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L_p – SATURATION THEOREM FOR A LINEAR COMBINATION OF INTEGRAL BASKAKOV TYPE OPERATORS

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ABSTRACT. In [1], Agrawal and Thamer introduced a sequence of linear positive operators named as integral Baskakov type operators and studied some direct results in L_p – approximation by a linear combination of this sequence. In [2] they established an inverse theorem in L_p – norm for the same operators. The present paper is a continuation of their work in [1] and [2]. Here we aim to discuss a saturation theorem in L_p – norm for the above combination of integral Baskakov type operators.

1. INTRODUCTION

For $f \in L_p[0,\infty)$, $1 \le p < \infty$, Agrawal and Thamer [1] introduced a new sequence of linear positive operators in the following way:

$$L_n(f(t);x) = \int_0^{\infty} K_n(t,x)\, f(t)\, dt, \tag{1.1}$$

where $K_n(t,x) = (n-1)\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\, p_{n,\nu-1}(t) + (1+x)^{-n}\,\delta(t)$, $\delta(t)$ being the Dirac-delta function and

$$p_{n,\nu}(x) = \binom{n+\nu-1}{\nu} x^{\nu}(1+x)^{-n-\nu},\ x \in [0,\infty).$$

May [3] and Rathore [4] have described a method for forming linear combinations of a sequence of linear positive operators so as to improve the order of approximation. Following their method, in [1] Agrawal and Thamer established some direct theorems for a linear combination of the operators (1.1). Later, they [2] also obtained an inverse theorem for these operators. The approximation process is described as follows:

KEYWORDS: Linear positive operators, Saturation theorem in L_p – space, Linear combination, Steklov mean.

For $k \in N^0$ (the set of nonnegative integers), the linear combination $L_n(f,k,x)$ of the operators $L_{d_jn}(f;x)$, $j = 0, 1, \dots, k$ is defined as:

(1.2) $$L_n(f,k,x) = \sum_{j=0}^{k} C(j,k)\, L_{d_jn}(f;x),$$

where $d_0, d_1, \dots, d_k \in N$ (the set of positive integers) are arbitrary and distinct but fixed and

(1.3) $$C(j,k) = \prod_{\substack{i=0 \\ i \neq j}}^{k} \frac{d_j}{d_j - d_i}, \; k \neq 0 \text{ and } C(0,0) = 1.$$

Throughout this paper, we denote by $C[a,b]$ the set of all continuous functions on the interval $[a,b]$, $C^m[a,b]$ the subset of $C[a,b]$ of $m-$times continuously differentiable functions, C_0 the set of continuous functions on $(0,\infty)$ having a compact support and C_0^k the subset of C_0 of $k-$times continuously differentiable functions. Also, we denote by $AC[a,b]$, the class of absolutely continuous functions on $[a,b]$, $0 < a, b < \infty$, $BV[a,b]$ the class of functions of bounded variation over $[a,b]$ and $\|.\|_{C[a,b]}$, the sup norm on the space $C[a,b]$.

Let $m \in N$ and $0 < a < b < \infty$ and $f \in L_p[a,b]$, $1 \leq p < \infty$, the m^{th} order integral modulus of smoothness of f is defined as:

$$\omega_m(f,\tau,p,[a,b]) = \sup_{0<\delta\leq\tau} \left\| \Delta_\delta^m f(t) \right\|_{L_p[a,b-m\delta]},$$

where $\Delta_\delta^m f(t)$ is the m^{th} order forward difference of the function f with step length δ and $0 < \tau \leq (b-a)/m$.

Throughout this paper, we assume that $0 < a_1 < a_3 < a_2 < b_2 < b_3 < b_1 < \infty$, $I_i = [a_i, b_i]$ $(i = 1, 2, 3)$, $\langle h,g \rangle = \int_0^\infty h(x)\, g(x)\, dx$ the inner product on the space $L_p[0,\infty)$ and C denotes a constant not necessarily the same in different cases.

