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MODULAR INEQUALITIES OF MAXIMAL OPERATORS IN ORLICZ SPACES

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Abstract

Given $p \geq 1$, we study modular inequalities for the operators $\mathcal{M}_p$ and $M^{-}_{1/p}$, related to p-averages and Cesàro means of order $1/p$, in the context of Orlicz Spaces establishing a comparison between their boundedness properties. We also analyze their behavior on weighted Orlicz spaces for weights in the class A_1 and A_1^{-}, respectively. We find out that, in both cases, conditions on the growth functions to have a modular inequalities, render unchangeable. Also, a converse inequality for $\mathcal{M}_p$ is given.

1 INTRODUCTION

Let (Ω, μ) be a finite measure space and $\mathfrak{M}(\Omega)$ be the space of measurable functions from Ω into $\overline{\mathbb{R}}$. Let $\Psi : [0, \infty] \mapsto [0, \infty]$ an increasing function such that $\Psi(0) = 0$. The set of functions

$$L^{\Psi}(\Omega) = \{f \in \mathfrak{M}(\Omega) : \int_{\Omega} \Psi(\epsilon\, |f|)\, d\mu < \infty \text{ for some } \epsilon > 0 \ \}$$

is called an *Orlicz space* associated to Ψ. We may write L^{Ψ} when the set Ω is known. If Ψ is convex we can define a norm on L^{Ψ} called the *Luxemburg norm* given by (see [7])

$$||f||_{\Psi} = \inf \left\{ s > 0 : \int_{\Omega} \Psi\left(\frac{|f|}{s}\right) d\mu \leq 1 \right\}.$$

Let T be a sublinear and positive homogeneous operator defined on a subspace $\mathfrak{D} \subset \mathfrak{M}(\Omega)$ and taking values on $\mathfrak{M}(\Omega)$. We assume that $\mathfrak{D}$ contains all the characteristic functions of sets of finite measure and has the property that whenever $f \in \mathfrak{D}$ and g is a truncation of f, then $g \in \mathfrak{D}$.
A such operator T is of *weak type* (p,p) if there exists a constant A such that for any measurable function $f \in \mathfrak{D}$,

$$\mu(\{Tf > s\}) \leq \left(\frac{A}{s}\, ||f||_p\right)^p \qquad \text{for all } s > 0.$$

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T is of *restricted weak type* (p,p) if there exists a constant A such that for any measurable function $f \in \mathfrak{D}$,

$$\mu(\{Tf > s\}) \leq \left(\frac{A}{s}\,\|f\|_{p,1}\right)^p \qquad \text{for all } s > 0,$$

with $\|f\|_{p,1} = \int_0^\infty \mu(\{f > r\})^{1/p}\,dr$, the seminorm in the Lorentz Space $L^{p,1}(\Omega)$ (for more details, see [8]).
Finally, T is of *type* (∞,∞) if there exists a constant B such that for any measurable function $f \in \mathfrak{D}$,

$$\|Tf\|_\infty \leq B\|f\|_\infty.$$

In the sequel we will work with functions Φ and Ψ given by

$$\Phi(t) = \int_0^t a(s)\,ds \qquad \text{and} \qquad \Psi(t) = \int_0^t b(s)\,ds$$

for all $t \geq 0$, where a and b be positive continuous functions defined on $[0,\infty)$.
We are interested in the study of two kinds of maximal operators. The first is related with the p-averages of a function. For $p \geq 1$, we define the $\mathcal{M}_p$ operator given by

$$\mathcal{M}_p f(x) = \sup_{I\in\mathcal{I},x\in I} \left(\frac{1}{|I|}\int_I |f|^p\right)^{1/p},$$

with $\mathcal{I}$ the family of all intervals contained in $[0,1]$. It is well known that $\mathcal{M}_p$ is of weak type (p,p).
The second is related with the Cesàro means and has two lateral forms. Let $0 < \alpha \leq 1$,

$$M_\alpha^+ f(x) = \sup_{x<c<1} \frac{\alpha}{(c-x)^\alpha} \int_x^c |f(s)|\,(c-s)^{\alpha-1}\,ds\,, \text{ for } x \in [0,1],$$

and

$$M_\alpha^- f(x) = \sup_{0<c<x} \frac{\alpha}{(x-c)^\alpha} \int_c^x |f(s)|\,(s-c)^{\alpha-1}\,ds\,, \text{ for } x \in [0,1].$$

It is also known that these operators are not of weak type $(1/\alpha, 1/\alpha)$ but of restricted weak type $(1/\alpha,1/\alpha)$ (see [2]).
In [1] the authors found conditions for the boundedness of these operators in terms of modular inequalities as follows.
We say that an operator T is (Ψ,Φ)*-bounded* on (Ω,μ) if there exists a constant C such that

$$\int_\Omega \Phi(|Tf|)\,d\mu \leq C + C\int_\Omega \Psi(C\,|f|)\,d\mu\,, \tag{1}$$

for all $f \in \mathfrak{D}$.

Theorem 1. *Let T be of weak type (p,p) with $p \geq 1$, and of type (∞,∞). If for some constant C, a and b satisfy*

$$t^{p-1}\int_1^t \frac{a(s)}{s^p}\,ds \leq C\,b(C\,t)\,, \qquad \textit{for all } t \geq 1, \tag{2}$$

then, T is (Ψ,Φ)-bounded on (Ω,μ).

Theorem 2. *Let $p \geq 1$ and b monotone on $[1, \infty)$. The operator $\mathcal{M}_p$ is (Ψ, Φ)-bounded on $([0,1], dx)$, i.e., there exists a constant C' such that*

$$\int_0^1 \Phi(|\mathcal{M}_p f(x)|)\, dx \leq C' + C' \int_0^1 \Psi(C'\, |f(x)|)\, dx\,, \tag{3}$$

for all $f \in \mathfrak{M}([0,1])$ if, and only if, (2) holds.

