m}|\sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1}c_{j,k}2^{j(m-s)}\Psi_{m;j,k}|_{H^{s-m}_{\mathrm{per}}}\\ &= (2\pi)^{-m}\|\sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1}c_{j,k}2^{j(m-s)}\Psi_{m;j,k}\|_{H^{s-m}_{\mathrm{per}}}\end{aligned}$$

with $|s-m|<m-1/2$. Using Proposition 1, we then get constants $c, C>0$ such that

$$c\sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1}|c_{j,k}|^2 \leq |\sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1}c_{j,k}2^{j(m-s)}\Theta_{m;j,k}|^2_{H^s_{\mathrm{per}}} \leq C\sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1}|c_{j,k}|^2.$$

Now, since the functions $\Theta_{m;j,k}$ all vanish at 0, we get

$$\left\|c_0+\sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1}c_{j,k}2^{j(m-s)}\Theta_{m;j,k}\right\|_s^2 = |c_0|^2+|\sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1}c_{j,k}2^{j(m-s)}\Theta_{m;j,k}|^2_{H^s_{\mathrm{per}}}.$$

Using these two relations, we obtain that the functions 1 and $2^{j(m-s)}\Theta_{m;j,k}$, $j\geq 0$, $0\leq k<2^j$, form a Riesz family.

The functions $\Theta_{m;j,k}$ are 1-periodic spline functions of degree $2m-1$ with respect to the mesh $\{k2^{-j-1}: k\in\mathbb{Z}\}$. The space $\mathcal{V}^{(2m)}_{j+1}$ of these functions has the dimension 2^{j+1} and we have exactly $1+1+2+\ldots+2^j=2^{j+1}$ elements of this space among the Riesz family. Since the union of these spaces is dense in $H^s_{\mathrm{per}}(]0,1[)$, this proves the proposition. □

It is now natural to ask for a characterization of the dual Riesz basis of the family

$$\{1\} \cup \{2^{j(m-s)}\Theta_{m;j,k},\ j \geq 0, 0 \leq k < 2^j\},$$

at least for $s = m$. If one remembers the link with the Chui-Wang wavelets, it is clear that this basis should be related to the dual of the Chui-Wang wavelets and constructed in the same way.

For any $m \in \mathbb{N}$, it is known that the function $\widetilde{\psi}_m$, called the dual Chui-Wang wavelet, defined as

$$\widehat{\widetilde{\psi}}_m(\xi) = \frac{\widehat{\psi}_m(\xi)}{\sum_{k=-\infty}^{+\infty} |\widehat{\psi}_m(\xi + 2k\pi)|^2}$$

has the following properties (see [6], [7]).

Proposition 4 *For every $j, k \in \mathbb{Z}$ we set $\widetilde{\psi}_{m;j,k}(x) = 2^{j/2}\widetilde{\psi}_m(2^j x - k)$, $x \in \mathbb{R}$.*
1) We have

$$\sum_{k=-\infty}^{+\infty} |\widehat{\psi}_m(\xi + 2k\pi)|^2 = \omega_m(\xi)\omega_m(\frac{\xi}{2})\omega_m(\pi + \frac{\xi}{2}),\ \xi \in \mathbb{R}.$$

It follows that $\widetilde{\psi}_m$ is exponentially decreasing and that there are $c_k^{(m)}$, $k \in \mathbb{Z}$ and $r_m > 0$ such that

$$\widetilde{\psi}_m(x) = \sum_{k=-\infty}^{+\infty} c_k^{(m)} \psi_m(x - k),\ x \in \mathbb{R}, \quad \sup_k e^{r_m|k|}|c_k^{(m)}| < +\infty.$$

2) For every $j \in \mathbb{Z}$, the family $\widetilde{\psi}_{m;j,k}$ $(k \in \mathbb{Z})$ is the $L^2(\mathbb{R})$ dual of the Riesz basis $\psi_{m;j,k}$ $(k \in \mathbb{Z})$ of $W_j^{(m)}$. Since the spaces $W_j^{(m)}$ are L^2-orthogonal to each other, we get that the family $\widetilde{\psi}_{m;j,k}$ $(j, k \in \mathbb{Z})$ is the dual Riesz basis of $\psi_{m;j,k}$ $(j, k \in \mathbb{Z})$.□

We define $\widetilde{\theta}_m$ as follows:

$$\widetilde{\theta}_m(x) = \frac{1}{(m-1)!}\int_{-\infty}^{x} (x-t)^{m-1}\widetilde{\psi}_m(t)\, dt, \quad x \in \mathbb{R}.$$

Here again, it is readily seen that this function has the following properties.

Proposition 5 *We have $D^m\widetilde{\theta}_m = \widetilde{\psi}_m$ on $\mathbb{R}$ and $\widetilde{\theta}_m(x) = \sum_{k=-\infty}^{+\infty} c_k^{(m)}\theta_m(x-k)$ for every $x \in \mathbb{R}$. It follows that $\widetilde{\theta}_m$ belongs to $V_1^{(2m)}$ and is exponentially decreasing.*□

For $0 \leq j$ and $0 \leq k < 2^j$, we also define

$$\widetilde{\Theta}_{m;j}(x) = 2^{-mj+j/2}\sum_{k=-\infty}^{+\infty} \widetilde{\theta}_m(2^j(x-k)), \quad \widetilde{\Theta}_{m;j,k}(x) = \widetilde{\Theta}_{m;j}(x - k2^{-j}), \quad x \in \mathbb{R}.$$

We see that these functions are continuous $1-$periodic functions and that they are related to the functions $\Theta_{m;j,p}$, $(j \geq 0, 0 \leq p \leq 2^j)$ by the following relations

$$\widetilde{\Theta}_{m;j}(x) = \sum_{k=-\infty}^{+\infty} c_k^{(m)} \Theta_{m;j}(x - 2^{-j}k) = \sum_{p=0}^{2^j-1} C_{j;p}^{(m)} \Theta_{m;j,p}(x)$$

with

$$C_{j;p}^{(m)} = \sum_{k=-\infty}^{+\infty} c_{p+2^j k}^{(m)}, \quad p = 0, \dots 2^j - 1.$$

