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The Adaptive Dynamics For The Randomly Alternatıng Prisoner's Dilemma Game

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Abstract.

We consider a game with two players and two choices for each one. In each round of the alternating model, there is one option for one player called leader. The two players have the same chance to be leader. In this model each two consecutive rounds represent one unit. We consider strategies realized by simple transition rules depending on the previous outcome. We consider homogeneous population of strategy S, and ask for the most favorable adaptation. Any parameter x in S changes according to the adaptive dynamics $\dot{x} = \frac{\partial F}{\partial x}$; F is the payoff for S-player.

1.Introduction.

Many examples of reciprocal altruism are modeled by an alternating Prisoner's Dilemma (PD) game see [11]. In this game we have two players I and II and two choices C to cooperate and D to defect. In each round of the game one of the two players choose his option to defect. In each round of the game one of the two players choose his option and the other player reply with his option in another round. And this means that for each round there is a single option for one of the two players. This player is called leader (or donor) and the other is called recipient. Consequently the leader control what the outcome is going to be.

The two players in this game alternate the leader role and both of them have the same chance to be leader. There are two types for this game, strictly alternating where the two players exchange the leader's role every round and the randomly alternating where the two players exchange the leader's role randomly. In this paper we shall be interested in the randomly alternating PD game. [1]

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2-The Transition Matrix For The Game.

According to the evolutionary game dynamics, each player receives points to represent the increments in fitness see [8]. For the randomly alternating PD game, leader decides between the two choices C and D option C yields α points to the donor and β points to the recipient, while option D yields χ points to the donor and δ points to the recipient. In a single round, option D is better than C for the leader. We shall assume that the cost to the donor is less than the benefit that brings to the recipient see [3]. The loss is $\chi - \alpha$ and the benefit is $\beta - \delta$, then we have

$$0 < \chi - \alpha < \beta - \delta \tag{1}$$

Now, we consider two consecutive rounds in which the players exchange the leader role in turn. If both play C, both earn $\alpha + \beta$, which we denote by R, if both play D, both earn $\chi + \delta$, which we denote by P, but if one plays C and the other D, then the cooperator earns $\alpha + \delta$ which we denote by S and the defectors earns $\chi + \beta$ which we denote by T. From (1) follows

$$T > R > P > S \tag{2}$$

and

$$2R > T + S \tag{3}$$

The inequalities (2) and (3) are usual conditions for the payoff for the simultaneous PD game. Adding, we obtain

$$T + S = P + R \tag{4}$$

Conditions (2), (3) and (4) describe the alternating PD game, while condition (4) means in the simultaneous PD that the cost from switching D to C is the same against a defector as against a cooperator ($P - S = T - R$). The four states of outcomes α, β, χ and δ represents the possible payoff obtained by one of the two players for one round. If we denote these outcomes by1, 2, 3 and 4 resp., then the possible strategies for each player can be represented by the quadruple (u_1, u_2, u_3, u_4), where u_i denotes the probability to play C after outcome i. These probabilities are independent of the random decision of who is going to be leader. If a player with strategy $P = (p_1, p_2, p_3, p_4)$ and probability y to play X in the first round match against a player with strategy $Q = (q_1, q_2, q_3, q_4)$ and probability y' to play C in the first round, then the transition matrix for the match from one round to the next is given by

$$M = \frac{1}{2}\begin{pmatrix} p_1 & q_2 & (1-p_1) & (1-q_2) \\ p_2 & q_1 & (1-p_2) & (1-q_1) \\ p_3 & q_4 & (1-p_3) & (1-q_4) \\ p_4 & q_3 & (1-p_4) & (1-q_3) \end{pmatrix} \qquad (5)$$

For instance the transition probability from state 1 to state 2 for the P-player equals the product of the probability that the Q-player is leader, which equals $\frac{1}{2}$ and the probability q_2 that the Q-player cooperates after receiving β in the previous round.

3-The Payoff of a Player in Randomly Alternating PD Game.

