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THE HADAMARD FRACTIONAL POWER IN MIKHLIN-BESOV INCLUSIONS

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This paper is dedicated to the memory of Angel Rafael Larotonda, deep mathematician and pioneer of Banach algebras in Argentina.

ABSTRACT. Banach algebras defined by fractional Mikhlin-type conditions are continuously contained in Besov spaces, in such a way that the difference between the corresponding degrees of derivation can be made arbitrarily small. In this note a proof of this inclusion is given which is based on the Hadamard fractional operator and its adjoint integration operator on the positive half-line.

§1. Introduction.

For $0 < \theta < \pi$, put $S_\theta := \{z \in \mathbf{C} \setminus \{0\} : |\arg z| < \theta\}$. Let $H^\infty(S_\theta)$ the usual Banach algebra of bounded, holomorphic functions on S_θ. Let X be a Banach space. An injective, closed operator A on X with domain and range dense is said to be a *sectorial operator of type* 0 if the spectrum $\sigma(A)$ is contained in $[0,\infty)$ and, for every $\theta > 0$,

$$\|\lambda(\lambda - A)^{-1}\| \leq C_\theta \qquad (\lambda \in S_\theta).$$

Let $\mathcal{L}(X)$ denote the Banach algebra of bounded operators on X. On the basis of the classical Dunford-Riesz formula, a certain mapping can be constructed, which takes functions f of $H^\infty(S_\theta)$ into closed, non necessarily bounded, operators $f(A)$ on X. When $f(A)$ is in $\mathcal{L}(X)$, then the correspondence $f \mapsto f(A)$ defines a bounded Banach algebra homomorphism $H^\infty(S_\theta) \to \mathcal{L}(X)$, and we say that A has an H^∞ functional calculus [CDMY], [M]. The existence of the (suitably scaled) H^∞ calculus for A is shown to be equivalent to the existence of a functional calculus defined on the Besov space $\Lambda^\alpha_{\infty,1}(\mathbf{R}^+)$ in the following terms.

For all $\xi \in \mathbf{R}$, let define

$$\phi_0(\xi) := (2 - 2|\xi|)_+ - (1 - 2|\xi|)_+; \quad \phi_1(\xi) = (1 - 2|\xi - 1|)_+ + (1/2 - |\xi - 3/2|)_+,$$

and $\phi_{k\varepsilon}(\xi) = \phi_1(2^{1-k}\ \varepsilon\ \xi)$ if $k \in \mathbf{N}$, $\varepsilon \in \{-1, +1\}$. Let $\alpha > 0$. The Besov space $\Lambda^\alpha_{\infty,1}(\mathbf{R}^+)$ is defined as the set of all bounded continuous functions f on $\mathbf{R}^+$ such that

$$\|f\|_{\Lambda,\alpha} := \sum_{k=-\infty}^{\infty} 2^{|k|\alpha} \|f_e * \check{\phi}_k\|_\infty < \infty$$

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where $f_e(x) := f(e^x)$, $x \in \mathbf{R}$, see [CDMY, p. 73]. Then $\Lambda^{\alpha}_{\infty,1}(\mathbf{R}^+)$ is a Banach space with respect to the norm $\|f\|_{\Lambda,\alpha}$. Moreover, it is a Banach algebra with respect to pointwise multiplication [BL].

Theorem A. *([CDMY, Theorem 4.10]) Suppose that A is an injective, sectorial operator of type 0. Then the following conditions are equivalent:*

(i) *There exist constants $\alpha, C > 0$ such that, for every $\theta > 0$ the operator A has a functional calculus $H^\infty(S_\theta) \to \mathcal{L}(X)$ with*
$$\|f(A)\| \le C\theta^{-\alpha}\|f\|_\infty$$
for every $f \in H^\infty(S_\theta)$.