Theorem 3. *Let T be of restricted weak type (p,p) with $p > 1$, and of type (∞, ∞). If for some constant C, a and b satisfy*

$$\sup_{t \geq 1} \left(\int_1^t \frac{a(s)}{s^p}\, ds \right)^{1/p} \left(\int_t^\infty b(C\, s)^{-p'/p}\, ds \right)^{1/p'} < \infty \tag{4}$$

then, T is (Ψ, Φ)-bounded on (Ω, μ).

Theorem 4. *Let $0 < \alpha < 1$ and b monotone on $[1, \infty)$. The operator M_α^- is (Ψ, Φ)-bounded on $([0,1], dx)$, i.e., there exists a constant C' such that*

$$\int_0^1 \Phi(|M_\alpha^- f(x)|)\, dx \leq C' + C' \int_0^1 \Psi(C'\, |f(x)|)\, dx\,, \tag{5}$$

for all $f \in \mathfrak{M}([0,1])$ if, and only if, condition (4) holds with $p = 1/\alpha$.
We have the same result for M_α^+.

Theorems 2 and 4 are useful for studying the mapping behavior of $\mathcal{M}_p$ and M_α^- obtaining more information than that derived from the Marcinkiewicz interpolation theorem. From these results some questions arise naturally:

(a) For $q > p$, it is known that $\mathcal{M}_p$ maps L^q into L^q; then, for which b does $\mathcal{M}_p$ map L^Ψ into itself?

(b) For which b does $M_{1/p}^+$ map L^Ψ into itself?

(c) Since conditions (2) and (4) are not the same, for which b does $\mathcal{M}_p$ and $M_{1/p}^+$ maps L^Ψ into the same L^Φ?

(d) What happens if we consider weighted Orlicz spaces?

(e) If we have a function f that $\mathcal{M}_p f$ or $M_{1/p}^+ f$ belongs to some L^Φ, what can we say about f?

In Section 2 we answer questions (a), (b) and (c), establishing a comparison between the two kinds of operators and their common properties. In Section 3 we generalize theorems 1 to 4 dealing with question (d). Section 4 is devoted to the converse inequalities and we find some answers to question (e).

2 COMPARISON OF MAPPING BEHAVIOR

Let $p > 1$ and consider the operators $\mathcal{M}_p$ and $M^-_{1/p}$. Using the Marcinkiewicz Interpolation Theorem we assert that both $M^-_{1/p}$ and $\mathcal{M}_p$ are bounded from L^q into L^q for $q > p$ and therefore, we have (Ψ,Ψ)-boundedness with $\Psi(t) = t^q$. Hence, we may expect (Ψ,Ψ)-boundedness for Ψ "greater" than t^q, for $q > p$.
According to theorems 2 and 4 we can study the (Ψ,Φ)-boudedness checking condition (2) for $\mathcal{M}_p$ and condition (4) for $M^+_{1/p}$. Comparing both conditions we find that (Ψ,Ψ)-boundedness is no longer true when we deal with domains "close" to L^p. For example, if we take $\Psi(t) = [t\log(t)]^p$ then, we do not have (Ψ,Ψ)-boundedness for any of the operators.
Also, $\mathcal{M}_p$ and $M^+_{1/p}$ do not have the same behavior near L^p, for example, if $\Psi(t) = t^p\log(t)$ and $\Phi(t) = t^p$, the operator $\mathcal{M}_p$ is (Ψ,Φ)-bounded, but $M^+_{1/p}$ is not.
Now, we present some known facts about real functions (see [4], p.6, and [6], p.131).

Lemma 1. *Let b be a non-negative and non-increasing function defined on $[0,\infty)$. The following statments are equivalent:*

(i) There exists a constant C such that

$$\int_t^\infty b^{-p'/p} \le C\, t b^{-p'/p}(t) \tag{6}$$

for all $t \ge 1$.

(ii) There exists a constant C such that

$$\int_1^t \frac{b(s)}{s^p}\, ds \le C\, t^{1-p}\, b(t) \tag{7}$$

for all $t \ge 1$.

(iii) There exists constants C and $\gamma > 1$ such that

$$b^{-p'/p}(s\,t) \le C\, s^{-\gamma} b^{-p'/p}(t)$$

for all $s \ge 1$ and $t \ge 1$.

(iv) There exists constants C and $\eta > p-1$ such that

$$b(s\,t) \le C\, s^{\eta} b(t) \tag{8}$$

for all $0 \le s \le 1$ and $st \ge 1$.

We recall that a function satisfying inequality (8) is said to be of *lower type* η at infinity.
The following corollaries 1 and 2 give answers to questions (a), (b). They are direct consequence of Theorem 2, Theorem 4 and Lemma 1.

Corollary 1. *The operator $\mathcal{M}_p$ is (Ψ,Ψ)-bounded if, and only if, Ψ has a lower type greater than p.*

Corollary 2. *The operator $M^+_{1/p}$ is (Ψ,Ψ)-bounded if, and only if, Ψ has a lower type greater than p.*

In order to answer question (c), we state the following corollary.

Corollary 3. *Given Ψ, the following statements are equivalent:*

(i) For all Φ, $\mathcal{M}_p$ is (Ψ,Φ)-bounded if, and only if, $M^+_{1/p}$ is (Ψ,Φ)-bounded.

(ii) Ψ has a lower type greater than p.

Proof. To prove that (i) implies (ii), let Ψ be fixed and suppose that for all Φ, if $\mathcal{M}_p$ is (Ψ,Φ)-bounded, then $M^+_{1/p}$ is (Ψ,Φ)-bounded. We may suppose that b' exists (if b is not differentiable we can always find an equivalent function having that property) and that $\dfrac{b(s)}{s^p}$ is increasing (otherwise $\mathcal{M}_p$ can not be (Ψ,Φ)-bounded for any Φ, see [1], p.7). Set $a(t)=tb'(t)$. Due to $b'\geq 0$, we have $a\geq 0$, and then a and b satisfy

$$t^{1-p}\,b(t)\leq\int_1^t\frac{a(s)}{s^p}\,ds\leq p\,t^{1-p}\,b(t). \tag{9}$$

By Theorem 2 we have that $\mathcal{M}_p$ is (Ψ,Φ)-bounded and by the hypothesis, $M^+_{1/p}$ is (Ψ,Φ)-bounded and Theorem 4 implies (4). From (9) and (4) we obtain

$$t^{1-p}\,b(t)\leq\left(\int_t^\infty b(C\,s)^{-p'/p}\,ds\right)^{-p/p'} \tag{10}$$

and this is (6). Finally, Lemma 1 implies that Ψ has a lower type greater than p. On the other hand, if we assume that Ψ has a lower type greater than p, by Lemma 1 we have (6) and therefore, inequality (2) implies (4) and then, the (Ψ,Φ)-boundedness of $\mathcal{M}_p$ implies that of $M^+_{1/p}$. Because inequality (4) is stronger than inequality (2), using Theorems 2 and 4, the (Ψ,Φ)-boundedness of $M^+_{1/p}$ always implies the (Ψ,Φ)-boundedness of $\mathcal{M}_p$. □

