REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Volumen 45, Número 1, 2004, Páginas 1–6

TRANSFERENCE OF A LITTLEWOOD-PALEY-RUBIO INEQUALITY AND DIMENSION FREE ESTIMATES

T.A. GILLESPIE AND J. L. TORREA

Dedicated to Professor Roberto Macías

Abstract. A well known result by Rubio de Francia asserts that for every finite family of disjoint intervals $\{I_k\}$ in $\mathbb{R}$, and p in the range $2 \le p < \infty$, there exists $C_p > 0$ such that

$$\| \sum_k r_k S_{I_k} f \|_{L^p_{L^p([0,1])}(\mathbb{R})} \le C_p \|f\|_{L^p(\mathbb{R})},$$

where the r_k's are the Rademacher functions.

In this note we prove that, given a compact connected abelian group G with dual group Γ and p in the range $2 \le p < \infty$, there is a constant C_p, independent of G and the particular ordering on Γ, such that for every sequence $\{I_k\}$ of disjoint intervals in Γ, we have

$$\| \sum_k r_k S_{I_k} f \|_{L^p_{L^p([0,1])}(G)} \le C_p \|f\|_{L^p(G)}.$$

We obtain the result by a transference approach that can be used for functions taking values in Banach spaces.

1. INTRODUCTION

In [RdeF2], Rubio de Francia proved the following remarkable result establishing a Littlewood-Paley property for disjoint intervals in $\mathbb{R}$.

Theorem 1.1. *Given an interval $I \subset \mathbb{R}$, denote by S_I the partial sum operator $(S_I f)\hat{} = \hat{f}\chi_I$, where $\hat{f}$ stands for the Fourier transform of the function f. For every p in the range $2 \le p < \infty$, there exists $C_p > 0$ such that, for every sequence $\{I_k\}$ of disjoint intervals, we have*

$$\left\| \Big(\sum_k |S_{I_k} f|^2\Big)^{1/2} \right\|_{L^p(\mathbb{R})} \le C_p \|f\|_{L^p(\mathbb{R})}, \quad f \in L^p(\mathbb{R}). \tag{1.2}$$

See also [J] for the n-dimensional analogue.

By Kintchine's inequality, an equivalent formulation of this result can be given in which the expression

$$\left\| \Big(\sum_k |S_{I_k} f|^2\Big)^{1/2} \right\|_{L^p(\mathbb{R})}$$

The authors were partially supported by the HARP network HPRN-CT-2001- 00273 of the European Commission and by grant BFM 2002-04013-C02-02 of Ministerio de Ciencia y Tecnología.

is replaced by

$$\left\|\sum_k r_k S_{I_k} f\right\|_{L^p_{L^p[0,1]}(\mathbb{R})},$$

where the r_k's are the Rademacher functions. In this form, the conclusion of Rubio de Francia's theorem allows the possibility of extension to functions taking values in a Banach space and led to the following definition (see [B,G,T2]).

Definition 1.3. *Let B be a Banach space and let p be in the range $2 \leq p < \infty$. We say that B satisfies the LPR_p property if there exists a constant $C_{p,B}$ such that for every finite family of disjoint intervals $\{I_k\}$ in $\mathbb{R}$, we have*

$$\|\sum_k r_k S_{I_k} f\|_{L^p_{L^p_B([0,1])}(\mathbb{R})} \leq C_{p,B}\|f\|_{L^p_B(\mathbb{R})}. \tag{1.4}$$

Although no examples are known of Banach spaces having the property LPR_p other than the classical scalar valued L^p spaces, it was shown in [B,G,T2] that such a Banach space B must satisfy $p(B) = 2$, where $p(B)$ denotes the supremum of the Rademacher type of B. Note also that a space having LPR_p must be a **UMD** space (see [Bou], [B] and [RdeF1] for the definition and properties of **UMD** spaces). It is possible to consider a version of LPR_p in which the group $\mathbb{R}$ is replaced by the circle group $\mathbb{T}$ and the partial sum operators S_{I_k} are relative to disjoint intervals in $\mathbb{Z}$. The required inequality would then take the form

$$\|\sum_k r_k S_{I_k} f\|_{L^p_{L^p_B([0,1])}(\mathbb{T})} \leq C_{p,B}\|f\|_{L^p_B(\mathbb{T})}$$

for functions f in $L^p_B(\mathbb{T})$ and we might refer to this property as LPR_p **relative to** $\mathbb{T}$.