(ii) *There exists a functional calculus $\Lambda^{\alpha}_{\infty,1}(\mathbf{R}^+) \to \mathcal{L}(X)$.*

The above result quantifies H^∞ holomorphy on all sectors in terms of Mikhlin-Hörmander conditions [CDMY, p. 75]. The connection between Mikhlin-Hörmander conditions and Besov spaces is exemplified in [CDMY, p. 73] by noticing that every N-differentiable function f on $\mathbf{R}^+$ obeying Mikhlin conditions like
$$\sup_{t>0} t^j |f^{(j)}(t)| < \infty \qquad (j = 0, 1, ..., N).$$
belongs to $\Lambda^{\alpha}_{\infty,1}$, provided that $N > \alpha$. A proof of this property relies on elementary, but not trivial, Fourier analysis as indicated in [CDMY]. This consists of considering the integration operator $\mathcal{I}f(x) = \int_{-\infty}^x f(y)dy$, and then using the estimate $\|\mathcal{I}^N \check{\phi}_k\|_1 \le C_N 2^{-|k|N}$ in the product $f_e * \check{\phi}_k = f_e^{(N)} * \mathcal{I}^N \phi_k$. In [GM], a closer look at the relationships between Mikhlin conditions and Besov spaces is done using fractional derivation. Let h be a locally integrable function on $\mathbf{R}^+$. Suppose that $\alpha > 0$, $n = [\alpha]$ and $\alpha = n + \delta$ with $0 < \delta < 1$. If $\omega > 0$, we put
$$I^{\delta}_{\omega} h(t) := \frac{1}{\Gamma(\delta)} \int_t^{\omega} (s-t)^{\delta-1} h(s)\, ds,$$
when $0 < t < \omega$, and $I^{\delta}_{\omega} h(t) := 0$ when $t \ge \omega$. Then, let define
$$h^{(\delta)}(t) := \lim_{\omega\to\infty} (-\frac{d}{dt})(I^{1-\delta}_{\omega} h)(t).$$
and
$$h^{(\alpha)}(t) := (\frac{d}{dt})^n h^{(\alpha-n)}(t), \qquad t > 0,$$
whenever the two right-hand side members of the equalities exist. This definition is from [C], see also [GT]. In the sequel, when we consider $h^{(\alpha)}$, we will be assuming that the above limit and derivatives exist and that $I^{1-\delta}_{\omega} h$, for $\omega > 0$, and $h^{(\delta)}, \dots, h^{(\alpha-1)}$ are locally absolutely continuous functions on $\mathbf{R}^+$.

Let $WBV_{\infty,\alpha}$ denote the set of functions so-called of *weak bounded variation*, formed by all functions h in $L^\infty \cap C(\mathbf{R}^+)$ for which there exists $h^{(\alpha)}$ and $\|h\|_{\infty,\alpha} := \|h\|_\infty + \|t^\alpha h^{(\alpha)}(t)\|_\infty < \infty$. Then $WBV_{\infty,\alpha}$ is a Banach space with respect to the norm $\|\cdot\|_{\infty,\alpha}$ and we have continuous inclusions $WBV_{\infty,\beta} \hookrightarrow WBV_{\infty,\alpha}$ if $0 < \alpha \le \beta$, see [GT].

Let m be an integer. Suppose that F is a bounded, $C^{(m)}$ function on $\mathbf{R}^+$ such that $\sup_{s>0} |F^{(m)}(s)s^m| < \infty$. Then, for every $\varepsilon > 0$, there exists $f \in C^{(\infty)}(\mathbf{R}^+)$ such that $\sup_{s>0} |F^{(k)}(s) - f^{(k)}(s)| s^k < \varepsilon$ for $k = 0, m$. To see this, take h in $C^{(\infty)}(\mathbf{R}^+)$ such that $|F^{(m)}(x) - h(x)| \le \varepsilon(1+x^2)^{-(m+1)}$ and define

$$\varphi(s) = (-1)^m \int_s^\infty \int_{s_2}^\infty \cdots \int_{s_m}^\infty \left[F^{(m)}(s_{m+1}) - h(s_{m+1}) \right] ds_{m+1} \ldots ds_2,$$

for every $s > 0$. It is not difficult to check that $\varphi \in C^{(m)}(\mathbf{R}^+)$, $\varphi^{(m)} = F^{(m)} - h$, and $|\varphi^{(k)}(s)| \le (\pi/2)^{m-k} \varepsilon (1+s^2)^{-(k+1)}$ for $k = 0, 1, \ldots, m$ and $s > 0$. If $f := F - \varphi$ then $f \in C^{(m)}(\mathbf{R}^+)$ and $f^{(m)} = F^{(m)} - \varphi^{(m)} = h$, so the function f is a $C^{(\infty)}$ function. Moreover,

$$\sup_{s>0} |(F-f)^{(m)}(s)| s^m = \sup_{s>0} |F^{(m)}(s) - h(s)| s^m \le \sup_{s>0} \left(\frac{s^m}{(1+s^2)^{m+1}} \right) \varepsilon < \varepsilon$$

and $\|F - f\|_\infty \le (\frac{\pi}{2})^m \varepsilon$.

