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THE RESOLVENT ON $\mathbb{R}^n$, A DEGENERATING METHOD.

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ABSTRACT. Let $G = SO(n,1)$ and $K = SO(n)$. We use a continuous family of Lie algebras isomorphic to the Lie algebra of G, $\mathfrak{g}$, to degenerate the hyperbolic real space $H^n \simeq G/K$ into the Euclidean space $\mathbb{R}^n$. This allows us to recover the resolvent of the Laplacian on $\mathbb{R}^n$ from the resolvent of the hyperbolic Laplacian.

1. INTRODUCTION

Let $H^n = G/K$ be the hyperbolic space, where $G = SO(n,1)$ and $K = SO(n)$. By using the Cartan decomposition of the Lie algebra of G, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, we change continuously the Lie bracket on $\mathfrak{g}$ in order to make it vanishes on $\mathfrak{p}$, and so degenerating $\mathfrak{g}$ to the Lie algebra $\mathfrak{g}_0$ of the group of isometries of $\mathbb{R}^n$. We then have a continuous family of Lie algebras $\mathfrak{g}_s$ (for $s \in [0,1]$), from $\mathfrak{g}_1 = \mathfrak{g}$ to $\mathfrak{g}_0 = \mathfrak{k} \oplus \mathbb{R}^n$ such that for $s \neq 0$, $\mathfrak{g}_s$ is isomorphic to $\mathfrak{g}$. For $s \neq 0$, the techniques we have in the hyperbolic case (see [5]) are applied to calculate the radial part of the Casimir element of $\mathcal{U}(\mathfrak{g}_s)$, and also we characterize the spherical functions as certain solutions of the following ordinary differential equation (note that it also depends on s)

$$\left(\frac{d^2}{dt^2} + s(n-1)\coth(st)\frac{d}{dt} - \lambda_s(\nu)\right) f(a_s(t)) = 0. \tag{1}$$

With all this, we obtain an expression for the resolvent kernel, given by a solution of the above equation with appropriate asymptotic behavior. Then by looking at the limits we obtain an expression for the kernel in $\mathfrak{g}_0$. Studding the differential equation this kernel satisfies, we also can express it in terms of special functions (see (15) and (16)).

Although an expression of the kernel of the resolvent of the Laplacian in this case is of course known, we think that this method may be used in other cases where one knows something about the resolvent, as for example the other rank one symmetric spaces of noncompact type, or more generally, the Damek-Ricci spaces. Also this approach might present some interest in considering the heat Kernel instead of the resolvent.

2. PRELIMINARIES

We begin by introducing notation that will be used throughout this paper. As is customary, we will denote a Lie group by an upper case letter and its Lie algebra by the corresponding lower case gothic letter.

Let $G = SO(n,1)$ be the Lie group of matrices in $Sl(n+1,\mathbb{R})$ leaving the quadratic form $-x_1^2 - x_2^2 - \ldots - x_n^2 + x_{n+1}^2$ invariant. Consider $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition

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of its Lie algebra $\mathfrak{g}$, associated to the Cartan involution $\theta(X) = JXJ$, where $J = \left[\begin{smallmatrix} -\mathrm{Id} & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Thus

$$\mathfrak{k} = \left\{ \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} : A \in \mathfrak{so}(n) \right\} \quad \text{and} \quad \mathfrak{p} = \left\{ \begin{bmatrix} 0 & b \\ -b^t & 0 \end{bmatrix} : b \in \mathbb{R}^n \right\}.$$

Also, if we put

$$H_0 = \left[\begin{smallmatrix} & & 1 \\ & 0 & \\ -1 & & \end{smallmatrix}\right] \in \mathfrak{p}$$

it is easy to see that $\mathfrak{a} = \mathbb{R}H_0$ is a maximal abelian subalgebra of $\mathfrak{p}$. Let B be a multiple of the Killing form of $\mathfrak{g}$ such that $B(H_0, H_0) = 1$, and we take the inner product $\langle \cdot, \cdot \rangle$ given by $B_\theta = -B(\cdot, \theta \cdot)$. Let $\{\alpha\}$ be the corresponding system of positive roots of the pair $(\mathfrak{g}, \mathfrak{a})$, such that $\alpha(H_0) = 1$, and as usual, let $\rho = \frac{n-1}{2}\alpha$.