Proposition 6 *For any integer $m \geq 1$, the family $\{1\} \cup \{(2\pi)^{2m}\widetilde{\Theta}_{m;j,k},\ j \geq 0, 0 \leq k < 2^j\}$, is the dual of the Riesz basis $\{1\} \cup \{\Theta_{m;j,k},\ j \geq 0, 0 \leq k < 2^j\}$ relatively to the scalar product*

$$\langle f, g \rangle_m = f(0)\overline{g}(0) + \sum_{k \neq 0} |k|^{2m} f_k \overline{g}_k.$$

Proof. We only have to show the orthonormality between these two families. We have

$$\begin{aligned}
\left\langle \Theta_{m;j,k}, \widetilde{\Theta}_{m;j',k'} \right\rangle_m &= \Theta_{m;j,k}(0)\widetilde{\Theta}_{m;j',k'}(0) + (2\pi)^{-2m} \left\langle \Psi_{m;j,k}, \widetilde{\Psi}_{m;j',k'} \right\rangle_0 \\
&= (2\pi)^{-2m} \left\langle \Psi_{m;j,k}, \widetilde{\Psi}_{m;j',k'} \right\rangle_0 \\
&= (2\pi)^{-2m} \delta_{j,j'} \delta_{k,k'}
\end{aligned}$$

and

$$\langle \Theta_{m;j,k}, 1 \rangle_m = 0.$$

Moreover, using $\Theta_m(l) = 0$ $(l \in \mathbb{Z})$, we get

$$\widetilde{\Theta}_{m;j,k}(0) = \sum_{p=0}^{2^j-1} C_{j;p}^{(m)} \Theta_{m;j,p}(-2^{-j}k) = 0$$

for every $j \geq 0, 0 \leq k < 2^j$. Hence

$$\left\langle 1, \widetilde{\Theta}_{m;j,k} \right\rangle_m = 0.$$

□

Remark. Other hierarchical Riesz bases of splines could be used for test and trial functions (the Chui-Wang periodic wavelets are the first example). To get collocation methods, test functions have to be splines of degree $2m-1$ but for the trial functions, according to Theorem 7, we could use splines of any degree. For example, we could also consider functions coming from $l < m$ primitivations of ψ_m i.e. the functions $S_{m;l}$ defined by $D^l S_{m;l} = \psi_m$. These functions have $m - l$ vanishing moments and the corresponding periodic functions

$$S_{m;l;j}(x) = 2^{-jl+j/2} \sum_{p=-\infty}^{+\infty} S_{m;l}(2^j(x-p))$$

are such that

$$1, 2^{j(l-s)} S_{m;l;j,k}, \; j \geq 0, 0 \leq k < 2^j,$$

form a Riesz basis of smooth 1-periodic splines of degree $m+l-1$ for $H^s_{\text{per}}(]0,1[)$ if l, s satisfy $1/2 - m + l < s < m + l - 1/2$. This result is directly obtained using Proposition 1.

Moreover, since the functions $S_{m;l}$ have at least one vanishing moment, the functions $S_{m;l;j}$ are $L^2(]0,1[)$ orthogonal to constants. This leads to the fact that we see immediately that the dual Riesz basis of the family

$$1, 2^{j(l-s)} S_{m;l;j,k}, \; j \geq 0, 0 \leq k < 2^j,$$

relatively to the scalar product

$$\langle f, g \rangle_{H^l_{\text{per}}} = f_0 \overline{g}_0 + \sum_{k \neq 0} |k|^{2l} f_k \overline{g}_k$$

is obtained by the same procedure as before, but with l primitivations of the dual Chui-Wang wavelet.

3 Collocation with spline wavelets

3.1 Presentation of the problem

Let Ω be a smooth bounded and connected open subset of $\mathbb{R}^2$ whose boundary $\partial\Omega$ is also connected. To solve the Dirichlet problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u_{|\partial\Omega} = f \end{cases}$$

two methods are widely used. First, we can use the single layer potential representation of u

$$u(x) = -\frac{1}{2\pi} \int_{\partial\Omega} v(y) \log|x-y| \, d\sigma(y), \quad x \in \Omega$$

where the boundary integral equation for v is simply $Vv = f$ with

$$(Vv)(x) = -\frac{1}{2\pi} \int_{\partial\Omega} v(y) \log|x-y| \, d\sigma(y), \quad x \in \partial\Omega.$$

We can also represent the solution as a double layer potential

$$u(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{(x-y).\nu_y}{|x-y|^2} w(y) \, d\sigma(y), \quad x \in \Omega,$$

where ν is the unit inward normal to the boundary and the boundary equation for w is $(\frac{1}{2} + K)w = f$ with

$$Kw(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{(x-y).\nu_y}{|x-y|^2} w(y) \, d\sigma(y), \quad x \in \partial\Omega.$$

It is known that $V : H^s(\partial\Omega) \to H^{s+1}(\partial\Omega)$ is an isomorphism for every s if and only if the analytic capacity of Ω is not 1. Moreover $\frac{1}{2} + K : H^s(\partial\Omega) \to H^s(\partial\Omega)$ is an isomorphism for every $s \in \mathbb{R}$. Since the boundary is smooth, K is a compact operator from $H^s(\partial\Omega)$ to $H^s(\partial\Omega)$.

3.2 A result of coercivity

These two boundary operators are particular cases of the classical pseudodifferential operators with constant coefficients in the periodic setting :

$$A = b_+ Q_+^\beta + b_- Q_-^\beta + K_0$$

where $b_+, b_- \in \mathbb{C}$, $\beta \in \mathbb{R}$,

$$Q_+^\beta u(x) = \sum_{k\neq 0} |k|^\beta u_k e^{2ik\pi x}, \quad Q_-^\beta u(x) = \sum_{k\neq 0} \operatorname{sgn}(k)|k|^\beta u_k e^{2ik\pi x}$$

and, for all $r \in \mathbb{R}$, K_0 is compact from H^r_{per} into $H^{r-\beta}_{\text{per}}$. Indeed, if we use a parameterization of $\partial\Omega$ proportional to arc length and defined in $[0,1]$, the boundary operators considered are of this form with :
- $\beta = 0$ and $b_- = 0, b_+ = \frac{1}{2}$, $K_0 u = \frac{u_0}{2} + Ku$ for the double layer potential
- $\beta = -1$ and $b_- = 0, b_+ = \frac{1}{4\pi}$, K_0 a compact operator from $H^r_{\text{per}}(]0,1[)$ into C^∞_{per} for the single layer potential.