This approximation property motivates our definition of Mikhlin-type space. For $\alpha > 0$, let $\mathcal{M}_\infty^{(\alpha)}$ denote the closure in $WBV_{\infty,\alpha}$ of $WBV_{\infty,\alpha} \cap C^{(\infty)}(\mathbf{R}^+)$. We call here $\mathcal{M}_\infty^{(\alpha)}$ the *Mikhlin* space of order α. Using the Leibniz formula of [GP, p. 316], it can be proved that $\mathcal{M}_\infty^{(\alpha)}$ is a Banach algebra for pointwise multiplication. This is given in [GM] together with the following theorem and corollary.

Theorem B. *For every $\alpha > \beta > 0$,*

$$\mathcal{M}_\infty^{(\alpha)} \hookrightarrow \Lambda_{\infty,1}^{\beta}(\mathbf{R}^+) \hookrightarrow \mathcal{M}_\infty^{(\beta)}.$$

On the other hand, it happens that $H^\infty(S_\theta) \hookrightarrow \mathcal{M}_\infty^{(\alpha)}$ for all $\theta, \alpha > 0$. This is also shown in [GM] using a sort of Cauchy formula for fractional derivatives of holomorphic functions.

Theorem B and the above remark tell us to what extent Besov calculus and Mikhlin calculus are close to each other.

Corollary. *Suppose that A is a sectorial operator of type 0. Let $\beta > 0$. Then the following assertions are equivalent.*

(i) *For all $\theta > 0$ there is a functional calculus for A, $H^\infty(S_\theta) \hookrightarrow \mathcal{L}(X)$, such that*

$$\|h(A)\| \le C_\alpha \theta^{-\alpha} \|h\|_\infty$$

for every $h \in H^\infty(S_\theta)$, if $\alpha > \beta$, where C_α is a constant which only depends on α.

(ii) *A has a functional calculus $\Lambda_{\infty,1}^{\alpha}(\mathbf{R}^+) \hookrightarrow \mathcal{L}(X)$ for every $\alpha > \beta$.*

(iii) *A admits a functional calculus $\mathcal{M}_\infty^{(\alpha)} \hookrightarrow \mathcal{L}(X)$ for every $\alpha > \beta$.*

The continuous inclusion $\mathcal{M}_\infty^{(\alpha)} \hookrightarrow \Lambda_{\infty,1}^{\beta}(\mathbf{R}^+)$ of Theorem B is proved in [GM] by considering a suitable integral expression of the Besov space, see [P, pp. 9, 11]. In the present paper, a different argument is given, where the key tool is the Hadamard fractional power $(-s\frac{d}{ds})^\alpha$ on $\mathbf{R}^+$ (let us recall that, for integer n, the operator $(s(d/ds))^n$ on $\mathbf{R}^+$ corresponds to usual derivation $(d/ds)^n$ on $\mathbf{R}$ through the change of variable $s = \exp(x)$, $x \in \mathbf{R}$).

In order to make the link between $\Lambda^{\beta}_{\infty,1}(\mathbf{R}^+)$ and $\mathcal{M}^{(\alpha)}_{\infty}$ explicit, we will work directly on $\mathbf{R}^+$. Thus we redefine $\Lambda^{\beta}_{\infty,1}(\mathbf{R}^+)$ as the Banach algebra of all functions in $L^\infty \cap C(\mathbf{R}^+)$ normed by

$$\|f\|_{\Lambda,\beta} := \sum_{k=-\infty}^{\infty} 2^{|k|\beta} \|f * \sigma_k\|_\infty$$

where $\sigma_k(s) := \check{\phi}_k(\log s)$ and $(f * \sigma_k)(s) := \int_0^\infty f(s/t)\ \sigma_k(t)\ \frac{dt}{t}$ for every $s > 0$ and $k \in \mathbf{N}$.