Proof of Lemma 1. The equivalence between (iii) and (iv) is trivial. To see that (iii) implies (i), let $t\geq 1$,

$$\begin{aligned}\int_t^\infty b^{-p'/p}(s)\,ds&=t\int_1^\infty b^{-p'/p}(tr)\,dr\\&\leq C\,tb^{-p'/p}(t)\int_1^\infty r^{-\gamma}\,dr\\&=\frac{C}{1+\gamma}tb^{-p'/p}(t)\end{aligned} \tag{11}$$

Now we prove that (i) implies (iii). Let $t\geq 1$ and $s\geq 1$. If we call $h=b^{-p'/p}$, we have by (i)

$$\frac{h(r)}{\int_r^\infty h}\geq\frac{1}{C\,r}, \tag{12}$$

for all $r > 1$. We first suppose $s > 2$. Integrating between t and $st/2$,

$$\log\left(\frac{\int_t^\infty h}{\int_{st/2}^\infty h}\right) \geq \frac{\log(s/2)}{C} \tag{13}$$

and exponentiating, we obtain

$$\int_t^\infty h \geq (s/2)^{1/C} \int_{st/2}^\infty h. \tag{14}$$

Using h non-increasing and inequalities (14) and (6),

$$\begin{aligned} ts^{1+1/C}\, h(s\,t) &\leq 2s^{1/C} \int_{st/2}^\infty h \\ &\leq 2^{1+1/C} \int_t^\infty h \\ &\leq 2^{1+1/C} C\, t\, h(t). \end{aligned} \tag{15}$$

Then, for $s > 2$,

$$h(s\,t) \leq 2^{1+1/C} C\, s^{-(1+1/C)}\, h(t). \tag{16}$$

If $1 \leq s < 2$, since h is non-increasing,

$$h(s\,t) \leq h(t) \leq 2^{1+1/C} s^{-(1+1/C)}\, h(t) \tag{17}$$

In a similar way, we obtain the equivalence between (ii) and (iv), then, the proof is finished. □

3 WEIGHTED INEQUALITIES

A measurable and nonnegative function $w : \Omega \mapsto \mathbb{R}$ is called a *weight* on Ω. Given a weight w on Ω and Ψ as above, we introduce the following generalization of Orlicz spaces. The set

$$L^\Psi(\Omega, w) = \{f \in \mathfrak{M}(\Omega) : \int_\Omega \Psi(\epsilon\,|f|)\, w\, d\mu < \infty \text{ for some } \epsilon > 0\ \}$$

will be called a *Weighted Orlicz Space*.
A weight w defined on the $[0,1]$ interval with the Lebesgue measure, is said to be in $A_1([0,1])$ if there exists a constant C such that for every interval $I \subset [0,1]$ we have

$$\frac{1}{|I|}\int_I w \leq C \inf_I w$$

It is well known that $A_1([0,1])$ are the weights which characterized the weak type $(1,1)$ of M, the Hardy-Littlewood maximal function on $[0,1]$, and since $\mathcal{M}_p f = (Mf^p)^{1/p}$, we see that $A_1([0,1])$ also characterized the weak type (p,p) of $\mathcal{M}_p$. The following theorem states for which a and b the operator $\mathcal{M}_p$ is (Ψ, Φ)-bounded on $([0,1], w)$.

Theorem 5. *Let w be a weight in $A_1([0,1])$. There exists a constant C' such that*

$$\int_0^1 \Phi(|\mathcal{M}_p f(x)|)\, w(x)\, dx \le C' + C' \int_0^1 \Psi(C'\,|f(x)|)\, w(x)\, dx \tag{18}$$

for all $f \in \mathfrak{M}([0,1])$ if, and only if, (2) holds.

Proof. Suppose that condition (2) holds. Since w is in $A_1([0,1])$, $\mathcal{M}_p$ is of weak type (p,p) and then, (18) follows since we are in a particular case of Theorem 1. We will now see that (2) is a consequence of (18). We use the notation $w(E) = \int_E w$ for any measurable set E. Without loss of generality we may suppose that $w([0,1]) = 1$ and that 0 is a Lebesgue point of w with $w(0) > 0$. Let $t \ge 1$ be fixed. Let $y_t \in [0,1]$ such that $w([0,y_t)) = \frac{1}{t^p}$. Let $f_t = t\chi_{[0,x_t)}$, with $x_t = \max\{y_t, \frac{1}{t^p}\}$. Since

$$w(\{f_t > s\}) = \begin{cases} 0 & \text{if } s \ge t \\ w([0,x_t)) & \text{if } 0 < s < t \end{cases}$$

we have

$$\int_0^1 \Psi(|f(x)|) w(x)\, dx = \int_0^\infty b(s) w(\{f_t > s\})\, ds \le w([0,x_t))\, t\, b(t).$$

If $x_t = y_t$,

$$w([0,x_t)) = w([0,y_t)) = \frac{1}{t^p}$$

and in the case $x_t = \frac{1}{t^p}$, we use that w is in $A_1([0,1])$ to see that

$$w([0,x_t)) \le \frac{1}{t^p} w([0,\frac{1}{t^p}])\, t^p \le \frac{1}{t^p} \inf\{0 \le x \le 1/t^p :\ w(x)\} \le \frac{w(0)}{t^p}.$$