However, the aim of the present note is to show by transference that LPR_p already implies LPR_p **relative to** $\mathbb{T}$. In a subsequent note, it will be shown that in fact the two properties are equivalent and, indeed, are also equivalent to the corresponding property for $L^p_B(\mathbb{Z})$ where the partial sum operators involve disjoint arcs in $\mathbb{T}$. This is similar to the situation regarding the definition of the **UMD** property in which the required boundedness of the Hilbert transform (for $\mathbb{R}$) is equivalent to the boundedness of both the conjugate function (for $\mathbb{T}$) and the discrete Hilbert transform (for $\mathbb{Z}$) on the corresponding L^p spaces of B-valued functions. We shall also transfer the LPR_p property from $\mathbb{T}$ to an arbitrary compact connected abelian group G, giving dimensionally independent constants when $G = \mathbb{T}^n$.

The technique of transference used in this note has proved effective in many harmonic analysis contexts and has its origins in the work of Calderón and Zygmund on singular integrals [CZ] and Cotlar on the ergodic Hilbert transform [Co]. An early account expository account was given by Calderón [C] and this was followed by a more comprehensive survey by Coifman and Weiss [C,W].

2. The main result

We prove the following result.

Theorem 2.5. *Let B be a Banach space with property LPR_p for some p in the range $2 \leq p < \infty$ and corresponding constant $C_{p,B}$ in (1.4). Then, for every finite sequence I_k of disjoint intervals in $\mathbb{Z}$,*

$$\|\sum_k r_k S_{I_k} f\|_{L^p_{L^p_B([0,1])}(\mathbb{T})} \leq C_{p,B} \|f\|_{L^p_B(\mathbb{T})}, \tag{2.6}$$

where S_I denotes the partial sum operator on $L^p_B(\mathbb{T})$ defined by $(S_I f)\hat{} = \hat{f}\chi_I$ for an interval I in $\mathbb{Z}$.

Before proving this result we note that a space B with property LPR_p is necessarily a **UMD** space and so the partial sum operators S_I are bounded on $L^p_B(\mathbb{T})$ by the boundedness of the conjugate function. We shall obtain Theorem 2.5 by combining a straightforward adaptation of the techniques in Chapter 3 of [C,W] to a vector valued setting with the following transference result from [B,G,T2].

Theorem 2.7. *Let G be a locally compact abelian group, let X, Y be Banach spaces and let K be a function in $L^1_{\mathcal{L}(X,Y)}(G)$. Assume that there exist strongly continuous representations R and $\tilde{R}$ of the group G such that:*

1. *For every $u \in G$, we have $R_u \in \mathcal{L}(X,X)$ and $\tilde{R}_u \in \mathcal{L}(Y,Y)$;*
2. *There exist constants c_1 and c_2 such that $\|R_u\|_{\mathcal{L}(X,X)} \leq c_1$ and $\|\tilde{R}_u\|_{\mathcal{L}(Y,Y)} \leq c_2,\ u \in G$;*
3. *R and $\tilde{R}$ intertwine K in the sense that*

$$K(u)R_v(x) = \tilde{R}_v K(u)(x),\ u, v \in G,\ x \in X.$$

We define the operator $T_K = \int_G K(u)R_{-u}du$. Then T_K is well defined as an element of $\mathcal{L}(X,Y)$ and

$$\|T_K\| \leq \inf_{1\leq p<\infty} (c_1 c_2 N_{p,X,Y}(K)),$$

where $N_{p,X,Y}(K)$ denotes the operator norm of the convolution operator defined by the kernel K from L^p_X into L^p_Y.