The following result gives an estimate of Céa type for strongly elliptic constant coefficients pseudodifferential operators. It concerns splines of any order (see also [17]). The technique used here presents a new expression of the condition leading to the estimate of Céa type. This expression gives then an easy description of the relations between the degree (r) of the splines and the meshes (see 1) and 2) of Theorem 7 below). It also leads to results on boundedness of condition number arising in numerical computations.

Theorem 7 *Let m be a strictly positive integer, $\delta \in [0,1[$,*

$$Au = b_+ Q_+^\beta u + b_- Q_-^\beta u + u_0$$

a pseudodifferential operator and assume that $s > \frac{1}{2}$ and $r + \frac{1}{2} > s + \beta$. Then there is $c > 0$ such that

$$\sup_{g \in \mathcal{V}_j^{(2m)},\, \|g\|_{H^{2m-s}_{\text{per}}} = 1} |\langle Af, g\rangle_{H^m_{\text{per}}}| \geq c \|f\|_{H^{s+\beta}_{\text{per}}}$$

for every j and every $f \in \mathcal{V}_{j,\delta}^{(r+1)}$ if and only if $b_+ \neq \pm b_-$ and the function

$$a \mapsto \sigma_A^{(r)}(a,\theta) = \int_0^{+\infty} \frac{t^{r-\beta} N^{(r)}(t,a,\theta)}{\cosh t - \cos\theta}\, dt$$

does not vanish in $]0,1[$ with $\theta = 2\pi\delta$. Here

$$N^{(r)}(t,a,\theta) = (b_+ + b_-)e^{-at}(e^t - e^{i\theta}) + (-1)^{r+1}(b_+ - b_-)e^{at}(e^{i\theta} - e^{-t}).$$

Defining $c_+ := b_+ + b_-$, $c_- := b_+ - b_-$ and $\gamma := \inf\{\mathcal{R}c_+, \mathcal{R}c_-\}$, we get the following particular cases of application.
1) Assume ($b_- = 0$ and $b_+ \neq 0$) or ($b_+, b_- \in \mathbb{R}$ and $\gamma > 0$). If r is odd (respectively r is even), the condition is satisfied if and only if $\delta \neq \frac{1}{2}$ (respectively $\delta \neq 0$).
2) Assume $\gamma > 0$. If r is odd (respectively r is even), the condition is satisfied in case $\delta = 0$ (respectively $\delta = \frac{1}{2}$).

Proof. Let $N = 2^j$. The Fourier coefficients

$$c_k = \int_0^1 g(x)e^{-2i\pi kx}\,dx$$

of $g \in \mathcal{V}_j^{(2m)}$ satisfy

$$k^{2m}c_k = (k+N)^{2m}c_{k+N}, \quad k \in \mathbb{Z}.$$

For $f \in \mathcal{V}_{j,\delta}^{(r+1)}$, we have

$$f(x) = \sum_{k=-\infty}^{+\infty} \alpha_k e^{-2ik\pi\delta/N}\, e^{2ik\pi x}$$

where the coefficients

$$\alpha_k = \int_0^1 f(x+\frac{\delta}{N})e^{-2i\pi kx}\,dx$$

satisfy

$$k^{r+1}\alpha_k = (k+N)^{r+1}\alpha_{k+N}, \quad k \in \mathbb{Z}.$$

Using this, we get

$$\begin{aligned}\langle Af, g\rangle_{H^m_{\mathrm{per}}} &= \alpha_0\overline{c_0} + \sum_{k=-\infty}^{+\infty} |k|^{2m+\beta}(b_+ + b_-\,\mathrm{sgn}(k))\alpha_k\overline{c_k}e^{-2ik\pi\delta/N} \\ &= \alpha_0\overline{c_0} + \sum_{k=1}^{N-1} k^{2m+r+1}\alpha_k\overline{c_k}e^{-2ik\pi\delta/N}d_k\end{aligned}$$

which leads to

$$\begin{aligned}S &:= \sup_{g\in\mathcal{V}_j^{(2m)},\ \|g\|_{H^{2m-s}_{\mathrm{per}}}=1} |\langle Af, g\rangle_{H^m_{\mathrm{per}}}| \\ &= \left(|\alpha_0|^2 + \sum_{k=1}^{N-1} k^{2(r+1)}\frac{|\alpha_k|^2|d_k|^2}{p_k}\right)^{1/2} = \left\langle Af, \frac{g_0}{\|g_0\|_{H^{2m-s}_{\mathrm{per}}}}\right\rangle_{H^m_{\mathrm{per}}}\end{aligned}$$

where, for $k = 1, \dots, N-1$

$$d_k = \sum_{p=-\infty}^{+\infty} e^{-2i\pi p\delta}(b_+ + b_-\,\mathrm{sgn}(k+pN))\frac{|k+pN|^\beta}{(k+pN)^{r+1}}, \quad p_k = \sum_{p=-\infty}^{+\infty}\frac{1}{(k+pN)^{2s}}$$

and $g_0 \in \mathcal{V}_{j,\delta}^{(2m)}$ is defined by

$$c_0 = S^{-1}\alpha_0, \quad c_k = S^{-1}e^{-2i\pi k\delta/N}k^{r+1-2m}\alpha_k\frac{d_k}{p_k}, \quad k = 1, \dots, N-1.$$