On the other hand, since

$$\mathcal{M}_p f_t(x) = \begin{cases} t & \text{if } x \in [0,x_t] \\ \dfrac{t x_t^{1/p}}{x^{1/p}} & \text{if } x \in (x_t, 1], \end{cases}$$

the distribution of $\mathcal{M}_p f_t$ with respect to w is given by

$$w(\{\mathcal{M}_p f_t > s\}) = \begin{cases} 0 & \text{if } t \le s \\ w([0, \frac{t^p x_t}{s^p}]) & \text{if } t x_t^{1/p} < s < t \\ 1 & \text{if } 0 < s < t x_t^{1/p}. \end{cases}$$

and then,

$$\begin{aligned} \int_0^1 \Psi(\mathcal{M}_p f_t(x)) w(x)\, dx &= \int_0^\infty a(s) w(\{\mathcal{M}_p f_t > s\})\, ds \\ &\ge \int_1^t a(s) w([0, \frac{t^p x_t}{s^p}])\, ds \\ &\ge \int_1^t a(s) w([0, \frac{1}{s^p}])\, ds \\ &\ge \inf\left\{0 \le x \le 1 :\ \frac{w([0,x])}{x}\right\} \int_1^t \frac{a(s)}{s^p}\, ds. \end{aligned}$$

As a consequence of the fact that 0 is a Lebesgue point of w and $w(0) > 0$, there exists a number $\delta > 0$ small enough such that for all $x \in [0, \delta)$, we have $\frac{w([0,x])}{x} > w(0)/2$. Therefore, $\inf\left\{\frac{w([0,x])}{x} : 0 \le x \le 1\right\} \ge \inf\{w(0)/2, w([0,\delta])\} > 0$ and this completes the proof. □

In [5] the authors characterized the weights for the restricted weak type $(\frac{1}{\alpha}, \frac{1}{\alpha})$ of the operators M_α^- and M_α^+. For M_α^- this class of weights is the $A_1^-([0,1])$ defined by the set of weights w such that

$$\frac{1}{b-a}\int_a^b w \le C\, w(a) \qquad \forall 0 \le a < b \le 1. \tag{19}$$

For the operator M_α^+, the class $A_1^+([0,1])$ is defined similarly (see [5]).

Theorem 6. *Let w be a weight in $A_1^-([0,1])$, then for some constant C'*

$$\int_0^1 \Phi(M_\alpha^- f(x))\, w(x)\, dx \le C' + C' \int_0^1 \Psi(C'\,|f(x)|)\, w(x)\, dx \tag{20}$$

for all $f \in \mathfrak{M}([0,1])$ if, and only if, condition (4) *holds.*

Proof. Since w is in $A_1^-([0,1])$, the operator M_α^- is simultaneously of restricted weak type $(1/\alpha, 1/\alpha)$ and of type (∞, ∞) (see [5]), from Theorem 3 we have that (4) implies (20).
For the converse, assume that (20) holds. Suppose that 0 is a Lebesgue point of w and that $w(0) > 0$. Due to inequality (19), if for some x, $w(x) = 0$, then $w(y) = 0$ for all $y > x$. Then, we may assume $w(x) > 0$ almost everywhere. Let $g : [0,1] \mapsto [0,1]$ defined as $g(x) = w([0,x])$. Since $w(x) > 0$ a.e., we have that g is strictly increasing and so g^{-1} is well defined.
Also, from inequality (19), we have

$$g(x) = w([0,x]) \le w(0)x \tag{21}$$

and

$$g^{-1}(x) \ge \frac{x}{w(0)}. \tag{22}$$

We first assume that b has the property

$$\int_1^\infty b(s)^{-p'/p}\, ds < \infty. \tag{23}$$

Let $t \ge 1$ be fixed. For $s > 0$, let

$$h_t(s) = A_t\, b(C\,s)^{-p'}$$

with

$$A_t = w(0)\left[t\, b(C\,t)^{-p'/p} + \int_t^\infty b(C\,s)^{-p'/p}\, ds\right]^{-p}$$

and $C > (C')^2$ such that $\int_1^\infty b(C\,s)^{-p'/p} < (C')^{-p'/p}$. Observe that since b is increasing, $\lim_{s\to\infty} b(s) = \infty$, h_t is decreasing and $\lim_{s\to\infty} h_t(s) = 0$, then $h_t^{-1}(r)$ is well defined for $r > 0$.
Now consider $f_t \in \mathfrak{M}([0,1])$ defined by

$$f_t(x) = h_t^{-1}(g(x))\chi_{(0,y_t)}(x),$$

with $y_t = \min\{g^{-1}(h_t(t)), 1\}$.
The distribution function of f_t is for $s > 0$

$$\begin{aligned} w(\{f_t > s\}) &= w(\{x \in (0,1] : \ f_t(x) > s\}) \\ &= w(\{x \in (0,1] : \ h_t^{-1}(g(x)) > s \text{ and } x < y_t\}) \\ &= w(\{x \in (0,1] : \ g(x) < h_t(s) \text{ and } x < y_t\}) \\ &= \min\{h_t(s), h_t(t), 1\}. \end{aligned}$$

From the last equation and the fact that b is increasing we get

$$\begin{aligned} C' \int_0^1 \Psi(C'\,|f_t(x)|)w(x)\,dx &= C'^2 \int_0^\infty b(C'\,s)w(\{f_t > s\})\,ds \\ &\le C\left[h_t(t)\int_0^t b(C\,s)\,ds + \int_t^\infty b(C\,s)h_t(s)\,ds\right] \\ &\le C\left[t\,b(C\,t)h_t(t) + \int_t^\infty b(C\,s)h_t(s)\,ds\right] \\ &= CA_t\left[t\,b(C\,t)^{-p'/p} + \int_t^\infty b(C\,s)^{-'p/p}\,ds\right] \\ &\le C\,w(0)\left[\int_t^\infty b(C\,s)^{-'p/p}\,ds\right]^{-p/p'}. \end{aligned}$$

