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CONNES' METRIC FOR STATES IN GROUP ALGEBRAS*

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Abstract

We follow the main idea of A. Connes for the construction of a metric in the state space of a C^*-algebra. We focus in the reduced algebra of a discrete group Γ, and prove some equivalences and relations between two central objects of this category: the word-length growth (connected with the degree of the extension of Γ when the group is an extension of $\mathbb{Z}$), and the topological relation between the ω^* topology and the one introduced with this metric in the state space of $C_r^*(\Gamma)$. Recent studies [Antonescu] of E Christensen and C Antonescu show that, using a variation of the distance introduced by Connes, these topologies are equivalent if the group is of rapid decay, a concept which is equivalent in discrete groups to the concept of polynomial growth for the word-length (there is an extensive survey by Jolissant [Jol] that settles this equivalence). In this article we prove with elementary techniques, that Connes' metric is finite and induces a topology which is equivalent to the ω^* topology in the state space, when the group Γ is a finite extension of $\mathbb{Z}$. This is not surprising at all, since M Rieffel established in [Rieffel2] (with a complete different approach) this equivalence for $\Gamma = \mathbb{Z}$.

1 INTRODUCTION

In [Connes1] and [Connes2], A. Connes introduced what he called *non commutative metric spaces*, which consist of a triples $(\mathcal{A}, D, H)$ where $\mathcal{A}$ is a C*-algebra, acting on the Hilbert space H, and D is an unbounded operator in H, called the Dirac operator, satisfying

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- $(D^2+1)^{-1}$ is compact
- the set $\{a \in \mathcal{A} : [D,a] \textit{ is bounded } \}$ is norm-dense in $\mathcal{A}$

We are interested in the case when Γ is a discrete group with identity element e and the algebra $\mathcal{A}$ is the reduced C*-algebra $C_r^*(\Gamma)$. The Hilbert space is $\ell^2(\Gamma)$, with $C_r^*(\Gamma)$ acting as left convoluters (i.e. the left regular representation). The *Dirac operator* is defined in terms of a length function on Γ. A *length function* is a map $L : \Gamma \to \mathbb{R}_+$ satisfying

1. $L(gh) \leq L(g) + L(h)$ for all $g, h \in \Gamma$.
2. $L(g^{-1}) = L(g)$ for all $g \in \Gamma$.
3. $L(e) = 0$.

If Γ is given by generators and relations, the prototypical length function is the map which assigns to each word its (minimal) length. We shall fix this data L, and we will make the further assumption that the sets

$$\{g \in \Gamma : L(g) \leq c\}$$

are finite for any $c > 0$. The *Dirac operator* [Connes2] is then defined as follows:

$$D(\delta_g) = L(g)\delta_g$$

where $\{\delta_g : g \in \Gamma\}$ is the canonical orthonormal basis of $\ell^2(\Gamma)$. As is custom, we shall denote by λ_g the element δ_g regarded as an operator in $\ell^2(\Gamma)$. The *metric (of the non commutative metric space)* is defined in the state space $\mathcal{S}(C_r^*(\Gamma))$ of $C_r^*(\Gamma)$ by means of the formula

$$d(\psi, \varphi) = \sup\{|\psi(a) - \varphi(a)| : a \in C_r^*(\Gamma) \text{ with } \|[a, D]\| \leq 1\}.$$

Here [,] denotes the usual conmutator of operators. This d is not necessarily finite. In this note we study situations in which it is finite, and consider a problem posed by M. Rieffel, asking under which assumptions the metric thus defined induces on the state space a topology which is equivalent to the w^* topology.
The basic example of this situation, which even justifies the name "non commutative metric space", occurs when $\mathcal{A}$ is $C(M)$, the algebra of continuous functions on a spin manifold M [Connes2], [GL]. M Rieffel found [Rieffel2] a natural triple associated to non commutative tori. Also he pointed out that one can find a positive answer for matrix algebras.