We shall apply this result in the following setting. Take $X = L^p_B(\mathbb{T})$, $Y = L^p_{L^p_B([0,1])}(\mathbb{T})$ and $G = \mathbb{R}$. For $u \in \mathbb{R}$, define R_u on X by $(R_u f)(e^{i\theta}) = f(e^{i(\theta+u)})$ for $f \in X$ and $\tilde{R}_u$ on Y by $(\tilde{R}_u g)(e^{i\theta}) = g(e^{i(\theta+u)})$ for $g \in Y$. Note that, in this case, $\|R_u\| = \|\tilde{R}_u\| = 1$. Let $k_j \in L^1(\mathbb{R})$ for $j = 1, \ldots, J$ and, for $u \in \mathbb{R}$ a.e., define $K(u) \in \mathcal{L}(X,Y)$ by $K(u)f = (\sum r_j k_j(u))f$. An easy computation gives that, in this situation, the transferred operator T_K satisfies $T_K e_n x = (\sum r_j \hat{k}_j(n)) e_n x$, where $x \in B$ and $e_n(e^{i\theta}) = e^{in\theta}$. Thus T_K is the operator from X to Y corresponding to the $\left(L^p_B(\mathbb{T}), L^p_{L^p_B([0,1])}(\mathbb{T})\right) = (X,Y)$ multiplier $m_K = \sum r_j \hat{k}_j|_{\mathbb{Z}}$. An application of Theorem 2.7 gives the following.

Theorem 2.8. *With the above notation, suppose that there is a constant C satisfying*

$$\| \sum r_j k_j * f \|_{L^p_{L^p_B([0,1])}(\mathbb{R})} \le C \|f\|_{L^p_B(\mathbb{R})} \tag{2.9}$$

for all $f \in L^p_B(\mathbb{R})$. Then

$$\|(m_K \hat{g})\check{}\,\|_{L^p_{L^p_B([0,1])}(\mathbb{T})} \le C \|g\|_{L^p_B(\mathbb{T})} \tag{2.10}$$

for all $g \in L^p_B(\mathbb{T})$.

Suppose now that B is a Banach space with the property LPR_p, where $2 \le p \le \infty$, and let $C_{p,B}$ be as in (1.4). For an interval I in $\mathbb{Z}$ of the form $[m,n]$, let ψ_I denote the function on $\mathbb{R}$ taking the value 1 on $(m-\frac{1}{4}, n+\frac{1}{4})$, $\frac{1}{2}$ at $m-\frac{1}{4}$ and at $n+\frac{1}{4}$, and 0 elsewhere. Also, let $\tilde{I}$ denote the interval $(m-\frac{1}{4}, n+\frac{1}{4})$ in $\mathbb{R}$. For a half-infinite interval I in $\mathbb{Z}$, define ψ_I and $\tilde{I}$ similarly. Note that each such function ψ_I is the normalized multiplier (in the sense of [C,W], p. 13) corresponding to $S_{\tilde{I}}$. Now fix disjoint intervals I_j in $\mathbb{Z}$ for $j = 1, \ldots, J$. By the property LPR_p for B, applied to the disjoint intervals $\tilde{I}_j$, we have that $m = \sum r_j \psi_{I_j}$ is an $\left(L^p_B(\mathbb{R}), L^p_{L^p_B([0,1])}(\mathbb{R})\right)$ multiplier with multiplier norm not exceeding $C_{p,B}$. It is straightforward to deduce that $m|_{\mathbb{Z}}$ is an $\left(L^p_B(\mathbb{T}), L^p_{L^p_B([0,1])}(\mathbb{T})\right)$ multiplier with multiplier norm not exceeding $C_{p,B}$ by adapting the arguments given in [C,W] to prove the multiplier restriction theorem ([C,W], Theorem 3.4). Since $\psi_I|_{\mathbb{Z}}$ equals the characteristic function of I, this gives (2.6) and completes the proof of Theorem 2.5.

3. An application to dimension free estimates

Let G be a compact connected abelian group with dual group Γ. Then Γ can be ordered in a non-canonical way so that it becomes an ordered group. Fix any such ordering $\le$ on Γ. Let B be a UMD space and let $1 < p < \infty$. It is well known that, for every interval I in Γ, the characteristic function χ_I is a multiplier for $L^p_B(G)$, see [Bo] and [B,G,M2] . Furthermore, the corresponding operator S_I has norm bounded by a constant dependent only on p and B. The result of the previous section can be extended to $L^p_B(G)$ as follows.

Theorem 3.11. *Let G and Γ be as above and let B have the property LPR_p for some p in the range $2 \le p < \infty$. Then there is a constant $C_{p,B}$ with the property that, for every finite family $\{I_k\}$ of disjoint intervals in Γ,*

$$\| \sum_k r_k S_{I_k} f \|_{L^p_{L^p_B([0,1])}(G)} \le C_{p,B} \|f\|_{L^p_B(G)} \tag{3.12}$$

for all $f \in L^p_B(G)$. In particular, the constant $C_{p,B}$ can be taken to be independent of the particular order on Γ and can be taken to be any constant that suffices for $L^p_B(\mathbb{T})$.