Our argument to prove $\mathcal{M}^{(\alpha)}_{\infty} \hookrightarrow \Lambda^{\beta}_{\infty,1}(\mathbf{R}^+)$, $\alpha > \beta$, follows the pattern indicated in [CDMY] and so it consists of, roughly speaking, showing that

$$\|\mathcal{J}^\alpha \sigma_k\|_1 \le C_{\alpha,k}(2^{-|k|\alpha})$$

and then applying this estimate in the convolution

$$\int_0^\infty f(s/t)\ \sigma_k(t)\ \frac{dt}{t} = \int_0^\infty (-t\frac{d}{dt})^\alpha f(s/.)(t)\ \mathcal{J}^\alpha \sigma_k(t)\ \frac{dt}{t}, \tag{1.1}$$

for every $f \in \mathcal{M}^{(\alpha)}_{\infty}$. Here, $\mathcal{J}^\alpha$ is the adjoint operator of $(-s\frac{d}{ds})^\alpha$ [SKM, p. 330] (see below), and $\|\mathcal{J}^\alpha h_k\|_1$ refers to the standard norm in $L^1(\mathbf{R}^+, ds/s)$.

The equality (1.1) is not a trivial one because $f(s/.), \sigma_k$ do not satisfy the general conditions which make formulae on integration by parts in $\mathbf{R}^+$ valid, see [BKT]. This note is devoted to prove equality (1.1). Doing so, the paper is partly intended as an example of the usage of fractional tools in Fourier analysis.

§2. Some Fourier analysis.

The results of this section are of a rather expected nature. Their proofs are included here for the sake of completeness.

Let φ be a finite linear combination of tent functions on $\mathbf{R}$ with $0 \notin \operatorname{supp} \varphi$.

Lemma 2.1. *For $\alpha \in \mathbf{R}$,*

$$F_\alpha(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi}\ \varphi(\xi)\ |\xi|^{-\alpha}\ d\xi$$

is an entire function in $x \in \mathbf{R}$ such that $F_\alpha(x) = O(x^{-2})$ as $|x| \to \infty$.

Proof: We can assume that φ is a single tent function. Suppose that $\operatorname{supp} \varphi = [a,b]$ with $0 < a < b$, and that φ has its pick at $c \in (a,b)$. Put $J_\alpha(x) := \int_I e^{ix\xi}\ \xi^{-\alpha}\ d\xi$, $(x, \alpha \in \mathbf{R})$ where either $I = [a,c]$ or $I = [c,b]$. Then J_α is bounded on $\mathbf{R}$ for every α whence, integrating by parts, we have that $J_\alpha(x) = O(x^{-1})$ as $|x| \to \infty$, for every α.

Analogously, there exist two constants $C, D > 0$ such that

$$ix\ F_\alpha(x)\ :=\ C\int_a^c e^{ix\xi}\ \xi^{-\alpha}\ d\xi + D\int_c^b e^{ix\xi}\ \xi^{-\alpha}\ d\xi + \alpha\ F_{\alpha+1}(x) \tag{2.1}$$

and therefore $F_\alpha(x) = O(x^{-1})$ as $|x| \to \infty$, for all α. Using this estimate once more in (2.1) we obtain that $F_\alpha(x) = O(x^{-2})$ as $|x| \to \infty$.

Finally, if $\operatorname{supp} \varphi \subset (-\infty, 0)$ then $F_\alpha(x) := \int_{-\infty}^{\infty} e^{-ix\eta}\ \varphi(-\eta)\ |\eta|^{-\alpha}\ d\eta$, so $F_\alpha(x) = O(x^{-2})$ as $|x| \to \infty$ from the above. ∎

From now on, g will denote the inverse Fourier transform of φ. We need to consider the integration operator, acting on g, which is defined by

$$\mathcal{I}g(x) := \int_{-\infty}^{x} g(y)\ dy; \quad \mathcal{I}^{\delta}g(x) := \frac{1}{\Gamma(\delta)}\int_{-\infty}^{x}(x-y)^{\delta-1}g(y)\ dy, \qquad x\in\mathbf{R},$$

and

$$\mathcal{I}^{\alpha} = \mathcal{I}^{n}\mathcal{I}^{\delta}; \qquad \mathcal{I}^{-\alpha} = \frac{d^{n+1}}{dx^{n+1}}\mathcal{I}^{1-\delta}$$

if $\alpha = n+\delta \geq 0$ with $n=[\alpha]$, $0\leq\delta<1$.