We will identify the complexified dual space $\mathfrak{a}_c^*$ with $\mathbb{C}$ under the correspondence $\nu = z\alpha \mapsto z$. In other words, since $\alpha(H_0) = 1$, we are identifying $\nu \in \mathfrak{a}_c^*$ with $\nu(H_0)$.

For each $s \in [0,1]$ let $\phi_s : \mathfrak{g} \mapsto \mathfrak{g}$ be defined by

$$\phi_s(X + Y) = X + sY \qquad X \in \mathfrak{k},\ Y \in \mathfrak{p}.$$

Define $\mathfrak{g}_s = (\mathfrak{g}, [\cdot,\cdot]_s)$ the metric Lie algebra with underlying inner product space $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ and Lie bracket given by

$$[X, Y]_s = \phi_s^{-1}[\phi_s X, \phi_s Y], \quad X, Y \in \mathfrak{g}.$$

Thus, $\phi_s : \mathfrak{g}_s \mapsto \mathfrak{g}$ is a Lie algebra isomorphism. It is easy to see that if $\mathfrak{g}_0 = \lim_{s \mapsto 0} \mathfrak{g}_s$, then $\mathfrak{g}_0 \simeq \mathfrak{k} \oplus \mathbb{R}^n$ is the Lie algebra of $M_0(\mathbb{R}^n)$, the group of isometries of $\mathbb{R}^n$.

For each $s \in [0,1]$, let G_s denote the connected Lie group with Lie algebra $\mathfrak{g}_s$. It is easy to see that

$$\begin{array}{ccc} \mathfrak{g}_s & \xrightarrow{\phi_s} & \mathfrak{g} \\ {\scriptstyle \exp_s}\downarrow & & \downarrow{\scriptstyle \exp} \\ G_s & \xrightarrow{\Phi_s} & G \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathfrak{g}_s & \xrightarrow{\mathrm{ad}_s} & \mathfrak{gl}(\mathfrak{g}_s) \\ {\scriptstyle \exp_s}\downarrow & & \downarrow{\scriptstyle \exp} \\ G_s & \xrightarrow{\mathrm{Ad}_s} & Gl(\mathfrak{g}_s) \end{array}$$

commute, that is

$$\Phi_s \circ \exp_s = \exp_s \circ \phi_s \ \text{ and } \ \mathrm{Ad}_s \circ \exp_s = e \circ \mathrm{ad}_s . \tag{2}$$

On the other hand, by the definition of $\mathfrak{g}_s$ we have that for $H \in \mathfrak{a}_s (= \mathfrak{a}\ \forall s)$, $[H, X]_s = \phi_s^{-1}[sH, \phi_s X]$. Therefore, if $X_\alpha \in \mathfrak{g}_\alpha$, and we take $X_{\alpha_s} = \phi_s^{-1} X_\alpha$, then we have that $[H, X_{\alpha_s}]_s = s\alpha(H) X_{\alpha_s}$. This implies that $\alpha_s = \alpha \circ \phi_s = s\alpha$ is the restricted root (system) of the pair $(\mathfrak{g}_s, \mathfrak{a}_s)$. We have also that the corresponding ρ_s is given by $\rho_s = \frac{s(n-1)}{2}$ due to our identification of $\mathfrak{a}_c^*$.

If for each s, $A_s^+ = \{\exp_s(tH_0) : t > 0\}$, we have the polar decomposition of G_s, $G_s = K\mathrm{Cl}(A_s^+)K$. We take on A_s, the Lie subgroup of G_s with Lie algebra $\mathfrak{a}_s$, the measure $da = dt$; on K we normalize the measure so that the total mass is one, and for the measure on G_s we have the following observation.