The object of the present paper is to prove the following (**saturation theorem**):

THEOREM 1. Let $f \in L_p[0,\infty)$, $1 \leq p < \infty$. Then, in the following statements, the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) hold:

(i) $\| L_n(f,k,.) - f \|_{L_p(I_1)} = O(n^{-(k+1)})$;

(ii) f coincides almost everywhere (a.e.) with a function F on I_3 having $2k+2$ derivatives such that:

(a) when $p > 1$, $F^{(2k+1)} \in AC(I_3)$ and $F^{(2k+2)} \in L_p(I_3)$,

(b) when $p = 1$, $F^{(2k)} \in AC(I_3)$ and $F^{(2k+1)} \in BV(I_3)$;

(iii) $\| L_n(f,k,.) - f \|_{L_p(I_2)} = O(n^{-(k+1)})$;

(iv) $\| L_n(f,k,.) - f \|_{L_p(I_1)} = o(n^{-(k+1)})$;

(v) f coincides a.e. with a function F on I_3, where F is $2k+2$ times continuously differentiable on I_3 and satisfies $\sum_{j=1}^{2k+2} p(j,k,x) F^{(j)}(x) = 0$, where $p(j,k,x)$ are the polynomials occurring in (2.1);

(vi) $\| L_n(f,k,.) - f \|_{L_p(I_2)} = o(n^{-(k+1)})$,

where $O(1)$ and $o(1)$ terms are with respect to n, when $n \to \infty$.

2. PRELIMINARY RESULTS

In order to prove the saturation theorem, we shall require the following results:
Let $f \in L_p[0,\infty)$, $1 \le p < \infty$. Then, for sufficiently small $\eta > 0$, the Steklov mean $f_{\eta,m}$ of m^{th} order corresponding to f is defined as follows:

$$f_{\eta,m}(t) = \eta^{-m} \left(\int_{-\eta/2}^{\eta/2} \right)^m \left\{ f(t) + (-1)^{m-1} \Delta^m_{\sum_{i=1}^{m} t_i} f(t) \right\} \prod_{i=1}^{m} dt_i \quad , t \in I_1 .$$

LEMMA 1 [6]. For the function $f_{\eta,m}(t)$ defined above, we have

(a) $f_{\eta,m}(t)$ has derivatives upto order m over I_1, $f_{\eta,m}^{(m-1)} \in AC(I_1)$ and $f_{\eta,m}^{(m)}$ exists a.e. and belongs to $L_p(I_1)$;

(b) $\left\| f_{\eta,m}^{(r)} \right\|_{L_p(I_2)} \le M_r \, \eta^{-r} \, \omega_r(f,\eta,p,I_1)$, $r = 1, 2, \ldots, m$;

(c) $\left\| f - f_{\eta,m} \right\|_{L_p(I_2)} \le M_{m+1} \, \omega_m(f,\eta,p,I_1)$;

(d) $\left\| f_{\eta,m} \right\|_{L_p(I_2)} \le M_{m+2} \left\| f \right\|_{L_p(I_1)}$;

(e) $\left\| f_{\eta,m}^{(m)} \right\|_{L_p(I_2)} \le M_{m+3} \, \eta^{-m} \left\| f \right\|_{L_p(I_1)}$,

where M_i's are certain constants that depend on i but are independent of f and η.

LEMMA 2 [1]. Let $m \in N^0$, the m^{th} order moment for the operators (1.1) be defined by:

$$U_{n,m}(x) = L_n((t-x)^m; x) = (n-1) \sum_{\nu=1}^{\infty} p_{n,\nu}(x) \int_0^{\infty} p_{n,\nu-1}(t)(t-x)^m \, dt + (-x)^m (1+x)^{-n} .$$

Then $U_{n,0}(x) = 1$, $U_{n,1}(x) = \dfrac{2x}{n-2}$ and

$$(n-m-2) U_{n,m+1}(x) = x(1+x) U'_{n,m}(x) + [(2x+1)m + 2x] U_{n,m}(x) + 2mx(1+x) U_{n,m-1}(x), \quad n > m+2 .$$

Further, we have the following consequences of $U_{n,m}(x)$:

(i) $U_{n,m}(x)$ is a polynomial in x of degree m;

(ii) for every $x \in [0,\infty)$ $U_{n,m}(x) = O(n^{-[(m+1)/2]})$, where $[\beta]$ denotes the integer part of β.