Lemma 2.2. *Let α, g be as above. Then $\mathcal{I}^{\alpha}g = c_{\alpha}F_{\alpha}$ for a certain constant c_{α}, and therefore $\mathcal{I}^{\alpha}g(x) = O(x^{-2})$ as $|x|\to\infty$.*

Proof: It is enough to assume that $\varphi=\hat{g}$ is a single tent. Suppose first that $\text{supp}(\phi)\subset(0,\infty)$. Take $0<\delta<1$. Then $\mathcal{I}^{\delta}g = \frac{1}{\Gamma(\delta)}y_{+}^{\delta-1} * g$ and therefore

$$(\mathcal{I}^{\delta}g)\widehat{\ }(\xi) \ = \ \frac{1}{\Gamma(\delta)}\widehat{y_{+}^{\delta-1}}(\xi)\hat{g}(\xi) \ = \ (i\xi)^{-\delta}\ \hat{g}(\xi),$$

for every $\xi\in\mathbf{R}$. Hence $\mathcal{I}^{\delta}g \ = \ e^{-i\delta(\pi/2)}F_{\delta}$.

Let $\alpha=n+\delta$ where $n=[\alpha]$. By Lemma (2.1), $\mathcal{I}^{\delta}g(x) = O(x^{-2})$ as $|x|\to\infty$ and then, integrating by parts, $\mathcal{I}(\mathcal{I}^{\delta+1}g') = \mathcal{I}^{\delta}g$. It follows that

$$(\mathcal{I}^{\delta+1}g)' = \mathcal{I}(\mathcal{I}^{\delta+1}g') = \mathcal{I}^{\delta}g = e^{-i\delta(\pi/2)}F_{\delta} = (e^{-i(\delta+1)(\pi/2)}F_{\delta+1})'.$$

Hence, there exists a constant C such that $\mathcal{I}^{\delta+1}g = e^{-i(\delta+1)(\pi/2)}F_{\delta+1}+C$. As $\mathcal{I}^{\delta+1}g(-\infty) = 0 = F_{\delta+1}(-\infty)$ we get $C=0$ and so $\mathcal{I}^{\delta+1}g = e^{-i(\delta+1)(\pi/2)}F_{\delta+1}$. Proceeding by induction we obtain that $\mathcal{I}^{\alpha}g = e^{-i\alpha(\pi/2)}F_{\alpha}$.

Now, through derivation under integral we have that $\mathcal{I}^{-\alpha}g = (d^{n+1}/dx^{n+1})\mathcal{I}^{1-\delta}g = e^{-i(1-\delta)(\pi/2)}(d^{n+1}/dx^{n+1})F_{\delta-1} = e^{-i(\delta+n)(\pi/2)}F_{-(\delta+n)} = e^{-i\alpha(\pi/2)}F_{-\alpha}$.

If supp $\varphi\subset(-\infty,0)$ then we obtain in a similar way that $\mathcal{I}^{\alpha}g \ = \ e^{i\alpha(\pi/2)}F_{\alpha}$. ■

REMARKS.- (i) We have used the well known fact that the distributional Fourier transform of $\frac{1}{\Gamma(\delta)}y_{+}^{\delta-1}$ is $(i\xi)^{-\delta}$, at each $\xi\in\mathbf{R}$. Such a calculation can be done with fractional calculus, via the fractional semigroup: Let $\mathcal{L}$ denote the Laplace transform. If ϕ is in the Schwarz class then

$$\begin{aligned}\langle\frac{1}{\Gamma(\delta)}y_{+}^{\delta-1},\hat{\phi}\rangle &= \lim_{\varepsilon\to0^{+}}\frac{1}{\Gamma(\delta)}\int_{0}^{\infty}y^{\delta-1}e^{-\varepsilon y}\hat{\phi}(y)dy\\ &= \lim_{\varepsilon\to0^{+}}\int_{-\infty}^{\infty}\mathcal{L}\left(\frac{y^{\delta-1}e^{-\varepsilon y}}{\Gamma(\delta)}\right)(i\xi)\phi(\xi)d\xi\\ &= \lim_{\varepsilon\to0^{+}}\int_{-\infty}^{\infty}(\varepsilon+i\xi)^{-\delta}\phi(\xi)\ d\xi = \int_{-\infty}^{\infty}(i\xi)^{-\delta}\phi(\xi)\ d\xi.\end{aligned}$$