If ω denote the probability that the game proceed to the next round, then we have two cases to study, when $\omega = 1$ (limiting case) and $\omega < 1$

1-For $\omega = 1$

The stationary distribution of transition matrix M is the eigen vector Π where

$$\Pi = (\pi_1, \pi_2, \pi_3, \pi_4) \ ; \ \sum_{i=1}^{4} \pi_i = 1,$$

which correspond to the left eigenvalue of M see[12]. Hence Π is given by

$$\Pi M = \Pi \qquad (6)$$

under the condition $\sum_{i=1}^{4} \pi_i = 1$, we get

$$\pi_3 = \frac{1}{2} - \pi_1, \quad \pi_4 = \frac{1}{2} - \pi_2 \qquad (7)$$

Let $\pi_1 = \pi$ and $\pi_2 = \pi'$, then from (7), we get

$$\Pi = (\pi, \pi', \frac{1}{2} - \pi, \frac{1}{2} - \pi')$$

Solving (6) and (7) in π, π' yeild

$$\begin{aligned} \pi &= \frac{(p_3 + p_4)(2 + q_3 - q_1) - (q_3 + q_4)(p_4 - p_2)}{2\ [(2 + p_3 - p_1)(2 + q_3 - q_1) - (p_4 - p_2)(q_4 - q_2)]} \\ \pi' &= \frac{(q_3 + q_4)(2 + p_3 - p_1) - (p_3 + p_4)(q_4 - q_2)}{2\ [(2 + p_3 - p_1)(2 + q_3 - q_1) - (p_4 - p_2)(q_4 - q_2)]} \end{aligned} \qquad (8)$$

Now, if F is the payoff for the P-player, then F is given by

$$F=\Pi\begin{bmatrix}\alpha\\ \beta\\ \chi\\ \delta\end{bmatrix}=\Pi\ \ (\alpha-\chi)+\pi'\ \ (\beta-\delta)+\frac{1}{2}(\chi+\delta)$$

Using (8) we get

$$F=\frac{1}{2}\ (\chi+\delta)+\ \frac{(\alpha-\chi)[(2+\upsilon')-\tau'\mu]+(\beta-\delta)[\tau'(2+\upsilon)-\tau\ \mu]}{2\ [(2+\upsilon)(2+\upsilon')-\mu\ \mu']} \tag{9}$$

where $\upsilon=p_3-p_1$, $\mu=p_4-p_2$, $\tau=p_3+p_4$, and similarly for υ',μ', and τ'. In this case we note that the initial probability y and y' play no role .

2- For $\omega<1$

The total payoff F for the P-player against Q-player in this case is given by

$$F=\frac{1}{2}[y\mathrm{A}+(1-y)\mathrm{X}+y'\mathrm{B}+(1-y')\Delta]$$

$$=\frac{1}{2}[\mathrm{X}+\Delta+y(\mathrm{A}-\mathrm{X})+y'(\mathrm{B}-\Delta)] \tag{10}$$

where A, B, X and $\varDelta$ are the expected payoffs for the P-player, given that the first round of the game resulted in α, β, χ and δ resp..

Then we have

$$A=\alpha+\frac{\omega}{2}[P_1A+q_2B+(1-P_1)X+(1-q_2)\varDelta] \tag{11}$$

where $\frac{1}{2}$ is the probability that P-player will be leader in this round. Equation (11) and the corresponding equations for the other expected payoffs B, X and $\varDelta$ can be written as

$$\mathcal{A}\begin{bmatrix}\mathrm{A}\\ \mathrm{B}\\ \mathrm{X}\\ \Delta\end{bmatrix}=\begin{bmatrix}\alpha\\ \beta\\ \chi\\ \delta\end{bmatrix} \tag{12}$$

where

$$\mathcal{A}=I-\omega M \tag{13}$$

and I is the 4-unit matrix has full rank, so that we can compute the payoff. From (12) we have

$$X + \Delta + P_1(A-X) + q_2(B-\Delta) + \frac{2}{\omega}(\alpha-A) = 0 \tag{14}$$

$$X + \Delta + P_2(A-X) + q_1(B-\Delta) + \frac{2}{\omega}(\beta-B) = 0 \tag{15}$$

$$X + \Delta + P_3(A-X) + q_4(B-\Delta) + \frac{2}{\omega}(\chi-X) = 0 \tag{16}$$

$$X + \Delta + P_4(A-X) + q_3(B-\Delta) + \frac{2}{\omega}(\delta-\Delta) = 0 \tag{17}$$