LEMMA 3 [1]. For $m \in N^0$, the function $V_{n,m}(t)$ be defined by:

$$V_{n,m}(t) = (n-1)\sum_{\nu=1}^{\infty} p_{n,\nu-1}(t)\int_0^{\infty} p_{n,\nu}(x)(x-t)^m\,dx.$$

Then $V_{n,0}(t) = 1$, $V_{n,1}(t) = \dfrac{2(1+t)}{(n-2)}$ and there holds the recurrence relation

$$(n-m-2)V_{n,m+1}(t) = t(1+t)V'_{n,m}(t) + \{(2t+1)(m+1)+1\}V_{n,m}(t) + 2mt(1+t)V_{n,m-1}(t),\ n > m+2,$$

Consequently:

(i) $V_{n,m}(t)$ is a polynomial in t of degree m;

(ii) for every $t \in [0,\infty)$, $V_{n,m}(t) = O(n^{-[(m+1)/2]})$.

LEMMA 4 [3]. If $C(j,k)$, $j = 0, 1, \ldots, k$ are defined as in (1.3), then

$$\sum_{j=0}^{k} C(j,k)\,d_j^{-m} = \begin{cases} 1, & m = 0 \\ 0, & m = 1,2,\ldots,k \end{cases}.$$

LEMMA 5 [1]. For $m \in N$ and n sufficiently large, there holds

$$L_n((t-x)^m, k, x) = n^{-(k+1)}\{q(m,k,x) + o(1)\},$$

where $q(m,k,x)$ is a certain polynomial in x of degree m and $x \in [0,\infty)$ is arbitrary but fixed. Further, we have

$$q(2k+1,k,x) = \frac{(-1)^k(2k+1)!}{k!\prod_{j=0}^{k} d_j}\{(k+1)(1+2x)-1\}\{x(1+x)\}^k \text{ and}$$

$$q(2k+2,k,x) = \frac{(-1)^k(2k+2)!}{(k+1)!\prod_{j=0}^{k} d_j}\{x(1+x)\}^{k+1}.$$

LEMMA 6. Suppose that $f \in C^{2k+2}(I_1)$ has a compact support. Then, there holds

$$L_n(f,k,x) - f(x) = n^{-(k+1)}\left\{\sum_{i=1}^{2k+2} p(i,k,x) f^{(i)}(x) + o(1)\right\},\ n \to \infty, \tag{2.1}$$

uniformly in $x \in I_1$, where $p(i,k,x)$ is a polynomials in x of degree i and does not vanish on $(0,\infty)$ for every $i = 1, 2, \ldots, 2k+2$.

Proof. We can write f by using Taylor's expansion of f on x as:

$$(2.2)\quad f(t)-f(x)=\sum_{i=1}^{2k+2}\frac{(t-x)^i}{i!}f^{(i)}(x)+\frac{(t-x)^{2k+2}}{(2k+2)!}\left(f^{(2k+2)}(\xi)-f^{(2k+2)}(x)\right),$$

where ξ lies between t and x.

Now, given any $\varepsilon>0$, there exists a $\delta>0$ such that $\left|f^{(2k+2)}(u)-f^{(2k+2)}(w)\right|<\varepsilon$, where $|u-w|<\delta$, $u,w\in I_1$. Thus, for all $t,x\in I_1$, we have

$$(2.3)\quad \left|(t-x)^{2k+2}\left(f^{(2k+2)}(\xi)-f^{(2k+2)}(x)\right)\right|<\varepsilon(t-x)^{2k+2}+\frac{2}{\delta^2}\left\|f^{(2k+2)}\right\|_{C(I_1)}(t-x)^{2k+4}.$$

Now, by the positivity of the operators L_n and the Lemma 2, we get

$$(2.4)\quad \left|L_n\left((t-x)^{2k+2}(f^{(2k+2)}(\xi)-f^{(2k+2)}(x));x\right)\right|<C\,n^{-(k+1)}(\varepsilon+n^{-1}).$$

Using Lemma 5, we have

$$(2.5)\quad L_n\left(\frac{(t-x)^i}{i!},k,x\right)=n^{-(k+1)}(p(i,k,x)+o(1)),\ n\to\infty,$$

for all $i=1,2,\dots,2k+2$ and o – term holds uniformly in $x\in I_1$.