In the same way

$$\begin{aligned}\|f\|^2_{H^{s+\beta}_{\rm per}} &= |\alpha_0|^2 + \sum_{k=-\infty}^{+\infty} |k|^{2(s+\beta)}|\alpha_k|^2\\ &= |\alpha_0|^2 + \sum_{k=1}^{N-1} k^{2(r+1)}|\alpha_k|^2 \sum_{p=-\infty}^{+\infty} \frac{1}{|k+pN|^{2(r+1)-2(s+\beta)}}.\end{aligned}$$

It follows that the stated bound holds if and only if there is $C>0$ such that

$$C\sum_{p=-\infty}^{+\infty} \frac{1}{|k+pN|^{2(r+1)-2(s+\beta)}} \leq \frac{|d_k|^2}{p_k}$$

for any N, k such that $1 \leq k < N$. Since both sides are homogeneous of degree $2(r+1-s-\beta)$ with respect to (k,N), this is equivalent to the existence of $C>0$ such that

$$\begin{aligned}&C\sum_{p=-\infty}^{+\infty} \frac{1}{(p+a)^{2s}} \sum_{q=-\infty}^{+\infty} \frac{1}{(q+a)^{2(r+1)-2(s+\beta)}}\\ &\qquad \leq \Big|\sum_{p=-\infty}^{+\infty} e^{-2ip\pi\delta}(b_+ + b_-\operatorname{sgn}(p+a))\frac{|p+a|^\beta}{(p+a)^{r+1}}\Big|^2 \qquad (*)\end{aligned}$$

for any $a \in]0,1[$. Lemma 8 below shows that, for $a \in]0,1[$, we have

$$\sum_{p=-\infty}^{+\infty} e^{-2ip\pi\delta}(b_+ + b_-\operatorname{sgn}(p+a))\frac{|p+a|^\beta}{(p+a)^{r+1}} = \frac{1}{2\Gamma(r+1-\beta)}\int_0^{+\infty} \frac{t^{r-\beta}N^{(r)}(t,a,\theta)}{\cosh t - \cos\theta}\,dt.$$

The left hand side of (*) is a continuous and strictly positive function of $a \in]0,1[$; moreover it has the behaviour of $a^{-2(r+1-\beta)}$ (resp. $(1-a)^{-2(r+1-\beta)}$) when $a \to 0+$ (resp. $1-$). Using the results of Lemma 8, the assertion readily follows. □

To simplify notations in the following lemma, we assume that we are in the conditions of Theorem 7 and we set

$$\begin{aligned}\mathcal{E}(a,\theta,r) &:= \sum_{p=-\infty}^{+\infty} e^{-ip\theta}(b_+ + b_-\operatorname{sgn}(p+a))\frac{|p+a|^\beta}{(p+a)^{r+1}}\\ &= c_+\mathcal{E}_+(a,\theta,r) \;+\; (-1)^{r+1}c_-\mathcal{E}_-(a,\theta,r)\end{aligned}$$

where

$$\mathcal{E}_+(a,\theta,r) = \sum_{p=0}^{+\infty} \frac{e^{-ip\theta}}{(p+a)^{r+1-\beta}},\quad \mathcal{E}_-(a,\theta,r) = \sum_{p=-\infty}^{-1} \frac{e^{-ip\theta}}{(-p-a)^{r+1-\beta}}.$$

Lemma 8 *We consider the function $a \in \,]0,1[\, \mapsto \mathcal{E}(a,\theta,r)$.*
1) This function is continuous, satisfies

$$\mathcal{E}(a,\theta,r) = \frac{1}{2\Gamma(r+1-\beta)} \int_0^{+\infty} \frac{t^{r-\beta} N^{(r)}(t,a,\theta)}{\cosh t - \cos\theta}\, dt$$

and

$$\lim_{a\to 0^+} a^{r+1-\beta}\mathcal{E}_+(a,\theta,r) \in \mathbb{C}\setminus\{0\}, \quad \lim_{a\to 1^-} (1-a)^{r+1-\beta}\mathcal{E}_-(a,\theta,r) \in \mathbb{C}\setminus\{0\}.$$

2) Assume ($b_- = 0$ and $b_+ \neq 0$) or ($b_+, b_- \in \mathbb{R}$ and $\gamma > 0$). If r is odd (respectively r is even), then

$$\mathcal{E}(a,\theta,r) \neq 0 \ \forall a \in \,]0,1[\, \Leftrightarrow \ \theta \neq \pi (\textit{respectively } \theta \neq 0).$$

3) Assume $\gamma > 0$. If r is odd (respectively r is even), then

$$\theta = 0 \ (\textit{respectively } \theta = \pi) \ \Rightarrow \ \mathcal{E}(a,\theta,r) \neq 0 \ \forall a \in \,]0,1[.$$

Proof. 1) For $a > 0$, $|z| \leq 1$, $\Re\alpha > 1$, we have

$$\sum_{p=0}^{+\infty} \frac{z^p}{(a+p)^\alpha} = \frac{1}{\Gamma(\alpha)} \sum_{p=0}^{+\infty} z^p \int_0^{+\infty} t^{\alpha-1} e^{-(a+p)t}\, dt = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} \frac{t^{\alpha-1}e^{-(a-1)t}}{e^t - z}\, dt.$$

It follows that

$$\begin{aligned}
\mathcal{E}(a,\theta,r) &= c_+ \sum_{p=0}^{+\infty} \frac{e^{-ip\theta}}{(p+a)^{r+1-\beta}} + (-1)^{r+1} e^{-i\theta} c_- \sum_{p=0}^{+\infty} \frac{e^{ip\theta}}{(p+1-a)^{r+1-\beta}} \\
&= \frac{1}{\Gamma(r+1-\beta)} \left(c_+ \int_0^{+\infty} \frac{t^{r-\beta}e^{(1-a)t}}{e^t - e^{-i\theta}}\, dt + (-1)^{r+1} c_- \int_0^{+\infty} \frac{t^{r-\beta}e^{(a-1)t}}{e^{-i\theta} - e^{-t}}\, dt \right) \\
&= \frac{1}{2\Gamma(r+1-\beta)} \int_0^{+\infty} \frac{t^{r-\beta}N^{(r)}(t,a,\theta)}{\cosh t - \cos\theta}\, dt
\end{aligned}$$

with

$$N^{(r)}(t,a,\theta) = c_+ e^{-at}(e^t - e^{i\theta}) + (-1)^{r+1} c_- e^{at}(e^{i\theta} - e^{-t}).$$

The behaviour of the functions $\mathcal{E}_\pm(a,\theta,r)$ is clear.