Solving equations (14)_(17) in $(A-X)$, $(B-\Delta)$ and $(X+\Delta)$, we get that (from (14) and and (16))

$$(2+\omega\upsilon)(A-X) + \omega\upsilon'(B-\Delta) = 2(\alpha-\chi) \tag{18}$$

and from (15) and (17) that

$$\omega\mu(A-X) + (2+\omega\upsilon')(B-\Delta) = 2(\beta-\delta) \tag{19}$$

Then using Cramer's rule for (18) and (19) ,we get

$$A-X = 2\,\frac{(\alpha-\chi)(2+\omega\upsilon') - (\beta-\delta)\omega\mu'}{(2+\omega\upsilon)(2+\omega\upsilon') - \omega^2\mu\mu'} \tag{20}$$

and

$$B-\Delta = 2\,\frac{(\beta-\delta)(2+\omega\upsilon) - (\alpha-\chi)\omega\mu}{(2+\omega\upsilon)(2+\omega\upsilon') - \omega^2\mu\mu'} \tag{21}$$

Adding (16) and (17), we get that

$$X + \Delta = \frac{1}{1-\omega}\left[(\chi+\delta) + \frac{\omega\tau}{2}(A-X) + \frac{\omega\tau'}{2}(B-\Delta)\right] \tag{22}$$

For the two cases of ω and, the previous computation, we get the following result. The total payoff F for an S-player is given by (10) for $\omega < 1$, where $X+\Delta$, $A-X$ and $B-\Delta$ are as in (22), (20) and (21). And for $\omega = 1$, F is given by (9).

4-The Adaptive Dynamics For The Alternating PD Game.

We consider a homogeneous population of S-players, and ask for the most favorable adaptation. If an individual was permitted a small deviation from strategy S, which direction would be most favorable (In a biological context, the small deviation would be produced by a mutation and natural selection). Now if x is any parameter belonging to S, where x can be y, p_1, p_2,..., or p_4, then x changes according to the adaptive dynamics, by

$$\dot{x} = \frac{\partial F}{\partial x} \tag{23}$$

where the right side of (23) evaluated at $S=S'$. The differentiation of F yields very cumbersome computation, so we shall use the implicit function theorem. Equations (12) can be written as

$$f_i(p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4, \mathrm{A}, \mathrm{B}, \mathrm{X}, \Delta) = 0, \qquad i = 1,2,3,4$$

for instance

$$f_1 = (1-\frac{\omega}{2}p_1)\mathrm{A} - \frac{\omega}{2}q_2\mathrm{B} - \frac{\omega}{2}(1-p_1)\mathrm{X} - \frac{\omega}{2}(1-q_2)\Delta$$

and see that

$$A = \frac{\partial(f_1, f_2, f_3, f_4)}{\partial(\mathrm{A}, \mathrm{B}, \mathrm{X}, \Delta)}$$

Hence, by the implicit function theorem, we have

$$\frac{\partial(\mathrm{A}, \mathrm{B}, \mathrm{X}, \Delta)}{\partial(p_1, p_2, p_3, p_4)} = -\left[\frac{\partial(f_1, f_2, f_3, f_4)}{\partial(\mathrm{A}, \mathrm{B}, \mathrm{X}, \Delta)}\right]^{-1} \frac{\partial(f_1, f_2, f_3, f_4)}{\partial(p_1, p_2, p_3, p_4)} = \frac{\omega}{2}(\mathrm{A}-\mathrm{X})E \tag{24}$$

where

$$E = A^{-1} \tag{25}$$

and from (13), we have

$$E = (I-\omega M)^{-1} = I + \omega M + (\omega M)^2 + \ldots \tag{26}$$

All elements e_{ij} of E are strictly positive. The formulae of these expressions (26) are rather cumbersome and for futur use.