Again, using Lemma 2 (the recurrence relation), we can see that $p(i,k,x)$ does not vanish on $(0,\infty)$ for every $i=1,2,\dots,2k+2$. By applying the linear combination $L_n(.,k,x)$ to (2.2), we get

$$(2.6)\quad L_n(f,k,x)-f(x)=\sum_{i=1}^{2k+2}\frac{f^{(i)}(x)}{i!}L_n((t-x)^i,k,x)+\frac{1}{(2k+2)!}L_n\left((t-x)^{2k+2}(f^{(2k+2)}(\xi)-f^{(2k+2)}(x)),k,x\right).$$

Finally, since $\varepsilon>0$ is arbitrary, using (2.4)-(2.6) the required result is immediate. ∎

LEMMA 7. Let $h\in L_p(I_1)$, $1\le p<\infty$, with $\operatorname{supp} h\subset I_1$. If h has $2k+1$ derivatives with $h^{(2k)}\in AC(I_1)$ and $h^{(2k+1)}\in L_p(I_1)$, then for each $g\in C_0^{2k+2}$ with $\operatorname{supp} g\subset(0,\infty)$, the following inequality holds:

$$\left|\langle L_n(h,k,x)-h(x),g(x)\rangle\right|\le\frac{C}{n^{k+1}}\sum_{r=0}^{2k}\left\|h^{(r)}\right\|_{C(I_1)}.$$

Proof. Clearly,

$$(2.7)\quad \langle L_n(h,k,x)-h(x),g(x)\rangle=\sum_{j=0}^{k}C(j,k)\left\langle L_{d_jn}(h(t);x),g(x)\right\rangle-\langle h,g\rangle.$$

By using Fubini's theorem and Taylor's expansion of g on t, we have

$$\left\langle L_{d_jn}(h(t);x), g(x)\right\rangle = \int_0^\infty\int_0^\infty K_{d_jn}(t,x)\,h(t)\,g(x)\,dt\,dx$$

$$= \int_0^\infty\int_0^\infty K_{d_jn}(t,x)\,h(t)\left[\sum_{i=0}^{2k+1}\frac{(x-t)^i}{i!}g^{(i)}(t) + \frac{(x-t)^{2k+2}}{(2k+2)!}g^{(2k+2)}(\xi_1)\right]dx\,dt,$$

(ξ_1 lying between x and t)

$$= \int_0^\infty\left(\int_0^\infty K_{d_jn}(t,x)\,dx\right)h(t)\,g(t)\,dt + \int_0^\infty\left(\int_0^\infty K_{d_jn}(t,x)(x-t)\,dx\right)h(t)\,g'(t)\,dt$$

$$+\sum_{i=2}^{2k+1}\frac{1}{i!}\int_0^\infty\int_0^\infty K_{d_jn}(t,x)(x-t)^i\,h(t)\,g^{(i)}(t)\,dx\,dt$$

$$+\frac{1}{(2k+2)!}\int_0^\infty\int_0^\infty K_{d_jn}(t,x)(x-t)^{2k+2}\,h(t)\,g^{(2k+2)}(\xi_1)\,dx\,dt$$

$$= \sigma_0 + \sigma_1 + \sum_{i=2}^{2k+1}\sigma_i + \sigma_{2k+2}.$$

Using Lemma 3, we have

$$\sum_{j=0}^{k} C(j,k)\,\sigma_0 = \langle h, g\rangle \text{ and}$$

$$\sum_{j=0}^{k} C(j,k)\,\sigma_1 = \sum_{j=0}^{k} C(j,k)\int_0^\infty\left(\int_0^\infty K_{d_jn}(t,x)(x-t)\,dx\right)h(t)\,g'(t)\,dt$$