2) If $b_- = 0$, we get

$$N^{(r)}(t,a,\theta) = b_+ \left(e^{-at}(e^t - \cos\theta) + (-1)^{r+1} e^{at}(\cos\theta - e^{-t}) - i \sin\theta (e^{-at} + (-1)^r e^{at}) \right).$$

Then, for $b_+ \neq 0$, a direct computation shows that if r is odd (respectively r is even)

$$\mathcal{E}(a,\theta,r) = 0 \ \Leftrightarrow \ \theta = \pi \text{ and } a = \frac{1}{2} (\text{respectively } \theta = 0 \text{ and } a = \frac{1}{2}).$$

If $b_+, b_- \in \mathbb{R}$, we separate real and imaginary parts and get

$$\begin{aligned} & N^{(r)}(t,a,\theta) \\ = & \; c_+e^{-at}(e^t - \cos\theta) + c_-(-1)^r e^{at}(e^{-t} - \cos\theta) - i\sin\theta(c_+e^{-at} + c_-(-1)^r e^{at}) \\ = & \; c_+e^{(-a+1)t} + (-1)^r c_- e^{(a-1)t} - \cos\theta(c_+e^{-at} + (-1)^r c_- e^{at}) \\ & \qquad -i\sin\theta(c_+e^{-at} + (-1)^r c_- e^{at}). \end{aligned}$$

Assume now $c_\pm > 0$. For r even, we get directly that if $\mathcal{E}(a,\theta,r) = 0$ for some $a \in]0,1[$, then $\delta = 0$. The converse also holds since

$$\lim_{a\to 0+} \mathcal{E}(a,0,r) = +\infty, \quad \lim_{a\to 1^-} \mathcal{E}(a,0,r) = -\infty.$$

Let us assume now that r is odd. If $a \in]0,1[$ is such that $\mathcal{E}(a,\theta,r) = 0$, a look at the real part of $N^{(r)}(t,a,\theta)$ shows immediately that $\theta \neq 0$; moreover, since

$$\sin\theta \, \Re \, N^{(r)}(t,a,\theta) + (1-\cos\theta)\Im \, N^{(r)}(t,a,\theta) = \sin\theta(c_+e^{-at}(e^t-1) + c_-e^{at}(1-e^{-t}))$$

we have

$$\begin{aligned} 0 &= \sin\theta \, \Re\mathcal{E}(a,\theta,r) \; + (1-\cos\theta) \, \Im\mathcal{E}(a,\theta,r) \\ &= \frac{\sin\theta}{2\Gamma(r+1-\beta)} \int_0^{+\infty} t^{r-\beta} \frac{c_+e^{-at}(e^t-1) + c_-e^{at}(1-e^{-t})}{\cosh t - \cos\theta} \, dt. \end{aligned}$$

This implies $\sin\theta = 0$ and finally $\theta = \pi$. As in the previous case, the converse also holds.

3) For r even and $\theta = \pi$ (respectively r odd and $\theta = 0$), we get

$$N^{(r)}(t,a,\theta) = c_+e^{-at}(e^t \pm 1) + c_-e^{at}(1 \pm e^{-t}).$$

Taking real parts proves the assertion. □

Remarks.

1) If we consider

$$A = b_+Q_+^\beta + b_-Q_-^\beta + K_0$$

where $b_+, b_- \in \mathbb{C}$, $\beta \in \mathbb{R}$,

$$Q_+^\beta u(x) = \sum_{k\neq 0} |k|^\beta u_k e^{2ik\pi x}, \quad Q_-^\beta u(x) = \sum_{k\neq 0} \operatorname{sgn}(k)|k|^\beta u_k e^{2ik\pi x}$$

and, for all $r \in \mathbb{R}$, K_0 is compact from H^r_{per} into $H^{r-\beta}_{\text{per}}$ and if the operator $u \mapsto b_+Q_+^\beta u + b_-Q_-^\beta u + u_0$ satisfies the inequality of Theorem 7, then there is a compact operator $K_1 : H^{s+\beta}_{\text{per}}(]0,1[) \to H^{s+\beta}_{\text{per}}(]0,1[)$ such that

$$\sup_{g\in\mathcal{V}_j^{(2m)},\, \|g\|_{H^{2m-s}_{\text{per}}}=1} |\langle Af, g\rangle_{H^m_{\text{per}}}| \geq c\|f\|_{H^{s+\beta}_{\text{per}}} - \|K_1 f\|_{H^{s+\beta}_{\text{per}}}$$

for every j and every $f \in \mathcal{V}_{j,\delta}^{(r+1)}$. If A is bijective from $H_{\text{per}}^{s+\beta}$ into H_{per}^{s}, a classical compactness argument shows that this inequality remains also valid with $K_1 = 0$ and a smaller constant c.

2) For $s > 1/2$, the norm $\|.\|_s$ related to the scalar product

$$\langle u, v\rangle_s = u(0)\overline{v}(0) + \sum_{|k|>0} |k|^{2s} u_k \overline{v}_k$$

is equivalent to the norm $\|.\|_{H_{\text{per}}^{s}}$. Under the assumptions of Theorem 7, if A is bijective and $s+\beta > 1/2$, $2m - s > 1/2$, we obtain that there is $c > 0$ such that

$$\sup_{g\in\mathcal{V}_j^{(2m)},\, \|g\|_{H_{\text{per}}^{2m-s}}=1} |\langle Af, g\rangle_{H_{\text{per}}^{m}}| \geq c\|f\|_{H_{\text{per}}^{s+\beta}}, \quad f \in \mathcal{V}_{j,\delta}^{(r+1)}$$

if and only if there is $c' > 0$ such that

$$\sup_{g\in\mathcal{V}_j^{(2m)},\, \|g\|_{2m-s}=1} |\langle Af, g\rangle_m| \geq c'\|f\|_{s+\beta}, \quad f \in \mathcal{V}_{j,\delta}^{(r+1)}.$$

This is essentially due to the fact that the difference between the two scalar products involves compact operators. The proof is straightforward and uses classical techniques.