The proof of this result uses the techniques developed in [B,G,M1] and [B,G,T1]. Firstly, the result for $G = \mathbb{T}^n$ is obtained from the result for $\mathbb{T}$ (that is, from Theorem 2.5) using a vector valued transference argument. The structure of the torsion-free abelian group Γ then yields the result for general G. In particular, the transference from $\mathbb{T}$ to $\mathbb{T}^n$ gives the following theorem.

Theorem 3.13. *Let B have the property LPR_p for some p in the range $2 \le p < \infty$. Then there is a constant $C_{p,B}$ with the property that, for all $n \in \mathbb{N}$, all orderings on $\mathbb{Z}^n$, and every finite family $\{I_k\}$ of disjoint intervals in $\mathbb{Z}^n$, we have*

$$\| \sum_k r_k S_{I_k} f \|_{L^p_{L^p_B([0,1])}(\mathbb{T}^n)} \le C_{p,B} \|f\|_{L^p_B(\mathbb{T}^n)} \tag{3.14}$$

for all $f \in L^p_B(\mathbb{T}^n)$. The constant $C_{p,B}$ can be taken to be any constant that suffices for $L^p_B(\mathbb{T})$.

References

[B,G,M1] E. Berkson, T.A. Gillespie and P.S. Muhly, *Analyticity and spectral decompositions of L^p for compact abelian groups*, Pacific J. Math 127 (1987), 247-260.

[B,G,M2] E. Berkson, T.A. Gillespie and P.S. Muhly, *Generalized analyticity in UMD spaces*, Arkiv för Matematik 27 (1989), 1-14.

[B,G,T1] E. Berkson, T.A. Gillespie and J.L. Torrea, *Proof of a conjecture of José L. Rubio de Francia,* to appear in Math. Z.

[B,G,T2] E. Berkson, T.A. Gillespie and J.L. Torrea, *Vector Valued Transference,* Proceedings of International Conference on Functional Space Theory and its applications, Wuhan 2003, 1-27. Research Information Ltd UK

[Bo] S. Bochner, *Additive set functions on groups*, Annals of Math. 40 (1939), 769-799.

[Bou] J. Bourgain, *Some remarks on Banach spaces in which martingale sequences are unconditional*, Arkiv för Matematik 21 (1983), 163-168.

[B] D. Burkholder *A geometric condition that implies the existence of certain singular integrals of Banach-spaces valued functions*, Conference on Harmonic Analysis in Honor of A. Zygmund 1981, Wadsworth International Group, Belmont, California (1983), 270-286.

[C] A. P. Calderón, *Ergodic Theory and translation invariant operators,* Proc. Nat. Acad. Sci. (U.S.A.) 59 (1968), 349-353.

[CZ] A. P. Calderón and A. Zygmund, *On singular integrals,* Amer. J. Math. 78 (1956), 289-309.

[C,W] R.R. Coifman and G. Weiss, *Transference methods in Analysis*, C.B.M.S. Regional Conf. Series in Math. No.31, Amer. Math. Soc., Providence, R.I., 1977, reprinted 1986.

[Co] M. Cotlar, *A unified theory of Hilbert transforms and ergodic theorems*, Rev. Mat. Cuyana 1 (1955), 105-167.

[J] J.L. Journé, *Calderón-Zygmund Operators on Product Spaces*, Rev. Mat. Iberoamericana 1 (1985), 55-93.

[RdeF1] J.L.Rubio de Francia, *Martingale and integral transforms of Banach space valued functions,* Lecture Notes in Mathematics 1221, Springer-Verlag, Berlin. 1985, pp.195-222.

[RdeF2] J.L.Rubio de Francia, *A Littlewood-Paley inequality for arbitrary intervals,* Rev. Mat. Iberoamericana 2 (1985), 1-14.

T. A. Gillespie
School of Mathematics
James Clerk Maxwell Building
University of Edinburgh
Edinburgh EH9 3JZ, Scotland.
t.a.gillespie@ed.ac.uk

José L. Torrea
Departamento de Matemáticas
Facultad de Ciencias
Universidad Autónoma de Madrid
28049 Madrid, Spain
Joseluis.torrea@uam.es

Recibido: 11 de marzo de 2005
Revisado: 20 de marzo de 2005