$$= \sum_{j=0}^{k} C(j,k)\int_0^\infty \frac{2(1+t)}{d_jn-2}\,h(t)\,g'(t)\,dt.$$

Thus, by using the compactness of g' and Lemma 4, we get:

$$\left|\sum_{j=0}^{k} C(j,k)\,\sigma_1\right| \le \frac{C}{n^{k+1}}\|h\|_{C(I_1)} \text{ and } \left|\sum_{j=0}^{k} C(j,k)\,\sigma_{2k+2}\right| \le \frac{C}{n^{k+1}}\|h\|_{C(I_1)}.$$

Now, let $h_i(t) = h(t)\,g^{(i)}(t)$, $2 \le i \le 2k+1$ then by using Taylor's expansion of h_i on x, we have $h_i(t) = \sum_{r=0}^{2k-1}\frac{(t-x)^r}{r!}h_i^{(r)}(x) + \frac{(t-x)^{2k}}{(2k)!}h_i^{(2k)}(\xi_2)$ where ξ_2 lies between t and x. Applying Fubini's theorem and Lemma 4, we have for each i

$$\left|\sum_{j=0}^{k} C(j,k)\,\sigma_i\right| \le \frac{C}{n^{k+1}}\left\{\sum_{r=0}^{2k}\left\|h_i^{(r)}\right\|_{C(I_1)}\right\} \le \frac{C}{n^{k+1}}\left\{\sum_{r=0}^{2k}\left\|h^{(r)}\right\|_{C(I_1)}\right\}.$$

Thus, $\left|\sum_{j=0}^{k} C(j,k)\left(\sum_{i=2}^{2k+1}\sigma_i\right)\right| \le \frac{C}{n^{k+1}}\sum_{r=0}^{2k}\left\|h^{(r)}\right\|_{C(I_1)}.$

Finally, combining the estimates of $\sum_{j=0}^{k} C(j,k)\sigma_i$, $i = 0, 1, \dots, 2k+2$ with (2.7), the required result follows. ∎

Our next result is the inverse theorem for $L_n(.,k,x)$ of the operators (1.1).

THEOREM 2 [2]. Let $0 < \alpha < 2k+2$, $f \in L_p[0,\infty)$, $1 \le p < \infty$ and

$$\| L_n(f,k,.) - f \|_{L_p(I_1)} = O(n^{-\alpha/2}) \text{ as } n \to \infty,$$

then $\omega_{2k+2}(f,\tau,p,I_3) = O(\tau^\alpha)$ as $\tau \to 0$.

THEOREM 3 [5]. Let $1 \le p < \infty$, $f \in L_p[a,b]$ and there hold

$$\omega_m(f,\tau,p,[a,b]) = O(\tau^{r+\alpha}),\ (\tau \to 0),$$

where $m, r \in N$ and $0 < \alpha < 1$. Then $f(x)$ coincides a.e. on $[c,d] \subset (a,b)$ with a function $F(x)$ possessing an absolutely continuous derivative $F^{(r-1)}(x)$, the r^{th} derivative $F^{(r)}(x) \in L_p[c,d]$, and there holds $\omega(F^{(r)},\tau,p,[c,d]) = O(\tau^\alpha),\ (\tau \to 0)$.

3. PROOF OF THEOREM 1.

We assume that $a_1 < x_1 < x_2 < a_3 < b_3 < y_2 < y_1 < b_1$, and $J_i = [x_i, y_i]$ $(i = 1, 2)$. We get from Theorems 2, 3 that f coincides a.e. on (x_1, y_1) with a function called F such that $F^{(2k)} \in AC(J_1)$ and $F^{(2k+1)} \in L_p(J_1)$. Moreover, for $0 < \beta < 1$,