3.3 Numerical approach

3.3.1 Introduction

If Ω is a smooth bounded and connected open subset of $\mathbb{R}^2$ whose boundary $\partial\Omega$ is also connected, the norm $\|.\|_m$ defines the usual topology on $H^m(\partial\Omega)$.

Let $\delta \in [0, 1[$, $r, m \in \mathbb{N}$ and

$$A = b_+ Q_+^\beta + b_- Q_-^\beta + K_0$$

be a bijective pseudodifferential operator as introduced before and such that the operator $u \mapsto b_+ Q_+^\beta u + b_- Q_-^\beta u + u_0$ satisfies the conditions of Theorem 7. To obtain an approximate solution of $Au = f$, one looks for $u_j \in \mathcal{V}_{j,\delta}^{(r+1)}$ such that the collocation equations

$$Au_j(2^{-j}k) = f(2^{-j}k), \quad k = 0, \ldots, 2^j - 1 \tag{1}$$

are satisfied. These equations are equivalent to

$$\langle Au_j, v\rangle_m = \langle f, v\rangle_m, \quad v \in \mathcal{V}_j^{(2m)}. \tag{2}$$

Indeed, both (2) and (1) define a $2^j \times 2^j$ linear system and $2m$ integrations by parts show that (2) is a consequence of (1). Moreover, for large j, it follows from Céa's lemma (see appendix) that the system (2) has full rank since Theorem 7 gives the strong ellipticity of the operators in the used spaces.

Using Theorem 7 and the fact that we deal with Riesz bases, we obtain that the l_2-condition numbers of the Galerkin matrices are bounded uniformly in j (Proposition 9). Hence, using the equivalence between the Galerkin and the collocation systems, the condition numbers of the collocation matrices have also a good asymptotic behaviour after preconditionning. Without this manipulation, we do not obtain a good condition number for the collocation matrices, but we do for the double layer potential (see Example 10).

3.3.2 Theoretical results on the l_2-condition number

Let us introduce some notations for the next result (Proposition 9). For every integer $j \geq 0$, we consider a Riesz basis $\{u_p^{(s+\beta)} : p \in \mathbb{Z}, p \geq 0\}$ of $H^{s+\beta}_{\text{per}}(]0,1[)$ (with bounds b, B) satisfying the following property: the functions $u_p^{(s+\beta)}$ $(p = 0, \ldots, 2^j - 1)$ form a basis of $\mathcal{V}^{(r+1)}_{j,\delta}$. In the same way, we consider a Riesz basis $\{v_p^{(s+\beta)} : p \in \mathbb{Z}, p \geq 0\}$ of $H^{2m-s}_{\text{per}}(]0,1[)$ (with bounds c, C) of splines of order $2m$ satisfying similar conditions. Wavelets bases fulfil these conditions.

Proposition 9 *Assume that* $\frac{1}{2} < s < 2m - \frac{1}{2}$, $\frac{1}{2} < s + \beta < r + \frac{1}{2}$. *If*

$$A = b_+ Q_+^\beta + b_- Q_-^\beta + K_0 \ : \ H^{s+\beta}_{\text{per}}(]0,1[) \to H^{s}_{\text{per}}(]0,1[)$$

is bijective and is such that the operator $u \mapsto b_+ Q_+^\beta u + b_- Q_-^\beta u + u_0$ *satisfies the conditions of Theorem7, then there is* $R > 0$ *such that the matrices* A_j *of dimension* 2^j *defined by*

$$A_j = (\langle A u_q^{(s+\beta)}, v_p^{(2m-s)} \rangle_m)_{0 \leq p,q < 2^j}$$

satisfy

$$\eta_2(A_j) = \|A_j\|_2 \|A_j^{-1}\|_2 \leq R \frac{BC}{bc}$$

if j *is large enough. The constant* R *depends on the norm of the operator* A *(hence on the boundary of the domain) and on the constant of coercivity* c *(Theorem 7).*

Proof. The conditions on s, r, m, β come from the fact they we use Theorem 7 and Riesz bases. The proof goes in the classical way using Riesz bases in the right spaces and Theorem 7. □

For the collocation case, we also introduce some notations.

If $\mathcal{B}_j = \{u_j^l,\ l = 0, \ldots, 2^j - 1\}$ is a basis of $\mathcal{V}^{(r+1)}_{j,\delta}$, we study the behaviour if $j \to +\infty$ of the condition numbers of the collocation matrices $A_j^{\mathcal{B}_j}$ defined as follows

$$\left(A_j^{\mathcal{B}_j}\right)_{l,k} = (A u_j^l)(k 2^{-j}), \qquad l, k = 0, \ldots 2^j - 1$$

where

$$A = b_+ Q_+^\beta + b_- Q_-^\beta + K_0.$$

We shall now discuss some examples that will be implemented numerically in Section 3.4. below. We only want to treat some examples, used in our numerical examples. First, we consider the following bases

$$\mathcal{B}_j = \{u_j^l(x) = u_j^0(x - l2^{-j}),\ l = 0, \ldots, 2^j - 1\},$$

$$\mathcal{B}'_j = \{1\} \cup \{\Psi_{\delta,r+1;i,k},\ 0 \le i < j, 0 \le k < 2^i\}$$

where

$$u_j^0(x) = 2^{j/2} \sum_{l=-\infty}^{+\infty} N_{\delta,r+1}(2^j(x-l)), \quad N_{\delta,r+1}(x) = N_{r+1}(x-\delta),$$

and where the functions $\Psi_{\delta,r+1;i,k}$ are constructed as usual starting from $\Psi_{\delta,r+1}$, which is the 1−periodization of $\psi_{r+1}(x - \delta/2)$. We shall also consider $r = 2m - 1$ and the bases

$$\mathcal{B}_j^s = \{1\} \cup \{2^{-is}\Psi_{2m;i,k} \quad 0 \le i < j, 0 \le k < 2^i\},$$

$$\mathcal{B}"_j = \{1\} \cup \{2^{i(m-s)}\Theta_{m;i,k} \quad 0 \le i < j, 0 \le k < 2^i\}.$$

In this case, because the order of the splines is even, we choose $\delta = 0$.