Then, by the choice of C,

$$C' + C' \int_0^1 \Psi(C'\,|f_t(x)|)w(x)\,dx \le (1 + w(0))\,C\left[\int_t^\infty b(C\,r)^{-p'/p}\,dr\right]^{-p/p'}. \tag{24}$$

On the other hand, we will see that

$$w(\{M_\alpha^- f_t > s\}) \ge \frac{c_0}{s^p} \quad \text{for all } s \in (1,t), \tag{25}$$

for some c_0 depending on w.
Therefore,

$$\begin{aligned} \int_0^1 \Phi(|M_\alpha^- f_t(x)|)w(x)\,dx &= \int_0^\infty a(s)w(\{M_\alpha^- f_t > s\})\,ds \\ &\ge c_0 \int_1^t \frac{a(s)}{s^p}\,ds. \end{aligned} \tag{26}$$

Then, from (26) and (24), we have

$$\begin{aligned} c_0 \int_1^t \frac{a(s)}{s^p}\,ds &\le \int_0^1 \Phi(|M_\alpha^- f_t(x)|)w(x)\,dx \\ &\le C' + C' \int_0^1 \Psi(C'\,|f_t(x)|)w(x)\,dx \\ &\le (1+w(0))\,C \left[\int_t^\infty b(C\,r)^{-p'/p}\,dr\right]^{-p/p'}. \end{aligned}$$

Since C and c_0 do not depend on t, we get (4).

It remains to prove (25). For this purpose we introduce the $\mathcal{H}_p$ operator with $p \ge 1$ defined for $f \in \mathfrak{M}([0,1])$ by

$$\mathcal{H}_p f(x) = \frac{1}{p\,x^{1/p}} \int_0^x |f(s)|\,s^{1/p-1}\,ds \qquad \text{for } x \in [0,1].$$

Since $\mathcal{H}_p \le M_\alpha^-$ point wise, we may prove equation (25) for $\mathcal{H}_p$ instead of M_α^-. Due to $\lim_{s\to\infty} h_t(s) = 0$, we have $\lim_{x\to 0} f_t(x) = \infty$ and hence $\lim_{x\to 0}\mathcal{H}_p f_t(x) = \infty$ (since for any decreasing function h, $\mathcal{H}_p h \ge h$). Also, $\mathcal{H}_p f_t$ is continuous and decreasing on $(0,1]$. Consequently, the image of $\mathcal{H}_p f_t$ is the interval $[\mathcal{H}_p f_t(1), \infty)$. For $\mathcal{H}_p f_t(1) < s < t$, we have

$$w(\{x : \ \mathcal{H}_p f_t(x) > s\}) = w([0, x_s)).$$

with $0 < x_s \le 1$ and such that $s = \mathcal{H}_p f_t(x_s) = \dfrac{1}{p\,x_s^{1/p}} \displaystyle\int_0^{x_s} f_t(x) x^{1/p-1}\,dx$. Then, $x_s = \left[\dfrac{1}{p\,s}\displaystyle\int_0^{x_s} f_t(x) x^{1/p-1}\,dx\right]^p$.

Since $\mathcal{H}_p f_t \ge f_t$, f_t is decreasing, $t > s$ and inequality (22), we have $x_s \ge f_t^{-1}(t) = g^{-1}(h_t(t)) \ge \frac{1}{w(0)} h_t(t)$, and due to inequality (21) and the fact that h_t^{-1} is decreasing, we have $f_t(x) \ge h_t^{-1}(g(x)) \ge h_t^{-1}(w(0)x)$. Then,

$$\begin{aligned} \int_0^{x_s} f_t(x) x^{1/p-1}\,dx &\ge \int_0^{\frac{h_t(t)}{w(0)}} h_t^{-1}(w(0)x) x^{1/p-1}\,dx \\ &= \frac{1}{w(0)^{1/p}} \int_0^{(h_t(t))^{1/p}} h_t^{-1}(y^p)\,dy \\ &= \frac{1}{w(0)^{1/p}} \left[t(h_t(t))^{1/p} + \int_t^\infty (h_t(r))^{1/p}\,dr\right] \\ &= \left(\frac{A_t}{w(0)}\right)^{1/p} \left[t\,b(C\,t)^{-p'/p} + \int_t^\infty b(C\,r)^{-p'/p}\,dr\right] \\ &= 1. \end{aligned}$$

If $\mathcal{H}_p f_t(1) < s < t$, we have $x_s \ge \dfrac{1}{(p\,s)^p}$. Therefore,

$$w([0,x_s)) \ge w([0, 1/(p\,s)^p]) = \frac{\int_0^{1/(p\,s)^p} w}{|[0, 1/(p\,s)^p]|}\,\frac{1}{(p\,s)^p} \ge \frac{c_1}{(p\,s)^p},$$

with $c_1 = \inf\left\{0 \le x \le 1 : \dfrac{w([0,x])}{x}\right\}$ a positive number, as we saw in the proof of Theorem 5. If $1 < s < \mathcal{H}_p f_t(1)$, obviously $w(\{\mathcal{H}_p f_t > s\}) = 1 > \dfrac{1}{s^p}$. Consequently, $w(\{\mathcal{H}_p f_t > s\}) \ge \dfrac{c_0}{s^p}$, with $c_0 = \min\{1, c_1\}$.

To finish the proof of the theorem it remains to deal with the case that

$$\int_1^\infty b(s)^{-p'/p}\, ds = \infty.$$

We will show that, in this situation, $\mathcal{H}_p$ does not map $L^\Psi([0,1])$ on $L^\Phi([0,1])$. For that, we consider the function

$$f = h^{-1}(g)\chi_{[0,1]}$$

on $\mathfrak{M}([0,1])$, where

$$h(x) = \frac{K\, b(x)^{-p'}}{\left(\int_{1/2}^x b^{-p'/p}\, ds\right)^p} \text{ for } x \ge 1$$

and K such that $h(1) = 1$. Note that h is decreasing and so f is well defined.