Lemma 2.1. *The Haar measure on G_s relative to the polar decomposition is given by $dx = J_s(a) dk_1\, da\, dk_2$. More precisely, if $f \in C_c(G_s)$,*

$$\int_{G_s} f(g) dg = \int_{KA_s^+K} f(k_1 a k_2) J_s(t) dk_1\, da dk_2$$

where $J_s(a) = \left(\frac{2\sinh(st)}{s}\right)^{n-1}$, *for* $a = \exp_s(tH_0)$.

Proof. The proof of this fact is basically given in [1, pp. 73], and we will only need the following two observations when s appears in the proof.

- Since $G_s = \mathfrak{k}\exp_s(\mathfrak{p}_s)$, we have that (see [1] pp.72)
$$dx = \det\left[\frac{\sinh\ ad_s(z)}{ad_s(z)}\right] dk\, dz \qquad \text{where } x = k\exp_s(z).$$
- The function $X \overset{\psi}{\mapsto} [X, H]$ induces an isomorphism from $\mathfrak{k}/\mathfrak{m}$ into $\mathfrak{a}_s^{\perp} \subset \mathfrak{p}$. It is easy to see that its determinant is given by $c\alpha(H)^{n-1}$, where c is a nonzero constant independent of H.

Then we have that $J_s(\exp_s(tH_0))$ is a multiple of
$$\det\left[\frac{\sinh\ ad_s(\mathrm{Ad}(k)tH_0)}{ad_s(\mathrm{Ad}(k)tH_0)}\right]\ \alpha(tH_0)^{n-1} = \det\left[\mathrm{Ad}(k)\frac{\sinh\ ad_s(tH_0)}{ad_s(tH_0)}\ \mathrm{Ad}(k)^{-1}\right]\ t^{n-1}$$
as asserted. □

For simplicity we will denote $J_s(t) = J_s(\exp_s(tH_0))$.

Definition 2.2. A function f on G is called *K-biinvariant* or *radial* if $f(kxk') = f(x)$ for all $k,\ k' \in K$.

Let $C(G_s/\!\!/K)$ be the space of continuous radial functions, and we will also denote by $C^\infty(G_s/\!\!/K)$ and $C_c^\infty(G_s/\!\!/K)$, the space of smooth radial functions and the compactly supported smooth radial functions, respectively.

Let f^- denote the restriction to A_s^+ of a function $f \in C(G_s/\!\!/K)$. It follows from the polar decomposition $G_s = K\mathrm{Cl}(A_s^+)K$ that f is determined by f^-. Moreover, if D is a differential operator on G_s invariant under the right and left action by elements of K, we can define the *radial component* of D, as a differential operator on A_s^+ such that (see [1, § 4.1])
$$(Df)^- = \Delta(D)f^-, \qquad \forall\, f \in C^\infty(G_s/\!\!/K)). \tag{3}$$

Let C denote the Casimir element of the complexification of $\mathfrak{g}_s$, $(\mathfrak{g}_s)_c$, with respect to B. We are interested in the radial component of C. To calculate it, we will use arguments analogous to those in [8, pp. 280].

Let $X_1, \ldots, X_{n-1}$, be a basis of $\mathfrak{g}_\alpha$, such that $-B(X_i, \theta(X_j)) = \delta_{i,j}$, i.e. orthonormal with respect to $\langle , \rangle$. For $i = 1, ..., n-1$, consider $X_i^s = \phi_s^{-1}X_i$. Thus, from the previous observations we have that $X_i^s \in \mathfrak{g}_{\alpha_s}$ for all i. If we define, as usual, for $j = 0, \ldots, (n-1)$
$$Z_j = 2^{-\frac{1}{2}}(X_j + \theta(X_j)), \qquad Y_j = 2^{-\frac{1}{2}}(X_j - \theta(X_j)),$$
and the corresponding Z_j^s and Y_j^s, it is easy to see that $Z_j^s = Z_j$ and $Y_j^s = s^{-1}Y_j$. On the other hand, since the inner product does not depend on s, we have that in each $\mathfrak{g}_s$ the Casimir operator is given by
$$C_s = H_0^2 + \sum_{j=1}^{n-1} Y_j^2 - \sum_{j=1}^{n-1} Z_j^2 \qquad \in \mathcal{U}(\mathfrak{g}_s).$$