Example 10 *Assume that* $\frac{1}{2} < s < 2m - \frac{1}{2}$, $\frac{1}{2} < s + \beta < r + \frac{1}{2}$ *and that the operator* $u \mapsto b_+Q_+^\beta u + b_-Q_-^\beta u + (u)_0$ *satisfies one of the conditions 1), 2) of Theorem 7. Then there are constants* $r_1, r_2 > 0$ *such that*

$$r_1 2^{j|\beta|} \le \left\{ \begin{array}{c} \eta_2(A_j^{\mathcal{B}_j}) \\ \\ \eta_2(A_j^{\mathcal{B}'_j}) \end{array} \right\} \le r_2 2^{j|\beta|}, \qquad r_1 2^{j|\beta|-s-\beta} \le \left\{ \begin{array}{c} \eta_2(A_j^{\mathcal{B}_j^s}) \\ \\ \eta_2(A_j^{\mathcal{B}"_j}) \end{array} \right\} \le r_2 2^{j|\beta|+s+\beta}$$

Proof. First we consider the case

$$Au = b_+Q_+^\beta u + b_-Q_-^\beta u + (u)_0$$

where $(u)_0$ is the zeroth Fourier coefficient of u. For $u \in \mathcal{V}_{j,\delta}^{(r+1)}$, some computations give

$$(Au)(k2^{-j}) = (u)_0 + \sum_{l=1}^{2^j-1} e^{-2i\pi\delta 2^{-j}l} e^{2i\pi 2^{-j}kl} |l|^{r+1} d_l\ (u)_l$$

with $(u)_l = \int_0^1 dx\ e^{-ix2\pi l} u(x + 2^{-j}\delta)$ (the Fourier coefficients of u are $(u)_l e^{-2i\pi\delta l 2^{-j}}$) and[1]

$$d_l = \sum_{p=-\infty}^{+\infty} e^{-2i\pi p\delta}(b_+ + b_-\ \mathrm{sgn}(l + p2^j)) \frac{|l + p2^j|^\beta}{(l + p2^j)^{r+1}}.$$

[1] the complex numbers d_l are the same as in Theorem 7.

The matrix S defined by

$$(S)_{lk} = 2^{-j/2} e^{-2i\pi 2^{-j}lk}, \quad l, k = 0, \ldots, 2^j - 1$$

is unitary and diagonalizes the collocation matrix $A_j^{\mathcal{B}_j}$. From this, we find that the eigenvalues $\lambda_k (k = 0, \ldots 2^j - 1)$ of $A_j^{\mathcal{B}}$ are

$$2^j (u_j^0)_0, \quad 2^j e^{-2i\pi\delta 2^{-j}k} (u_j^0)_k d_k |k|^{r+1} \; k = 1, \ldots 2^j - 1$$

and

$$\eta_2(A_j^{\mathcal{B}_j}) = \frac{\sup\{|\lambda_k| \; : \; k = 0, \ldots 2^j - 1\}}{\inf\{|\lambda_k| \; : \; k = 0, \ldots 2^j - 1\}}.$$

Using the result on the behaviour of d_k and $(u_j^0)_k$ we finally obtain the annouced result.

Now, we simply consider change of bases in $\mathcal{V}_{j,\delta}^{(r+1)}$. If C_j denotes the matrix used to change the bases, we get

$$A_j^{\mathcal{B}'_j} = \widetilde{C_j} A_j^{\mathcal{B}_j}$$

with $\eta_2(C_j)$ bounded uniformly for $j \in \mathbb{N}$. In case $r = 2m - 1$, the matrix change of bases $\mathcal{B}'_j = \mathcal{B}_j^0$ to $\mathcal{B}_j^{s+\beta}$ has a condition number equals to $2^{(s+\beta)(j-1)}$; the matrix change of bases $\mathcal{B}_j^{s+\beta}$ to $\mathcal{B}_j^{"}$ has a bounded condition number. Hence the conclusion.

For the general expression of the operator, we use the previous result and the one on the condition number of the Galerkin matrices (Proposition 9). Let us denote by K_j the collocation matrices constructed using the compact operator K_0. We use the notation A_j for any of the collocation matrices constructed in the first part of this example (i.e. with the particular expression of the operator); finally, G_j (resp. G'_j) denote the Galerkin matrices corresponding to the particular expression of the operator (resp. for the general expression).

Because collocation and Galerkin systems are equivalent (for j large enough), there are inverse matrices L_j, independent of the chosen basis for the collocation matrices, such that

$$G_j = L_j A_j \quad (1) \qquad G'_j = L_j(A_j + K_j) \quad (2).$$

From the first relations (1) and the fact that the Galerkin matrices G_j have a bounded condition number we obtain that the behaviour of the condition numbers of the matrices L_j and A_j are the same. Now, the second relations (2) and the fact that the Galerkin matrices G'_j have a bounded condition number show finally that the behaviour of the condition numbers of the matrices A_j and $A_j + K_j$ are the same. □

3.4 Some numerical examples

We have performed some numerical experiments to test the asymptotic convergence and stability obtained using the $\Theta_{m;j,k}$ functions. We used these functions because they have a very short support; this fact, and also the fact that bases of wavelets are hierachical ones, lead to computations that are very easy to handle. We do not try to use the dual space because the functions are obtained from functions that are not compactly supported, hence the exact computations are rather heavy. The collocation matrices are preconditioned to get the Galerkin matrices and then a bounded condition number.