We note that

$$\det A = \frac{1-\omega}{\mu}(2+\omega(\upsilon+\mu))(2+\omega(\upsilon-\mu)) \tag{27}$$

and as one can easily deduce from the special form of A. So by straight forward computation we can write E in the form

$$E = \begin{pmatrix} a & b & \chi & d \\ b & a & d & \chi \\ \varepsilon & f & g & h \\ f & e & h & g \end{pmatrix} \tag{28}$$

Using (25), we get

$$a+b=\frac{\omega}{4\det A}(2+\omega(\upsilon-\mu))(\frac{2}{\omega}-2+p_3+p_4), \tag{29}$$

$$c+d=\frac{\omega}{4\det A}(2+\omega(\upsilon-\mu))(2-p_1-p_2), \tag{30}$$

$$e+f=\frac{\omega}{4\det A}(2+\omega(\upsilon-\mu))(p_3+p_4), \tag{31}$$

$$g+h=\frac{\omega}{4\det A}(2+\omega(\upsilon-\mu))(\frac{2}{\omega}-p_1-p_2) \tag{32}$$

and hence

$$(a+b)-(c+d)=\frac{\omega}{4\det A}(2+\omega(\upsilon-\mu))(\frac{2}{\omega}-4+p_1+p_2+p_3+p_4), \tag{33}$$

$$(e+f)-(g+h)=\frac{\omega}{4\det A}(2+\omega(\upsilon-\mu))(-\frac{2}{\omega}+p_1+p_2+p_3+p_4), \tag{34}$$

Now from (10) and (23), we get the following

$$\dot{y}=\frac{1}{2}(\mathrm{A}-\mathrm{X}), \tag{35}$$

$$\begin{aligned}\dot{p}_1&=\frac{1}{2}[y(\frac{\partial \mathrm{A}}{\partial p_1}+\frac{\partial \mathrm{B}}{\partial p_1})+(1-y)(\frac{\partial \mathrm{X}}{\partial p_1}+\frac{\partial \Delta}{\partial p_1})]\\&=\frac{\omega}{4}[y(a+b)+(1-y)(c+d)](\mathrm{A}-\mathrm{X})\end{aligned} \tag{36}$$

$$\begin{aligned}\dot{p}_3&=\frac{1}{2}[y(\frac{\partial \mathrm{A}}{\partial p_3}+\frac{\partial \mathrm{B}}{\partial p_3})+(1-y)(\frac{\partial C}{\partial p_3}+\frac{\partial \Delta}{\partial p_3})]\\&=\frac{\omega}{4}[y(e+f)+(1-y)(g+h)](\mathrm{A}-C)\end{aligned} \tag{37}$$

We note that the sign of $\dot{p}_1$ and $\dot{p}_3$ is the same as the sign of $\mathrm{A}-C$ which has been computed in (20). Using the previous expressions together with the equations (29)__(32), we get the following results.

Theorem. The adaptive dynamics for the randomly alternating PD game is given by

(1) For $\omega<1$

$$\dot{y}=\frac{1}{2}(\mathrm{A}-\mathrm{X})$$

$$\dot{p}_1=\dot{p}_2=\frac{\omega^2}{4(1-\omega)}(2+\omega(\upsilon+\mu))^{-1}[2-p_1-p_2+y(\frac{2}{\omega}-4+p_1+p_2+p_3+p_4)](\mathrm{A}-\mathrm{X}) \tag{38}$$

$$\dot{p}_3 = \dot{p}_4 = \frac{\omega^2}{4(1-\omega)}(2+\omega(\upsilon+\mu))^{-1}\left[2 - p_1 - p_2 + y\left(\frac{2}{\omega} + p_1 + p_2 + p_3 + p_4\right)\right](A-X) \quad (39)$$

where

$$A - X = 2\frac{(\alpha-\chi)(2+\omega\upsilon)-(\beta-\delta)\omega\mu}{(2+\omega(\upsilon-\mu))(2+\omega(\upsilon+\mu))} \quad (40)$$

(2) For $\omega = 1$ (limiting case)

$$\dot{p}_1 = \dot{p}_2 = \frac{\tau[(\alpha-\chi)(2+\upsilon)-(\beta-\delta)\mu]}{2(2+\upsilon+\mu)^2(2+\upsilon-\mu)}, \quad (41)$$

$$\dot{p}_3 = \dot{p}_4 = \frac{(2+\upsilon+\mu-\tau)[(\alpha-\chi)(2+\upsilon)-(\beta-\delta)\mu]}{2(2+\upsilon+\mu)^2(2+\upsilon-\mu)} \quad (42)$$

For every case we note that

(1) $\dot{p}_1 = \dot{p}_2$ and $\dot{p}_3 = \dot{p}_4$ and this means that the optimal adaptation depends only on whether there was a C or a D in the previous round, and not on who actually implemented it.