Therefore the action of C_s on $C^\infty(G_s/\!\!/K)$ is given by the following ordinary differential equation.

Proposition 2.3. *If $f \in C^\infty(G_s/\!/K)$ then*

$$C_s f(a_s(t)) = \left(\frac{d^2}{dt^2} + s(n-1)\coth(st)\frac{d}{dt}\right) f(a_s(t)).$$

In particular, we have that in the limit

$$\lim_{s\mapsto 0} C_s f(a_s(t)) = \left(\frac{d^2}{dt^2} + \frac{n-1}{t}\frac{d}{dt}\right) f(tH_0). \tag{4}$$

which is the action of the Laplacian on $\mathbb{R}^n$ acting on radial functions (see [2, pp. 266]).

Proof. From the definition of the Z_i^s and (2) we have that

$$\begin{aligned}
\mathrm{Ad}_s(a_s(-t))\, Z_i &= \mathrm{Ad}_s(a_s(-t))\, Z_i^s \\
&= \tfrac{1}{\sqrt{2}}\left(e^{\mathrm{ad}_s(-tH_0)X^i_{\alpha_s}} + e^{\mathrm{ad}_s(-tH_0)X^i_{-\alpha_s}}\right) \\
&= \tfrac{1}{\sqrt{2}}\left(e^{-st}X^i_{\alpha_s} + e^{st}X^i_{-\alpha_s}\right) \\
&= \cosh(st)\, Z_i^s - \sinh(st)\, Y_i^s \\
&= \cosh(st)\, Z_i - \sinh(st)s^{-1}Y_i.
\end{aligned}$$

Thus

$$Y_i = s\coth(st)\, Z_i + s\sinh^{-1}(st)\, \mathrm{Ad}_s(a_s(-t))\, Z_i, \tag{5}$$

and then we get

$$\begin{aligned}
Y_i^2 &= s^2\coth^2(st)\, Z_i^2 + s^2\sinh^{-2}(st)\Big(\mathrm{Ad}_s(a_s(-t))\, Z_i\Big)^2 \\
&\quad -s^2\coth(st)\sinh^{-1}(st)\Big(Z_i\cdot \mathrm{Ad}_s(a_s(-t))\, Z_i + \mathrm{Ad}_s(a_s(-t))\, Z_i\cdot Z_i\Big).
\end{aligned}$$

Note that (5) also implies that

$$[Z_i, \mathrm{Ad}_s(a_s(-t))\, Z_i] = -s^{-1}\sinh(st)[Z_i, Y_i],$$

and then we have that

$$\begin{aligned}
Y_i^2 &= s^2\coth^2(st)\, Z_i^2 - s^2\sinh^{-2}(st)\,[\mathrm{Ad}_s(a_s(-t))\, Z_i]^2 \\
&\quad -2s^2\coth(st)\sinh^{-1}(st)\, \mathrm{Ad}_s(a_s(-t))\, Z_i\cdot Z_i + s\coth(st)H_0.
\end{aligned}$$

Therefore if f is biinvariant for the action of K, we have that

$$Z_i\, f(g) = \frac{d}{dt}_{|t=0} f(g\exp_s(tZ_i)) = 0,$$

and then

$$Y_i^2 f(a_s(t) = s\coth(st)H_0 f(a_s(t)),$$

as was to be shown. $\square$

3. Spherical functions and the Resolvent

Definition 3.1. If ϕ is a complex valued radial continuous function on G_s, then ϕ is said to be a *K-spherical* (or simply *spherical*) function if $\phi(e) = 1$ and $C_s\phi = \lambda\phi$ for some $\lambda \in \mathbb{C}$.