In this note we consider this problem for group algebras arising from discrete groups and triples arising from length functions. Instead of dealing directly with the d metric, we refer it to two metrics, d_∞ and d_2, related with the asymptotic behaviour of the family $\{\frac{1}{L(g)} : e \neq g \in \Gamma\}$:

$$d_\infty(\varphi, \psi) = \sup_{g \neq e} \frac{|\varphi(\lambda_g) - \psi(\lambda_g)|}{L(g)}, \tag{1.1}$$

and

$$d_2(\varphi, \psi) = (\sum_{g \neq e} \frac{|\varphi(\lambda_g) - \psi(\lambda_g)|^2}{L(g)^2})^{1/2}. \tag{1.2}$$

First note that d_∞ is a well defined metric and that $d_\infty(\varphi, \psi) \leq d(\varphi, \psi)$. The first fact is apparent. To prove the second, note that $\|[D, \lambda_g]\| = L(g)$, and therefore

$$d_\infty(\varphi, \psi) = \sup_{a = \frac{1}{L(g)}, g \neq e} |\varphi(a) - \psi(a)| \leq d(\varphi, \psi).$$

Also note that d_2 may fail to be finite. Indeed, consider $\Gamma = \mathbb{Z} \times \mathbb{Z}$. Then the family $\{\frac{1}{L(g)} : g \neq e\}$ does not belong to $\ell^2(\mathbb{Z} \times \mathbb{Z})$. Consider the positive definite functions $f(g) = 1$ for all g and $h = \delta_e$. These functions induce states φ_f and φ_h on $C_r^*(\mathbb{Z} \times \mathbb{Z})$ satisfying $\varphi_f(\lambda_g) = f(g)$ and $\varphi_h(\lambda_g) = h(g)$. It follows that

$$d_2(\varphi_f, \varphi_h) = \sum_{e \neq g \in \mathbb{Z} \times \mathbb{Z}} \frac{1}{L(g)^2} = \infty.$$

Denote by $K(\Gamma)$ the group algebra of Γ, i.e. the set of elements of the form $\sum_{g \in F} \alpha_g \lambda_g$, where $\alpha_g \in \mathbb{C}$ and $F \subset \Gamma$ is a finite set.

Lemma 1.1 *The topology on $\mathcal{S}(C_r^*(\Gamma))$ induced by d_∞ is the w^*-topology.*

Proof. If $d_\infty(\varphi_n, \varphi) \to 0$, then clearly $\varphi_n(\lambda_g) \to \varphi(\lambda_g)$ for all $g \neq e$. Since φ_n,φ are states, $\varphi_n(\lambda_e) = \varphi_n(1) = 1 = \varphi(\lambda_e)$. It follows that $\varphi_n(a) \to \varphi(a)$ for all $a \in K(\Gamma)$. Since φ_n, φ have their norms bounded (by 1), and since $K(\Gamma)$ is dense in $C_r^*(\Gamma)$, it follows that $\varphi_n \to \varphi$ in the w^* topology. Conversely, suppose that $\varphi_n(a) \to \varphi(a)$ for all $a \in C_r^*(\Gamma)$ and fix $\epsilon > 0$. Let $F = \{g \in \Gamma : L(g) < 4/\epsilon\}$, which is a finite set, say $F = \{g_1, ..., g_k\}$. If $g \in F$, one has

$$\frac{|\varphi_n(\lambda_g) - \varphi(\lambda_g)|}{L(g)} \leq \frac{|\varphi_n(\lambda_g)| + |\varphi(\lambda_g)|}{4/\epsilon} \leq \epsilon/2.$$

On the other hand, there exists n_0 such that for all $n \geq n_0$,
$|\varphi_n(\lambda_{g_i}) - \varphi(\lambda_{g_i})| < \frac{\epsilon}{2} \min\{L(g_1), ..., L(g_k)\}$, for $i = 1, ..., k$. It follows that

$$\frac{|\varphi_n(\lambda_{g_i}) - \varphi(\lambda_{g_i})|}{L(g_i)} < \epsilon/2.$$

Therefore, if $n \geq n_0$, then

$$\sup_{g \neq e} \frac{|\varphi_n(\lambda_g) - \varphi(\lambda_g)|}{L(g)} = d_\infty(\varphi_n, \varphi) \to 0.$$

□

In this article we discuss necessary and sufficient conditions for the following two main problems:

- Is the pseudo-metric d finite?

- Is the topology induced by d the w^*-topology in the state space?

The main result of section 2 is Proposition 2.4, which states a sufficient condition on the growth of the length function L, which answers affirmatively both questions.

In section 3 we consider the von Neumann algebra of Γ. We show that d is a metric, which induces the w^*-topology in the set of normal states which are bounded respect to the trace (Proposition 3.1).

2 COMPARISON BETWEEN d,d_∞ AND d_2

Here we establish the basic inequality for these metrics, namely $d_\infty \leq d \leq d_2$.