(2) $\dot{y}$ and all $\dot{p}_i$ has positive sign and this gives the zone of cooperation, which is defined by

$$(\alpha-\chi)(2+\omega\upsilon) > (\beta-\delta)\omega\mu \quad (43)$$

and it is independent of y.

If $\alpha = \dfrac{\beta-\delta}{\chi-\alpha}$, which is just $\dfrac{T-P}{T-R}$, and hence greater than one *i.e* $\alpha > 1$, we get from (43)(after substituting with υ and μ) that the zone of cooperativity is given by

$$\frac{2}{\omega} < p_1 - p_3 + \alpha(p_2 - p_4) \quad (44)$$

Condition (44) implies to the following.

The zone of cooperativity is non empty if and only if it contains the Tit For Tat strategy, which is given by $(p_1, p_2, p_3, p_4) = (1,1,0,0)$, i.e if and only if the condition

$$\alpha > \frac{2-\omega}{\omega} \quad (45)$$

holds. This condition agree with $\alpha > 1$.

In the limiting case $\omega = 1$, we see that condition (45) is always satisfied.

Cooperation is easier to achieve the larger of the zone, i.e the temptation *T-R* to defect unilaterally is smaller compared with the gain *R-P* obtained by mutual cooperation.
In particular, if

$$\alpha > \frac{2}{\omega} \tag{46}$$

Then from (44), the zone of cooperation contains the strategy given by $(p_1,p_2,p_3,p_4) = (1,1,1,0)$, which is always ready to cooperate except if it has been played for a suker,i.e if it has experienced a δ in the last round. In this case, it defects if it is leader in the next round, but it defects only once.We note that in the limiting case $\omega = 1$, condition (46) simply means that the cost to the donor $\chi-\alpha$ is twice as large as the benefit to the recipient $\beta-\delta$.
It is easy to find the consensus strategy with highest payoff which immune to defection. If all members of the population adopt this strategy, then the exploiters with a lower propensity to cooperate cannot invade, and the overall payoff for the population is maximal (subject to this non-invadability condition).This payoff is given by

$$F(s,s) = \frac{(\alpha-1)(\chi-\alpha)}{2(2+\omega(\upsilon+\mu))}(2y+\frac{\omega\tau}{(1-\omega)})+\frac{\chi+\delta}{2(1-\omega)}.$$

REFRENCES.

1- Axelrod, R. , The Evolution Of Cooperation, Basic Books, New york, 1988.

2- El Sedy, E. and El Shobaky, E. ,The perturbed payoff matrix of the repeated Prisoner's Dilemma game , Far East J. Appl. Math. 1(3) (1997), 255-270.

3-El Sedy, E. and Kanaya, Z., The perturbed payoff matrix of strictly alternating Prisoner's Dilemma game, Far East J. Appl. Math. 2(2) (1998), 125-146.

4- El Sedy, E. and Kanaya, Z., On Simultaneous repeated 2 □~~2~~games, Revista de la Unión Matemática Argentina vol.41, 3, 1999.

5- Hofbauer, J. and Sigmund, K., Dynamical systems and the Theory Of Evolution, Cambridge University Press 1988.

6- Hofbauer, J. and Sigmund, K., Adaptive dynamics and evolutionary stability of Appl. Math. lett. vol. 3, No 4, pp.75-79, 1990.

7-Lindgrn, K., Evolutionary phenomena in simple dynamics, in Artificial lif II(ed.c.G. Longtonet.al), Santa Fe Institute for Studies in the Sciences of Complexity X (1991)

295-312.

8-Maynard Smith,J., Evolution and the Theory of Games, Cambridge University Press 1982.

9-Nowak, M. and Sigmund, K.,The evolution of stochastic strategies in the Prisoner's Dilemma, Acta App. Math.20:247-265,1990.

10-Nowak, M. and Sigmund, K., Choas and the evolution of cooperation, Proc. Nat. Acad. Science USA 90 (1993a), 5091-5094.

11-Nowak, M. and Sigmund, K., The alternating Prisoner's Dilemma, J. Theor.Biol. 168 (1994), 219-226 .

12-Nowak, M. and Sigmund, K. and El Sedy, E., Automata repeated games and noise, J. Math. Biol. 33(1995), 703-72

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