First we see that f is in $L^\Psi([0,1])$. Since $\int_1^\infty b^{-p'/p} = \infty$ and $b^{-p'/p}$ is decreasing, there exists a sequence $\{x_n\}_{n=1}^\infty$ such that $\int_1^{x_n} b^{-p'/p} = n$ and $\lim_{n\to\infty} x_n = \infty$.

If we call $x_0 = 1$,

$$\begin{aligned}
\frac{1}{K}\int_1^\infty b(s)\, h(s)\, ds &= \int_1^\infty \frac{b(s)^{-p'/p}}{\left(\int_{1/2}^s b^{-p'/p}\right)^p}\, ds \\
&= \sum_{n=0}^\infty \int_{x_n}^{x_{n+1}} \frac{b(s)^{-p'/p}}{\left(\int_{1/2}^s b^{-p'/p}\right)^p}\, ds \\
&\le \int_1^{x_1} \frac{b(s)^{-p'/p}}{\left(\int_{1/2}^1 b^{-p'/p}\right)^p}\, ds + \sum_{n=1}^\infty \int_{x_n}^{x_{n+1}} \frac{b(s)^{-p'/p}}{\left(\int_1^{x_n} b^{-p'/p}\right)^p}\, ds.
\end{aligned}$$

The first term of the last expression is bounded by

$$\frac{1}{\left(\int_{1/2}^1 b^{-p'/p}\right)^p} < \infty$$

and the second by

$$\sum_{n=1}^\infty \frac{\int_1^{x_{n+1}} b^{-p'/p} - \int_1^{x_n} b^{-p'/p}}{\left(\int_1^{x_n} b^{-p'/p}\right)^p} = \sum_{n=1}^\infty \frac{1}{n^p} < \infty.$$

Hence, from the fact that $w(\{f > s\}) = h(s)$ for $s > 1$,

$$\begin{aligned}
\int_0^1 \Psi(|f(x)|)\, w(x)\, dx &= \int_0^\infty b(s) w(\{f > s\})\, ds \\
&\le \int_0^1 b(s)\, ds + \int_1^\infty b(s)\, h(s)\, ds < \infty.
\end{aligned}$$

Now we will see that $\mathcal{H}_p f$ is not in $L^\Phi([0,1],v)$, even more, we will show that $\mathcal{H}_p f(x) = \infty$ for all $x \in [0,1]$. Since $\mathcal{H}_p f$ is decreasing, it is enough to show $\mathcal{H}_p f(1) = \infty$. In fact,

$$\begin{aligned}\frac{1}{K^{1/p}}\mathcal{H}_p f(1) &= \frac{1}{pK^{1/p}}\int_0^1 h^{-1}(g(r))r^{1/p-1}\,dr \\ &= \frac{p}{K^{1/p}}\int_0^1 h^{-1}(g(t^p))\,dt \\ &\geq \frac{1}{K^{1/p}}\int_1^\infty \left[g^{-1}(h(r))\right]^{1/p}\,dr\end{aligned}$$

and from inequality (22),

$$\begin{aligned}\frac{1}{K^{1/p}}\int_1^\infty \left[g^{-1}(h(r))\right]^{1/p}\,dr &\geq \frac{1}{(w(0)K)^{1/p}}\int_1^\infty h(r)^{1/p}\,dr \\ &= \frac{1}{w(0)^{1/p}}\int_1^\infty \frac{b(s)^{-p'/p}}{\int_{1/2}^s b^{-p'/p}}\,ds\,.\end{aligned}$$

Finally, we have

$$\begin{aligned}\int_1^\infty \frac{b(s)^{-p'/p}}{\int_{1/2}^s b^{-p'/p}}\,ds &= \sum_{n=0}^\infty \int_{x_n}^{x_{n+1}} \frac{b(s)^{-p'/p}}{\int_{1/2}^s b^{-p'/p}}\,ds \\ &\geq \sum_{n=0}^\infty \int_{x_n}^{x_{n+1}} \frac{b(s)^{-p'/p}}{\int_{1/2}^{x_{n+1}} b^{-p'/p}}\,ds \\ &= \sum_{n=0}^\infty \frac{\int_1^{x_{n+1}} b^{-p'/p} - \int_1^{x_n} b^{-p'/p}}{\int_{1/2}^1 b^{-p'/p} + \int_1^{x_{n+1}} b^{-p'/p}} \\ &= \sum_{n=0}^\infty \frac{1}{\int_{1/2}^1 b^{-p'/p} + 1 + n} = \infty.\end{aligned}$$

□

4 CONVERSE INEQUALITY

It is well known that when we have a function f whose maximal Mf belongs to $L^1(\mathbb{T})$ we can assure that f is in $L\log L(\mathbb{T})$. This is generalized in [3] where the author finds that under appropriate assumptions on a and b, there exist constants c_1 and c_2 such that

$$\int_{\mathbb{T}} \Psi\left(c_1|f|\right) \leq c_2 + c_2 \int_{\mathbb{T}} \Phi\left(Mf\right) \tag{27}$$

for all f with $\|f\|_{L^1(\mathbb{T})} = 1$ if, and only if, there exists a constant c_3 such that

$$b(c_3 t)c_3 \leq \int_1^t \frac{a(s)}{s}\,ds \qquad \text{for all } t \geq 1.$$

In this section we will analyze inequalities of the type (27) when we replace M by $\mathcal{M}_p$ or $M^-_{1/p}$.
First we deal with the operator $\mathcal{M}_p$ and we start with the following lemma.