Lemma 2.1 *Let* $a = \sum_{g \in F} \alpha_g \lambda_g \in K(\Gamma)$, *then*

$$\|[D,a]\| \geq (\sum_{g \in F} |\alpha_g|^2 L(g)^2)^{1/2}.$$

Proof. Note that

$$[D,a]\delta_e = \sum_{g \in F} \alpha_g [D, \lambda_g]\delta_e = -\sum_{g \in F} \alpha_g L(g) \delta_g,$$

because $D\lambda_g \delta_e = D\delta_g = L(g)\delta_g$, and in particular $D\delta_e = 0$. Therefore

$$\|[D,a]\delta_e\|_2^2 = \sum_{g \in F} |\alpha_g|^2 L(g)^2.$$

It follows that $\|[D,a]\| \geq \|[D,a]\delta_e\|_2 = (\sum_{g \in F} |\alpha_g|^2 L(g)^2)^{1/2}$. □

Proposition 2.2

$$d_\infty(\varphi, \psi) \leq d(\varphi, \psi) \leq d_2(\varphi, \psi).$$

Proof. Pick $a = \sum_{g \in F} \alpha_g \lambda_g \in K(\Gamma)$, with $\|[D,a]\| \leq 1$ (note that for any $a \in K(\Gamma)$, $[D,a]$ is a bounded operator). Then

$$|\varphi(a) - \psi(a)| = |\sum_{g \in F} \alpha_g (\varphi(\lambda_g) - \psi(\lambda_g))| = |\sum_{e \neq g \in F} \alpha_g L(g) \frac{(\varphi(\lambda_g) - \psi(\lambda_g))}{L(g)}|,$$

which by the Cauchy-Schwarz inequality is less than or equal to

$$(\sum_{e \neq g \in F} |\alpha_g|^2 L(g)^2)^{1/2} (\sum_{e \neq g \in F} \frac{|\varphi(\lambda_g) - \psi(\lambda_g)|^2}{L(g)^2})^{1/2} \leq \|[D,a]\| d_2(\varphi, \psi) \leq d_2(\varphi, \psi).$$

The proof finishes by observing that the set of elements $a \in K(\Gamma)$ with $\|[D,a]\| \leq 1$ is dense in the set of elements $b \in C_r^*(\Gamma)$ with $\|[D,b]\| \leq 1$. Indeed, let $b = \sum_{g \in \Gamma} \beta_g \lambda_g \in C_r^*(\Gamma)$ with $\|[D,b]\| \leq 1$. For finite sets $F \subset \Gamma$, the truncated elements $b_F = \sum_{g \in F} \beta_g \lambda_g \in K(\Gamma)$ converge in norm to b. Clearly also the (bounded) commutants $[D, b_F]$ converge in norm to $[D,b]$.
Denote by $N_F = \|[D,b]\| \|[D,b_F]\|^{-1}$ (after deleting the elements b_F with $[D,b_F] = 0$). Then $N_F b_F$ lies in $K(\Gamma)$, the commutants $[D, N_F b_F]$ have norm less than or equal to 1, and converge to b. □

We emphasize that d_2 might be infinite. It is finite, for example, if the family $\{\frac{1}{L(g)} : e \neq g \in \Gamma\}$ lies in $\ell^2(\Gamma)$. This imposes a strong condition on Γ, namely, that the group Γ has linear growth (polinomial growth with degree 1), see [Gromov] and [Connes2]. This means that there exist constants k, l such that $\#\{g \in \Gamma : L(g) \leq c\} \sim kc + l$.

Example 2.3 *Consider the following groups* Γ, *wich satisfy the condition that* $\{\frac{1}{L(g)} : e \neq g \in \Gamma\}$ *lies in* ℓ^2.

1. *Let* $\Gamma = \mathbb{Z}$. *Here the length function is* $L(m) = |m|$, $m \in \mathbb{Z}$. *The group* C^**-algebra equals in this case* $C(S^1)$.