(3.1) $$\omega(F^{(2k+1)},\tau,p,J_1) = O(\tau^\beta),\ \tau \to 0.$$

Let $q \in C_0^{2k+2}$ with $\operatorname{supp} q \subset (a_1, b_1)$ and $q(x) = 1$ if $x \in J_1$. Put $\hat{f}(x) = F(x)q(x)$, $x \in [0,\infty)$ then

$$\left\| L_n(\hat{f},k,.) - \hat{f} \right\|_{L_p(J_2)} \le \| L_n(f,k,.) - f \|_{L_p(J_2)} + \left\| L_n(\hat{f} - f,k,.) \right\|_{L_p(J_2)}.$$

Because of $\hat{f} = f$ on J_1, the contribution of the second term of the right hand side can be made arbitrary small as $n \to \infty$. Hence, assuming (i), it follows that

$$\left\| L_n(\hat{f},k,.) - \hat{f} \right\|_{L_p(J_2)} = O(n^{-(k+1)}),\ n \to \infty.$$

Now, if $p > 1$, using Alaoglu's theorem there exists a function $h(x) \in L_p(J_2)$ such that for some subsequence $\{n_j\}$ and for every $g \in C_0^{2k+2}$ with $\operatorname{supp} g \subset (a_1, b_1)$,

(3.2) $$\lim_{n_j \to \infty} n_j^{k+1} \left\langle L_{n_j}(\hat{f},k,.) - \hat{f}, g \right\rangle = \langle h, g \rangle.$$

When $p = 1$, the functions $\phi_n(x)$ defined by:

$$\phi_n(u) = \int_{x_2}^{u} n^{k+1} \left\{ L_n(\hat{f},k,x) - \hat{f}(x) \right\} dx$$

are uniformly bounded and are of uniformly bounded variation. Making use of Alaoglu's theorem, it follows that there exists a function $\phi_0(x) \in AC(J_2)$ such that

(3.3) $$\lim_{n_j \to \infty} n_j^{k+1} \left\langle L_{n_j}(\hat{f}, k, .) - \hat{f}, g \right\rangle = -\langle \phi_0, g' \rangle .$$

Now, suppose that $\hat{f}_{\eta,2k+2}$ is the Steklov mean of $(2k+2)^{th}$ order corresponding to $\hat{f}$, we have

$$\begin{aligned}\left\langle L_{n_j}(\hat{f}, k, x) - \hat{f}(x), g(x) \right\rangle &= \left\langle L_{n_j}(\hat{f} - \hat{f}_{\eta,2k+2}, k, x) - (\hat{f} - \hat{f}_{\eta,2k+2})(x), g(x) \right\rangle \\ &\quad + \left\langle L_{n_j}(\hat{f}_{\eta,2k+2}, k, x) - \hat{f}_{\eta,2k+2}(x), g(x) \right\rangle \\ &= \left\langle L_{n_j}(\hat{f} - \hat{f}_{\eta,2k+2}, k, x) - (\hat{f} - \hat{f}_{\eta,2k+2})(x), g(x) \right\rangle \\ &\quad + \frac{1}{n_j^{k+1}} \left\langle P_{2k+2}(D) \hat{f}_{\eta,2k+2}(x), g(x) \right\rangle + o\left(\frac{1}{n_j^{k+1}}\right)\end{aligned}$$

in view of Lemma 6, where $P_{2k+2}(D) = \sum_{i=1}^{2k+2} p(i,k,x) D^i$ and $D \equiv \frac{\partial}{\partial x}$.

Let $P^*_{2k+2}(D) = \sum_{i=1}^{2k+2} p^*(i,k,x) D^i$ denote the differential operator adjoint to $P_{2k+2}(D)$, thus,

$$\begin{aligned} n_j^{k+1} \left\langle L_{n_j}(\hat{f}, k, x) - \hat{f}(x), g(x) \right\rangle - \left\langle \hat{f}_{\eta,2k+2}(x), P^*_{2k+2}(D) g(x) \right\rangle \\ = n_j^{k+1} \left\langle L_{n_j}(\hat{f} - \hat{f}_{\eta,2k+2}, k, x) - (\hat{f} - \hat{f}_{\eta,2k+2})(x), g(x) \right\rangle + o(1) \\ \leq C \left\{ \sum_{r=0}^{2k} \left\| \hat{f}^{(r)} - \hat{f}^{(r)}_{\eta,2k+2} \right\|_{C(I_1)} \right\} + o(1) \end{aligned}$$

(in view of Lemma 1 (property (a)) and Lemma 7).