For each $s \in [0,1]$ and $\nu \in \mathbb{C}$, let $\phi_s(\nu, \cdot)$ be the spherical function on G_s with eigenvalue $\lambda_s(\nu) = \nu^2 - \rho_s^2$.

Remark 3.2. We note that this eigenvalue is not arbitrary, and in order to calculate it for $s \neq 0$, since $\mathfrak{g}_s$ is isomorphic to $\mathfrak{g}$, one can proceed exactly as in $SO(n,1)$ to see that such functions are given by a matrix entry of the spherical principal series of G_s associated to the character $\chi_\nu^s(man) = a^{\nu+\rho_s}$ (see [1, pp.103]).

We then have, for each $s \neq 0$, that $\phi_s(\nu, \cdot)$ is the solution of the following differential equation

$$\left(\frac{d^2}{dt^2} + s(n-1)\coth(st)\frac{d}{dt} - \lambda_s(\nu)\right) f(a_s(t)) = 0 \tag{6}$$

continuous at $t = 0$, and such that $f(a_s(0)) = 1$.

Note that since $\lim_{t \mapsto 0} ts(n-1)\coth(st) = n-1$, this equation has a regular singular point at $t = 0$. Moreover, if we set $z = st$ it is easy to see that $f(z)$ satisfies the equation (6) if and only if

$$\left(s^2\frac{d^2}{dz^2} + s^2(n-1)\coth(z)\frac{d}{dz} - \lambda_s(\nu)\right) f(z) = 0,$$

or equivalently

$$\left(\frac{d^2}{dz^2} + (n-1)\coth(z)\frac{d}{dz} - \left[\left(\frac{\nu}{s}\right)^2 - \rho^2\right]\right) f(z) = 0. \tag{7}$$

This is a Jacobi equation with parameters $\lambda = \mathbf{i}\frac{\nu}{s}$, $\alpha = \frac{n-2}{2}$, $\beta = -\frac{1}{2}$, and $\gamma = \rho$ (see [3]), and therefore, we have that for each $s \neq 0$, the spherical functions ϕ_s are given by Jacobi functions in the following way:

$$\phi_s(\nu, a_s(t)) = \varphi_{\frac{\nu}{s}}^{(\frac{n-2}{2}, -\frac{1}{2})}(st).$$

Equivalently, in terms of the Gauss hypergeometric functions we have that

$$\phi_s(\nu, a_s(t)) = {}_2F_1\left(\frac{\rho_s - \nu}{2s}; \frac{\rho_s + \nu}{2s}; n; -\sinh^2(st)\right).$$

It can also be seen (see [3, pp. 7]) that for $s \neq 0$ and $\nu \notin -s\mathbb{N}$, a second solution of the equation (6) in $(0, +\infty)$ is given, in terms of the hypergeometric function, by

$$\tilde{Q}_s(\nu, t) = (2\cosh(st))^{-(\frac{\nu}{s}+\rho)}\ {}_2F_1\left(\frac{\rho_s + \nu}{2s}; \frac{s(n+1) + 2\nu}{4s}; \frac{\nu}{s} + 1; \cosh^{-2}(st)\right).$$

If $\mathrm{Re}\,\nu > 0$, the asymptotic behavior of these functions as $t \to \infty$, is given by:

$$\phi(\nu, a_s(t)) = c(\nu, s)\, e^{t(\nu - \rho_s)}, \qquad \tilde{Q}_s(\nu, t) \sim e^{-(\nu+\rho_s)t},$$

where

$$c(\nu, s) = \frac{2^{\rho - \frac{\nu}{s}}\Gamma(\frac{\nu}{s})\,\Gamma(\frac{n}{2})}{\Gamma(\frac{\rho_s+\nu}{2s})\,\Gamma(\frac{s(n+1)+2\nu}{4s})}.$$