We consider the double layer potential ($\beta = 0, m \geq 1$) and the single layer potential ($\beta = -1, m \geq 2$). For the experiments, we use the same basis for the test and trial bases although lower order splines can be used as test functions without great loss of accuracy. We always find a bounded sequence $\eta(A_j)$ and optimal order of convergence.

The computation of the elements of the matrices A_j can be performed easily using the Gauss-Legendre method with weights. It allows us to deal with a logarithm singularity in the case of the single layer potential.

We choose the connected open subset of $\mathbb{R}^2$ represented on Figure 1.

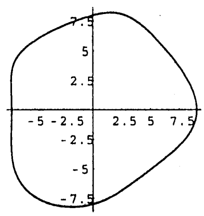

Figure 1

Its boundary is smooth, connected and given by

$$\gamma(t) = \Big((\cos(6\pi t) + 8)\cos(2\pi t), (\sin(4\pi t) + 8)\sin(2\pi t)\Big), \quad t \in [0,1].$$

To use as efficiently as possible the almost orthogonality of the functions $\Theta_{m,jk}$ defined on $[0,1]$, we work with a parameterization by arc length. This requires to solve an autonomous differential equation. This can be done using a standard integration method before the computation of the matrices and takes a very short time since it depends linearly on the number of points.

Figure 2 gives the ℓ^2-condition number η of the matrix A_j for the double and the single layer potential on γ. This figure also gives the L^2 norm of the error for the right-hand side $f(x) = 2\sin(2\pi x)$ and the estimated exponent of convergence (eoc).

For the double layer potential, we use the Riesz basis $\{1, \Theta_{m;j,k} : j \geq 0, 0 \leq k < 2^j\}$ and we treat the cases $m = 1$ (linear splines), and $m = 2$ (cubic splines).

For the single layer potential, we use the Riesz basis $\{1, 2^{\frac{j}{2}}\Theta_{m;j,k} : j \geq 0, 0 \leq k < 2^j\}$. Since the trial and test functions are chosen to be the same, we cannot use here the linear splines $\Theta_{1;j,k}$ because they do not satisfy the theoretical conditions of Proposition 9.

	Double layer potential						Simple layer potential		
j	$m=1$			$m=2$			$m=2$		
	η	error	eoc	η	error	eoc	η	error	eoc
1	5.99	7.1 e-01		2.15	2.6 e-01		6.04	4.4 e-02	
2	6.90	2.0 e-01	1.82	33.73	9.0 e-02	1.55	24.68	4.1 e-02	0.10
3	7.16	5.2 e-02	1.96	50.29	6.0 e-03	3.91	73.57	6.8 e-03	2.58
4	7.23	1.3 e-02	1.95	55.32	5.4 e-04	3.47	81.71	7.1 e-04	3.27
5	7.25	3.3 e-03	2.01	55.62	1.9 e-05	4.83	82.66	2.0 e-05	5.16
6	7.25	8.0 e-04	2.06	55.64	8.3 e-07	4.52	82.89	7.6 e-07	4.70
7	7.25	1.7 e-04	2.27	55.64	4.0 e-08	4.37	82.95	4.3 e-08	4.13

Figure 2

3.5 Appendix

Céa's lemma

Let X, Y *be Banach spaces,* let $A \in L(X,Y)$ *be bijective and let* $V_j \subset X$ $(j \in \mathbb{N})$, $T_j \subset Y'$ $(j \in \mathbb{N})$, *be sequences of subspaces such that* $dim(V_j) = dim(T_j) < +\infty$ *for every* j. *Assume that*
(i) there are $P_j \in L(Y', T_j)$ $(j \in \mathbb{N})$ such that $\lim_{j\to+\infty} P_j(f) = f$ in Y' for every $f \in Y'$,
(ii) there are $\delta > 0$ and a compact operator $K \in L(X,X)$ such that, for every j and $u \in V_j$:

$$\sup_{v\in T_j, \|v\|_{Y'}=1} |v(Au)| \;\geq\; \delta\|u\|_X - \|Ku\|_X.$$

Then there is $N_0 > 0$ *such that, for every* $j \geq N_0$ *and* $u \in X$, *the equation*

$$v(Au_j) \;=\; v(Au), \quad v \in T_j$$

has a unique solution $u_j \in V_j$. *Moreover, there is* $C > 0$ *such that*

$$\|u - u_j\|_X \;\leq\; C \inf_{w\in V_j} \|u - w\|_X.$$

Here we use this lemma with

$$V_j = \mathcal{V}_{j,\delta}^{(r+1)}, \quad T_j = \{< ., g >_{H^m_{per}(\mathbb{R})}: g \in \mathcal{V}_j^{(2m)}\}.$$

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Symbols

1. $\mathbb{Z}$ is the set of integers
2. $\mathbb{N}$ is the set of strictly positive integers
3. $V_j^{(m)}$ $(j \in \mathbb{Z}, m \in \mathbb{N})$ is the set of functions on $\mathbb{R}$ which are smoothest splines of degree $m-1$ with respect to the mesh $\{2^{-j}k : k \in \mathbb{Z}\}$ and belong to $L^2(\mathbb{R})$
4. $V_{j,\delta}^{(m)}$ $(\delta \in [0,1[, j \in \mathbb{Z}, m \in \mathbb{N})$ is the same (as $V_j^{(m)}$) set of splines but with respect to the mesh $\{2^{-j}(k+\delta) : k \in \mathbb{Z}\}$
5. The corresponding sets of 1-periodic splines are respectively denoted by $\mathcal{V}_j^{(m)}$, $\mathcal{V}_{j,\delta}^{(m)}$
6. $H^s_{\text{per}}(]0,1[)$ $(s \in \mathbb{R})$ is the Sobolev space of order s of 1-periodic distributions on $\mathbb{R}$
7. ψ_m is the classical Chui-Wang spline biwavelet $(\psi_m \in V_1^{(m)})$
8. θ_m is the spline function in $V_1^{(2m)}$ such that $D^m\theta_m = \psi_m$ and $\text{supp}(\theta_m) \subset [0, 2m-1]$

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Note: P. Laubin died in 2001

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