Therefore,

$$\lim_{n_j \to \infty} n_j^{k+1} \left\langle L_{n_j}(\hat{f}, k, x) - \hat{f}(x), g(x) \right\rangle - \left\langle \hat{f}_{\eta,2k+2}(x), P^*_{2k+2}(D) g(x) \right\rangle \leq C \sum_{r=0}^{2k} \left\| \hat{f}^{(r)} - \hat{f}^{(r)}_{\eta,2k+2} \right\|_{C(I_1)} .$$

Taking limit as $\eta \to 0$ and using (3.1) we get

(3.4) $$\lim_{n_j \to \infty} n_j^{k+1} \left\langle L_{n_j}(\hat{f}, k, x) - \hat{f}(x), g(x) \right\rangle - \left\langle \hat{f}(x), P^*_{2k+2}(D) g(x) \right\rangle = 0 .$$

Combining (3.2) and (3.4), we have $\left\langle \hat{f}(x), P^*_{2k+2}(D) g(x) \right\rangle = \langle h(x), g(x) \rangle$ and hence

(3.5) $$h = P_{2k+2}(D) \hat{f}$$

as generalized functions. Now, in view of Lemmas 5 and 6, we have

$$p(2k+2, k, x) = \frac{q(2k+2, k, x)}{(2k+2)!} = \frac{(-1)^k}{(k+1)! \prod_{j=0}^{k} d_j} \{ x(1+x) \}^{k+1} \neq 0 .$$

Hence, regarding (3.5) as a generalized first order linear differential equation for $\hat{f}^{(2k+1)}$ with the non-homogeneous terms linearly depending on $\hat{f}^{(i)}$, $0 \le i \le 2k$ and h with polynomial coefficients, as $\hat{f}^{(i)} \in C(J_2).(0 \le i \le 2k)$ and $h \in L_p(J_2)$ we conclude that $\hat{f}^{(2k+1)} \in AC(J_2)$ and therefore that $\hat{f}^{(2k+2)} \in L_p(J_2)$. Since $\hat{f}$ coincides with F on J_2 it follows that $F^{(2k+1)} \in AC(I_2)$ and that $F^{(2k+2)} \in L_p(I_2)$.

When $p=1$, proceeding as in the case of $p>1$ with (3.2) replaced by (3.3) we find that $F^{(2k)} \in AC(I_2)$ and $F^{(2k+1)} \in BV(I_2)$. This completes the proof of implication (i) $\Rightarrow$ (ii).

The implication (ii) $\Rightarrow$ (iii) follows from proposition of [1] for the case $p>1$ and $p=1$.

Assuming (iv), since $n^{k+1} \| L_n(f,k,.) - f \|_{L_p(I_1)} \to 0$ as $n \to \infty$, proceeding as in the proof of (i) $\Rightarrow$ (ii) it follows that $n^{k+1} \left\| L_n(\hat{f},k,.) - \hat{f} \right\|_{L_p(J_2)} \to 0$ as $n \to \infty$ and hence we find that $h(x)$ and $\phi_0(x)$ are zero functions.

Thus, $P^*_{2k+2}(D)\hat{f}(x) = 0$. This implies that $\hat{f}$ is $2k+2$ times continuously differentiable function. Now, applying Lemma 6 for the function $\hat{f}$

(3.6) $$\lim_{n_j \to \infty} n_j^{k+1} \left\langle L_{n_j}(\hat{f},k,.) - \hat{f}, g \right\rangle = \left\langle P_{2k+2}(D)\hat{f}, g) \right\rangle .$$

Comparing (3.4) and (3.6), we have $P_{2k+2}(D)\hat{f}(x) = 0$. Hence, over I_2, F is $2k+2$ times continuously differentiable function and $P_{2k+2}(D)F(x) = 0$.

Finally, (v) $\Rightarrow$ (vi) follows from Lemma 6. ∎

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