2. *Let* Γ *be a finite extension of* $\mathbb{Z}$, *i.e. a group* Γ *which has a copy of* $\mathbb{Z}$ *inside, as a normal subgroup, and the quotient* $\mathcal{F} = \Gamma/\mathbb{Z}$ *is finite. Then, as a set,* Γ *is* $\mathbb{Z} \times \mathcal{F}$. *Let* $\mathcal{F} = \{f_1, ..., f_n\}$. *Then the classes of* $(1, f_1), ..., (1, f_n)$ *(i.e. these elements regarded as elements of* Γ*) are generators for* Γ. *Let us consider the length function* L *given by word length with respect to this set of generators. Note that for this* L, *there are at most* $2n$ *elements of* Γ *with any given length. It follows that* $\{\frac{1}{L(g)} : e \neq g \in \Gamma\}$ *lies in* ℓ^2. *The (reduced)* C^**-algebra of such* Γ *can be computed. They consist of algebras of* $n \times n$ *matrices with entries in* $C(S^1)$ *(see chapter VIII of [Davidson] for a complete descrition of this computation). Let us point out two special examples of this type*

   (a) $\Gamma = \mathbb{Z} \times \mathcal{F}$ *with the usual product for pairs. In this case the* C^**-algebra is* $C_r^*(\mathbb{Z} \times \mathcal{F}) \simeq C(S^1) \otimes C_r^*(\mathcal{F})$. *The algebra* $C_r^*(\mathcal{F})$ *is finite dimensional, therefore in this case* $C_r^*(\Gamma)$ *consists of a direct sum of full matrix algebras with entries in* $C(S^1)$. *In particular, if* $\mathcal{F} = S_k$ *the group of permutations of order* k, *then* $C_r^*(\Gamma) = M_k(C(S^1))$.

   (b) *Consider the (unique) nontrivial automorphism of* $\mathbb{Z}$, $\theta(m) = -m$. *Then there is a* $\mathbb{Z}_2$ *extension of* $\mathbb{Z}$, $\Gamma = \mathbb{Z} \times_\theta \mathbb{Z}_2$, *and the corresponding* C^**-algebra* $C_r^*(\Gamma)$ *is the cross product* $C(S^1) \times_\theta \mathbb{Z}_2$, *which identifies with the algebra of* 2×2 *matrices with entries in* $C(S^1)$ *of the form*

$$\begin{pmatrix} f(z) & g(z) \\ f(\overline{z}) & g(\overline{z}) \end{pmatrix},$$

   where f *and* g *are continuous functions in* S^1.

Proposition 2.4 *If* Γ *has a length function* L *such that* $\{\frac{1}{L(g)} : e \neq g \in \Gamma\}$ *is square summable, then the metric* d *is finite and it induces the* w^**-topology on the state space of* $C_r^*(\Gamma)$.

Proof. Since $\{\frac{1}{L(g)}\} \in \ell^2$, $d \leq d_2 < \infty$. By the above results it suffices to prove that if a sequence φ_n converges to φ in the w^* topology, then it converges in the d

metric. We claim that it converges in the d_2 metric. Fix $\epsilon > 0$. There exists a finite set $F = \{g_1, ..., g_k\}$ such that

$$(\sum_{g\in\Gamma-F} \frac{|\varphi_n(\lambda_g) - \varphi(\lambda_g)|^2}{L(g)^2})^{1/2} \leq 2(\sum_{g\in\Gamma-F} \frac{1}{L(g)^2})^{1/2} < \epsilon/2.$$

Put $c = (\sum_{i=1}^k \frac{1}{L(g_i)^2})^{1/2}$. There exists n_0 such that for all $n \geq n_0$, one has $|\varphi_n(\lambda_{g_i}) - \varphi(\lambda_{g_i})| < \epsilon/2c$. Therefore

$$(\sum_{i=1}^k \frac{|\varphi_n(\lambda_{g_i}) - \varphi(\lambda_{g_i})|^2}{L(g_i)^2})^{1/2} < \epsilon/2.$$

Then $d_2(\varphi_n, \varphi) \to 0$. □

Corollary 2.5 *If Γ is a finite extension of $\mathbb{Z}$, then in $S(C_r^*(\Gamma))$ the d metric is well defined and induces the ω^*-topology.*

3 NORMAL STATES WHICH ARE BOUNDED WITH RESPECT TO THE TRACE

We shall prove that the metric d is finite on the set of normal states of $C_r^*(\Gamma)$ which are bounded with respect to the trace of $C_r^*(\Gamma)$, i.e. the states φ which extend to normal states of the von Neumann algebra $\mathcal{L}_\Gamma$ of Γ, and verify that there exists a constant $\kappa > 0$ such that

$$\varphi(a^*a) \leq \kappa\tau(a^*a),$$

or shortly, $\varphi \leq \kappa\tau$. Recall that the trace τ is given by $\tau(a) = \langle a\delta_e, \delta_e\rangle$. There is a Radon-Nykodim derivative for all such φ [Araki]. Namely, there exists an element $\rho_\varphi \geq 0$ in $\mathcal{L}_\Gamma$ such that

$$\varphi(a) = \tau(\rho_\varphi a) \ , \quad \text{with } \|\rho_\varphi\| \leq \kappa^{1/2}.$$