It is proved in [5] (see also [6]) that for $s = 1$, the function $Q_1(\nu, a_1(t)) = \frac{\tilde{Q}_1(\nu,t)}{2\nu c(\nu,1)}$ is a solution of the equation (6) (with $s = 1$), such that

$$\lim_{t\mapsto 0^+} J_1(t)Q_1(\nu, a_1(t)) = 0,$$

$$\lim_{t\mapsto 0^+} J_1(t)\,\frac{d}{dt}Q_1(\nu, a_1(t)) = 1,$$

and moreover, it is also proved that the resolvent of the Laplacian on G_1/K (in certain half plane of $\mathbb{C}$) is given by convolution with the K-biinvariant function on G extending this function. Analogously we can generalize this, in the following theorem, for the other values of s.

Theorem 3.3. *For each $s \in (0, 1]$ and $\nu \in \mathbb{C}$, $\nu \notin -s\mathbb{N}$, there exists a function $Q_s(\nu, \cdot) \in C^\infty(G_s - K/\!/K)$ with the following properties:*

(a) $C_sQ_s(\nu, \cdot) = \lambda(\nu, s)Q_s(\nu, \cdot)$.

(b) $\lim_{t\mapsto 0^+} J_s(t)Q_s(\nu, a_s(t)) = 0$ *and* $\lim_{t\mapsto 0^+} J_s(t)\,\frac{d}{dt}Q_s(\nu, a_s(t)) = 1$.

(c) *Where defined, $Q_s(\nu, g) \in L^1_{\text{loc}}(G_s)$, and if $\operatorname{Re}\nu > \rho_s$, $Q_s(\nu, g) \in L^1(G_s)$.*

(d) *If $f \in C_c^\infty(G_s/\!/K)$ and $\nu \notin -s\mathbb{N}$ then*

$$\int_{G_s} Q_s(\nu, x^{-1}y)(C_s - \lambda(\nu, s)\mathrm{Id})\mathrm{f(y)dy} = f(x). \tag{8}$$

Proof. The proof of this theorem is essentially the same as that of Theorem 2.2 in [6] (see also the references given there) and therefore we will just make some observations.

Let $Q_s(\nu, \cdot)$ be a K-radial function on G_s such that restricted to A_s is given by

$$Q_s(\nu, a_s(t)) = -\frac{\tilde{Q}_s(\nu, t)s^{2\rho}}{2\nu c(\nu, s)}.$$

It is easy to see from the above remarks that this function satisfies (a). To see (b), recall that this function is a solution of equation (6) and this equation has a regular singular point at $t = 0$. The corresponding indicial equation is given by $\alpha\,(\alpha + (n - 2)) = 0$, with solutions $\alpha = 0$ and $\alpha = -(n - 2)$. It is clear that $\phi_s(\nu, \cdot)$ is the solution corresponding to $\alpha = 0$. If $\nu \notin -s\mathbb{N}$, we know that $\tilde{Q}_s(\nu, t)$ is a linearly independent solution, and therefore, by the general theory of regular singular points, we have that, when $t \mapsto 0$ $\tilde{Q}_s(\nu, t) \sim d_s(\nu)t^{n-2}|\log(t)|^{\delta_{n,2}}$ where $d_s(\nu)$ is a meromorphic function of ν. Hence, $\lim_{t\mapsto 0^+} J_s(t)Q_s(\nu, a_s(t)) = 0$.