Lemma 2. *If* $p \geq 1$,

$$\frac{1}{2t^p}\int_t^\infty \mu_f(s)s^{p-1}\,ds \leq \mu_{\mathcal{M}_p f}(t) \qquad \text{for all } \ t \geq \|f\|_{L^p([0,1])}.$$

Proof. For $p=1$ see [9], p.93. If $p>1$, let f be in $L^p([0,1])$ and $t \geq \|f\|_{L^p([0,1])} = \left(\int_{[0,1]} f^p\right)^{1/p}$. Since the assertion is true for the maximal function $M = \mathcal{M}_1$, we have

$$\begin{aligned}\frac{1}{2t^p}\int_t^\infty \mu_f(s)s^{p-1}\,ds &= \frac{1}{2t^p}\int_{t^p}^\infty \mu_{f^p}(s)\,ds\\ &\leq \mu_{Mf^p}(t^p)\\ &= \mu_{\mathcal{M}_p f}(t).\end{aligned}$$

□

Theorem 7. *There exists a constant C' such that*

$$\int_{[0,1]} \Psi(|f|) \leq C' + C'\int_{[0,1]} \Phi(C'\,\mathcal{M}_p f) \tag{28}$$

for all $f \in \mathfrak{M}([0,1])$ with $\|f\|_{L^p([0,1])} = 1$ if, and only if, for some constant C

$$b(t) \leq C\,t^{p-1}\int_1^{C\,t} \frac{a(s)}{s^p}\,ds \qquad \text{for all } t \geq 1. \tag{29}$$

Proof. Suppose that inequality (29) holds. Let f be a function in $\mathfrak{M}([0,1])$ such that $\int_{[0,1]} f^p = 1$,

$$\begin{aligned}\int_{[0,1]} \Psi(|f|) &= \int_0^\infty b(t)\mu_f(t)\,dt\\ &= \left(\int_0^1 + \int_1^\infty\right) b(t)\mu_f(t)\,dt\\ &\leq \Psi(1) + \int_1^\infty b(t)\mu_f(t)\,dt.\end{aligned}$$

From inequality (29), Fubini's Theorem and Lemma 2, we get

$$\begin{aligned}\int_1^\infty b(t)\mu_f(t)\,dt &\leq C\int_1^\infty \left(t^{p-1}\int_1^{C\,t}\frac{a(s)}{s^p}\,ds\right)\mu_f(t)\,dt\\ &\leq C\int_1^\infty \frac{a(s)}{s^p}\left(\int_{s/C}^\infty t^{p-1}\mu_f(t)\,dt\right)ds\\ &\leq 2C\int_1^\infty a(s)\mu_{\mathcal{M}_p f}(s/C)\,ds\\ &\leq 2C\int_{[0,1]} \Phi(C\,\mathcal{M}_p f),\end{aligned}$$

proving that (29) implies (28).
We now see that (29) is a consequence of (28). For $t \geq 1$, let $f_t = t\chi_{(0,1/t^p)}$.

$$\begin{aligned}\int_{[0,1]} \Psi(|f_t|) &= \int_0^\infty b(s)\,\mu_{f_t}(s)\,ds \\ &= \frac{1}{t^p}\int_0^t b(s)\,ds \\ &\geq \frac{1}{2t^{p-1}}\,b(t/2),\end{aligned} \tag{30}$$

where we used that b is increasing.
On the other hand,

$$\begin{aligned}\int_{[0,1]} \Phi(\mathcal{M}_p f_t) &= \int_0^\infty a(s)\,\mu_{\mathcal{M}_p f_t}(s)\,ds \\ &\leq \Phi(1) + \int_1^t a(s)\,\mu_{\mathcal{M}_p f_t}(s)\,ds \\ &\leq \Phi(1) + A\int_1^t \frac{a(s)}{s^p}\,ds,\end{aligned} \tag{31}$$

Where in the last inequality we used the weak type (p,p) of $\mathcal{M}_p$.
The proof is completed, since $\int_1^t \frac{a(s)}{s^p}\,ds$ increases with t. □

For the operators M_α^+ and M_α^- with $\alpha = 1/p$ things are different. We are interested in what happens when a and b satisfy an inequelity opposite to (4), i.e.,

$$\left(\int_t^\infty b(s)^{-p'/p}\,ds\right)^{-p/p'} \leq C\int_1^{Ct} \frac{a(s)}{s^p}\,ds \qquad \text{for all } t \geq 1, \tag{32}$$

and some constant C.
If f is increasing then $M_\alpha^- f = f$. This implies that a result analogous to Theorem 7 is not possible. In fact, the pair $a(s) = s^p$ and $b(s) = s^{p-1}\log^p(s)$ satisfies (32), but we may find an increasing function f on $[0,1]$ such that $\int_{[0,1]} \Psi(|f|) < \infty$ and $\int_{[0,1]} \Phi(c|f|) = \infty$ for all $c > 0$ (take fo instance $f(x) = h^{-1}(1-x)\chi_{[0,1]}(x)$, with $h(t) = \frac{1}{(t\log t)^p}$).
The lateral nature of M_α^- implies that the operator does not enlarge increasing functions. However, for decreasing functions, an analogous of Lemma 2 is valid.

Lemma 3. *Let f be positive and decreasing function defined on $[0,1]$. Then,*

$$\left[\frac{1}{pt}\int_{\{f>t\}} f(x)x^{1/p-1}\,dx\right]^p \leq \left|\{x:\ M_{1/p}^- f(x) > t\}\right|, \tag{33}$$

for all $t > \|f\|_{p,1} = \|f\|_{L^{p,1}([0,1])}$.

Proof. Since f is decreasing,

$$\|f\|_{p,1} = \frac{1}{p}\int_0^1 f(x)x^{1/p-1}\,dx.$$

Let $t > \|f\|_{p,1}$. Since $M^-_{1/p}f(x) = \frac{1}{px^{1/p}}\int_0^x f(y)y^{1/p-1}\,dy$ is decreasing and continuous, we have $\{x : M^-_{1/p}f(x) > t\} = (0, x_t)$. For $t > \|f\|_{p,1} = M^-_{1/p}f(1)$,

$$t = \frac{1}{px_t^{1/p}}\int_0^{x_t} f(y)y^{1/p-1}\,dy \tag{34}$$

or

$$x_t = \left[\frac{1}{p\,t}\int_0^{x_t} f(y)y^{1/p-1}\,dy\right]^p. \tag{35}$$

We olso have $f(x) \le M^-_{1/p}f(x)$ for all x, then

$$\begin{aligned}\left[\frac{1}{p\,t}\int_{\{f>t\}} f(x)x^{1/p-1}\,dx\right]^p &\le \left[\frac{1}{p\,t}\int_{\{M^-_{1/p}f>t\}} f(x)x^{1/p-1}\,dx\right]^p \\ &= \left[\frac{1}{p\,t}\int_0^{x_t} f(x)x^{1/p-1}\,dx\right]^p \\ &= x_t = |\{x : M^-_{1/p}f(x) > t\}|.\end{aligned} \tag{36}$$