Denote by $\mathcal{S}_\kappa$ the set

$$\mathcal{S}_\kappa = \{\varphi \in \mathcal{S}(\mathcal{L}_\Gamma) : \varphi \leq \kappa\tau\}.$$

First note that a state which lies in $\mathcal{S}_\kappa$ is necessarily normal. Indeed, let $\{p_i : i \in I\}$ be an arbitrary family of pairwise orthogonal projections in $\mathcal{L}_\Gamma$. Fix $\epsilon > 0$ and let $J \subset I$ be a finite set such that $\tau(\sum_{i\in I-J} p_i) = \sum_{i\in I-J} \tau(p_i) < \epsilon/\kappa$. Then $\varphi(\sum_{i\in I-J} p_i) < \epsilon$. Therefore $0 \leq \varphi(\sum_{i\in I} p_i) = \sum_{j\in J} \varphi(p_j) + \varphi(\sum_{i\in I-J} p_i) \leq \sum_{j\in J} \varphi(p_j) + \epsilon$. That is, $\sum_{i\in I} \varphi(p_i) = \varphi(\sum_{i\in I} p_i)$, and φ is normal.

Also it is apparent that $\mathcal{S}_\kappa$ is w^* compact and convex.

Proposition 3.1 *The metrics d and d_2 are well defined on $\mathcal{S}_\kappa$ and both induce the w^* topology.*

Proof. Note that if $\varphi \in \mathcal{S}_\kappa$ then

$$\varphi(\lambda_g) = \tau(\rho_\varphi \lambda_g) = \langle \rho_\varphi \delta_g, \delta_e \rangle = \rho_\varphi(g^{-1}),$$

where $\rho_\varphi(g^{-1})$ denotes the g^{-1}-coordinate of ρ_φ regarded as an element of $\ell^2(\Gamma)$. In particular, it follows that the family $\{\varphi(\lambda_g) : g \in \Gamma\}$ is square summable. Moreover,

$$(\sum_{g\in\Gamma} |\varphi(\lambda_g)|^2)^{1/2} = \|\rho_\varphi\|_2 \leq \|\rho_\varphi\| \leq \kappa^{1/2}.$$

It follows that if $\varphi, \psi \in \mathcal{S}_\kappa$, them

$$d(\varphi,\psi) \leq d_2(\varphi,\psi) = (\sum_{e\neq g\in\Gamma} \frac{|\varphi(\lambda_g) - \psi(\lambda_g)|^2}{L(g)^2})^{1/2} \leq 2\kappa.$$

Suppose now that $\varphi_n \to \varphi$ in the w^* topology. Fix $\epsilon > 0$. Then the set $F = \{g \in \Gamma : L(g) \leq 2\kappa/\epsilon\}$ is finite. Say $F = \{g_1, ..., g_n\}$. If g lies outside F one has

$$(\sum_{e\neq g\in\Gamma - F} \frac{|\varphi(\lambda_g) - \psi(\lambda_g)|^2}{L(g)^2})^{1/2} \leq \frac{\epsilon}{2\kappa}(\sum_{g\in\Gamma-F} |\varphi_n(\lambda_g) - \varphi(\lambda_g)|^2)^{1/2} \leq \epsilon.$$

Let $C = \sum_{i=1}^n \frac{1}{L(g_i)^2}$. There exists n_0 such that if $n \geq n_0$ then

$$|\varphi_n(\lambda_{g_i}) - \varphi(\lambda_{g_i})| < \epsilon/C \quad \text{for } i = 1, ..., n.$$

It follows that $d(\varphi_n, \varphi) \leq d_2(\varphi_n, \varphi) < \epsilon$ if $n \geq n_0$. □

Remark 3.2 *1. The first part of the proof in fact shows that if φ and ψ are normal states of $\mathcal{L}_\Gamma$ whose Radon-Nykodim derivatives with respect to the trace τ lie in $\ell^2(\Gamma)$, then $d(\varphi,\psi) \leq d_2(\varphi,\psi) < \infty$.*

2. It is apparent that the metrics d and d_2 are also finite on the set $\cup_{\kappa>0}\mathcal{S}_\kappa$

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