Finally, to prove the second part of (b), we note that $s(n - 1)\coth(st) = \frac{J_s'(t)}{J_s(t)}$ (see Lemma (2.1)). This fact gives us an analogue of formula $(\star)$ in [5] pp. 669, and then we can proceed as in [5, lemma 1.3]. □

4. The Resolvent kernel

We first note that part (d) of the above theorem implies that in the limit, for $\mathrm{Re}(\nu) > 0$, $(\Delta - \nu^2\mathrm{Id})^{-1}$ is given by convolution with $Q_0(\nu, \cdot)$, where $Q_0(\nu, \cdot)$ is a radial function on $\mathbb{R}^n$ such that $Q_0(\nu, t)$ is the solution of the following differential equation (see (4))

$$\left(\frac{d^2}{dt^2} + \frac{n-1}{t}\frac{d}{dt} - \nu^2\right) f(t) = 0, \tag{9}$$

such that

$$\lim_{t\mapsto 0} J_0(t)Q_0(\nu,t) = 0 \quad \text{and} \quad \lim_{t\mapsto 0} J_0(t)\frac{d}{dt}Q_0(\nu,t) = 1.$$

Here $J_0(t) = (2t)^{n-1}$.

In order to have explicit solutions of (9), we will introduce now the Bessel equations. The following differential equation

$$u''(t) + \frac{1}{t}u'(t) + \left(1 - \frac{\eta^2}{t^2}\right) u(t) = 0 \tag{10}$$

is called a Bessel equation. It is easy to see that this equation is equivalent to

$$f''(t) + \frac{2\alpha+1}{t}f'(t) + \left(\mu^2 - \frac{\eta^2 - \alpha^2}{t}\right) f(t) = 0, \tag{11}$$

where $u(t) = t^\alpha f(\mu^{-1}t)$.

Therefore, if we take $\eta = \alpha = \frac{n-2}{2}$ and $\mu = \mathrm{i}\nu$ then we have that the equation (9) is equivalent to the Bessel equation

$$u''(t) + \frac{1}{t}u'(t) + \left(1 - \left(\frac{n-2}{2t}\right)^2\right) u(t) = 0, \tag{12}$$

where $u(t) = t^{\frac{n-2}{2}} f(\frac{t}{\mathrm{i}\nu})$.

The solutions of this differential equation are well known. We will now summarize some known results on them, following [7, Ch.3, sec 6].

First, let $\mathcal{J}_\eta$ be the Bessel function. It is given by

$$\mathcal{J}_\eta(z) = \left[\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\eta + \frac{1}{2}\right)\right]^{-1} \left(\frac{z}{2}\right)^\eta \int_{-1}^{1} (1-t^2)^{\eta-\frac{1}{2}} e^{zt\mathrm{i}}\, dt.$$

Note that the integral in the above formula is meromorphic in η with simple poles at $\eta + \frac{1}{2} \in -\mathbb{N}$, and these poles are cancelled by the factor $\Gamma(\eta+\frac{1}{2})^{-1}$. Thus, these functions give smooth solutions of the Bessel equation, analytic in η. We also have that for $\eta = k + \frac{1}{2}$ this integral can be calculated explicitly, since it involves $(1-t^2)^k$, and then $\mathcal{J}_{k+\frac{1}{2}}$ is an elementary function for each $k \in \mathbb{N}$.

Other type of solutions of (10) are given by the so called Hankel functions, $\mathcal{H}_\eta^{(1)}$ and $\mathcal{H}_\eta^{(2)}$. These functions form a basis of the space of solutions of (10) and they are linearly independent with $\mathcal{J}_\eta$. One can see that for $\mathrm{Re}\,\eta > -\frac{1}{2}$ and $\mathrm{Im}\, z > 0$, $\mathcal{H}_\eta^{(1)}$ is given by

$$\mathcal{H}_\eta^{(1)}(z) = \frac{2e^{-\pi\mathrm{i}\eta}}{\mathrm{i}\pi\Gamma(\eta+\frac{1}{2})}\left(\frac{z}{2}\right)^\eta \int_1^\infty (t^2-1)^{\eta-\frac{1}{2}} e^{\mathrm{i}zt}\, dt.$$