□

Having proved Lemma 3 we would expect a result analogous to Theorem 7 for decreasing functions to be true: if a and b satisfy (32) then

$$\int_{[0,1]} \Psi(|f|) \le C + C\int_{[0,1]} \Phi(M^-_\alpha f) \tag{37}$$

for all decreasing f with $\|f\|_{p,1} = 1$.
However, this is not true as the following example shows.
Consider the functions $\Phi(t) = \alpha t^{1/\alpha}$ and $\Psi(t) = t^{1/\alpha}[\log(1+t)]^{1/\alpha}$. For $n \ge 1$, let $f_n = n\chi_{[0,1/n^{1/\alpha}]}$. We have $\|f_n\|_{1/\alpha,1} = \alpha\int_0^1 f_n(x)x^{\alpha-1}\,dx = 1$, for all $n \ge 1$. We will see that inequality (37) can not be true for all f_n.
If $x \in [0, 1/n^{1/\alpha})$, we have $M^-_\alpha f_n(x) = n$; in fact, if $0 < c < x$,

$$\begin{aligned}\frac{\alpha}{(x-c)^\alpha}\int_c^x f_n(y)(y-c)^{\alpha-1}\,dy &= \frac{\alpha n}{(x-c)^\alpha}\int_c^x (y-c)^{\alpha-1}\,dy \\ &= \frac{\alpha}{(x-c)^\alpha}\int_0^{x-c} y^{\alpha-1}\,dy \\ &= n\end{aligned}$$

If $x \in [1/n^{1/\alpha}, 1)$, we have $M_\alpha^- f_n(x) = \dfrac{1}{x^\alpha}$. To see this, let $0 < c < x$. If $1/n^{1/\alpha} < c < x$, then $\dfrac{\alpha}{(x-c)^\alpha} \displaystyle\int_c^x f_n(y)(y-c)^{\alpha-1}\,dy = 0$. If $c < 1/n^{1/\alpha}$, we have

$$\begin{aligned}\frac{\alpha}{(x-c)^\alpha}\int_c^x f_n(y)(y-c)^{\alpha-1}\,dy &= \frac{\alpha n}{(x-c)^\alpha}\int_c^{1/n^{1/\alpha}} (y-c)^{\alpha-1}\,dy \\ &= \frac{\alpha n}{(x-c)^\alpha}\int_0^{1/n^{1/\alpha}-c} y^{\alpha-1}\,dy \\ &= \frac{n(1/n^{1/\alpha}-c)^\alpha}{(x-c)^\alpha} \le \frac{1}{x^\alpha}\end{aligned}$$

and

$$\frac{\alpha}{x^\alpha}\int_0^x f_n(y)y^{\alpha-1}\,dy = \frac{1}{x^\alpha}.$$

Therefore,

$$M_\alpha^- f_n(x) = \begin{cases} n & \text{if } x \in (0, 1/n^{1/\alpha}] \\ \frac{1}{x^\alpha} & \text{if } x \in (1/n^{1/\alpha}, 1] \end{cases}$$

and its distribution function is

$$\mu_{M_\alpha^- f_n}(t) = \begin{cases} 0 & \text{if } t > n \\ \frac{1}{t^{1/\alpha}} & \text{if } 1 < t < n \\ 1 & \text{if } 0 < t < 1, \end{cases}$$

while the distribution function of f_n is given by

$$\mu_{f_n}(t) = \begin{cases} 0 & \text{if } t > n \\ 1/n^{1/\alpha} & \text{if } 1 < t < n \\ 1 & \text{if } 0 < t < 1. \end{cases}$$

Now,

$$\int_{[0,1]} \Psi(f_n) = \frac{\Psi(n)}{n^{1/\alpha}} = (\log(1+n))^{1/\alpha} \tag{38}$$

and

$$\begin{aligned}\int_{[0,1]} \Phi(C\, M_\alpha^- f_n) &= C\int_0^\infty a(C\,s)\mu_{M_\alpha^- f_n}(s)\,ds \\ &= \Phi(C) + C^{\alpha-1}\int_1^n \frac{1}{s}\,ds = \Phi(C) + C^{\alpha-1}\log(n).\end{aligned} \tag{39}$$

Therefore, by letting n go to infinity, we see that there is no constant C independent of f_n such that

$$\int_{[0,1]} \Psi(f_n) \le C + C\int_{[0,1]} \Phi(C\, M_\alpha^- f_n) \qquad \text{for all } n \ge 1. \tag{40}$$

References

[1] B. Bongioanni, E. Harboure and L. Forzani *Weak type and restricted weak type (p,p) operators in Orlicz spaces* to appear in the Real Analysis Exchange, Research Articles sec., Vol. 28, No. 2, August, 2003.

[2] W.B Jurkat, J.L. Troutman, *Maximal Inequalities Related to Generalized a.e. Continuity*, Trans. AMS, V 252, Aug. 1979, 49-64.

[3] Hiro-0 Kita, *On a converse inequality for maximal functions in Orlicz spaces*, Studia Math. 118 (1) (1996).

[4] M. Krbec and V. Kokilashvili, *Weighted inequalities in Lorentz and Orlicz Spaces*, World Scientific Publishing Co., Singapore 1991.

[5] F. J. Martin-Reyes y A. de la Torre, *Some weighted inequalities for general one-side maximal operator*, Studia Math. 122 (1) 1997.

[6] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. AMS, V. 165 (1972), 207-226.

[7] M. M. Rao and Z. D. Ren, *Orlicz Spaces*, Marcel Dekker Inc., New York, Basel, Hong Kong, 1991.

[8] E. M. Stein and Guido Weiss, *Fourier Analysis on Euclidean Spaces*, Princenton University Press, 1971.

[9] A. Torchinski, *Real-variable methods in harmonic analysis*, Academic Press, Inc.; Orlando 1986.

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