We also have that for $\eta = k + \frac{1}{2}$, $\mathcal{H}_{k+\frac{1}{2}}$ are elementary functions for $k \in \mathbb{N}$. Finally, if we let

$$\mathcal{K}_\eta(r) = \frac{1}{2}\left(\frac{r}{2}\right)^\eta \int_0^\infty t^{-1-\eta} e^{\frac{r^2}{4t}-t}\, dt,$$

it can be seen that it is a solution of (10), and in fact (see [7] pp. 233), for $r > 0$ we have that

$$\mathcal{K}_\eta(r) = \frac{1}{2}\pi\mathrm{i}e^{\pi\mathrm{i}\eta/2}\mathcal{H}_\eta^{(1)}(\mathrm{i}r). \tag{13}$$

We then have that any solution of (9) is given by

$$f_\nu(t) = a\, t^{-\frac{n}{2}+1}\mathcal{J}_{\frac{n}{2}-1}(\mathbf{i}\nu t) + b\, t^{-\frac{n}{2}+1}\mathcal{H}^{(1)}_{\frac{n}{2}-1}(\mathbf{i}\nu t) \tag{14}$$

for some $a, b, \in \mathbb{C}$. From the properties of $\mathcal{J}_\eta$ and $\mathcal{H}^{(1)}_\eta$ listed above, it can be seen that $\phi_0(\nu, t)$ is a constant multiple of

$$(\nu t)^{1-\frac{n}{2}}\mathcal{J}_{\frac{n}{2}-1}(\mathbf{i}\nu t).$$

On the other hand, in order to have an explicit expression for the resolvent kernel, we first note that since the term in (14) corresponding to $\mathcal{J}_\eta$ leads to a C^∞ eigenfunction of Δ, we could set

$$Q_0(\nu, t) = b\, t^{-\frac{n}{2}+1}\mathcal{H}^{(1)}_{\frac{n}{2}-1}(\mathbf{i}\nu t).$$

We also have that the asymptotic behavior of $\mathcal{H}^{(1)}_{\frac{n}{2}-1}$ as $t \mapsto 0$ is given by

$$\mathcal{H}^{(1)}_{\frac{n}{2}-1}(\mathbf{i}\nu t) \sim -\mathbf{i}\frac{\Gamma(\frac{n}{2}-1)}{\pi}\left(\frac{2}{\mathbf{i}\nu t}\right)^{\frac{n}{2}-1},$$

and therefore, by straightforward calculation we have that

$$Q_0(\nu, t) = -\frac{t^{\frac{n}{2}+1}}{\Gamma(\frac{n}{2})}\left(\frac{2}{\mathbf{i}\nu}\right)^{1-\frac{n}{2}}\frac{\pi\mathbf{i}}{2}\mathcal{H}^{(1)}_{\frac{n}{2}-1}(\mathbf{i}\nu t).$$

It is easy to see from (13) that the above formula is equivalent to

$$Q_0(\nu, t) = -\frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})}\left(\frac{t}{\nu}\right)^{1-\frac{n}{2}}\mathcal{K}_{\frac{n}{2}-1}(\nu t), \qquad t > 0. \tag{15}$$

Recall that if n is odd, $\mathcal{H}^{(1)}_{\frac{n}{2}-1}$ is an elementary function.

Remark 4.1. We would like to point out that the difference between (15) and the corresponding formula [7, (6.49) pp. 232] (see also [4, (1.26) pp. 7]), is due to the fact that in our case the Haar measure on K is normalized (i.e. K has total mass 1). It is easy to see that with this normalization, we would have to consider the kernel $R_0(\nu, t) = \frac{1}{\mathrm{Vol}(S^{n-1})}Q_0(\nu, t)$, where $\mathrm{Vol}(S^{n-1}) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$, and then we obtain

$$R_0(\nu, t) = -(2\pi)^{-n/2}\left(\frac{t}{\nu}\right)^{1-\frac{n}{2}}\mathcal{K}_{\frac{n}{2}-1}(\nu t) \tag{16}$$

as in [7, (6.49) pp. 232].

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