(ii) Incidentally, it follows from Lemma 2.2 that $\mathcal{I}^{\alpha+\beta}g = \mathcal{I}^{\alpha}\mathcal{I}^{\beta}g$ for all $\alpha,\beta\in\mathbf{R}$.

To finish this section we give the following inversion result.

Lemma 2.3. *For $0<\delta<1$ and g as above,*

$$g(x) = \frac{1}{\Gamma(-\delta)}\int_{0}^{\infty}\frac{(\mathcal{I}^{\delta}g)(x-y)-(\mathcal{I}^{\delta}g)(x)}{y^{1+\delta}}dy$$

for every $x \in \mathbf{R}$.

Proof: Take $x \in \mathbf{R}$. Let $\mathcal{D}\mathcal{I}^\delta g(x)$ denote the integral of the statement ($\mathcal{D}\mathcal{I}^\delta$ is a Marchaud type derivative, see [SKM]). By Lemma 2.1 and Lemma 2.2, $\mathcal{I}^\delta g(x) = O(x^{-2})$ as $|x| \to \infty$ and then we have

$$\mathcal{D}\mathcal{I}^\delta g(x) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dx} \int_{-\infty}^{x} (x-y)^{-\delta} (\mathcal{I}^\delta g)(y) dy,$$

see [SKM, pp. 109]. Thus

$$\begin{aligned} \mathcal{D}\mathcal{I}^\delta g(x) &= \tfrac{d}{dx}\left(\mathcal{I}^{1-\delta}(\mathcal{I}^\delta g)\right)(x) = \tfrac{d}{dx}(\mathcal{I}^1 g)(x) = \tfrac{d}{dx}\int_{-\infty}^{0} g(x+u)du \\ &= \int_{-\infty}^{0} g'(x+u)du = g(x) - g(-\infty) = g(x). \end{aligned}$$

■

§3. The Hadamard fractional operator.

Let g be as above and put $h(s) := g(\log s)$ if $s > 0$. For $\alpha = n + \delta$, where $n = [\alpha]$, let define

$$\mathcal{J}^\alpha h(s) := \frac{1}{\Gamma(\delta)} \int_0^s \int_0^{s_1} \cdots \int_0^{s_n} \left(\log\frac{s}{t}\right)^{\delta-1} h(t)\, \frac{dt}{t} \frac{ds_n}{s_n} \cdots \frac{ds_1}{s_1} \frac{ds}{s}, \tag{3.1}$$

see [SKM, p. 330], [BKT]. Note that $\mathcal{J}^\alpha h(s) = \mathcal{I}^\alpha g(\log s)$ if $s > 0$. Hence $\mathcal{J}^\alpha h(s) = O\left((\log s)^{-2}\right)$ as $s \to 0^+, \infty$.

The *Hadamard fractional power* $(-s\frac{d}{ds})^\alpha$ is the adjoint operator of $\mathcal{J}^\alpha$, and it is explicitely given by

$$(-s\frac{d}{ds})^\delta f(s) = \frac{1}{\Gamma(-\delta)} \int_s^\infty \frac{f(t) - f(s)}{(\log(t/s))^{1+\delta}} \frac{dt}{t}$$

and

$$(-s\frac{d}{ds})^\alpha f = (-s\frac{d}{ds})^n\, (-s\frac{d}{ds})^\delta f,$$

for $f \in \mathcal{M}_\infty^{(\alpha)} \cap C^{(n+1)}(\mathbf{R}^+)$ in particular [SKM, p. 332]. The interest of the Hadamard operator in this paper relies on the following result.

Lemma 3.1. *([GM, Corollary 2.3]) Let* $\alpha > 0$. *Then* $\sup_{0 \le \gamma \le \alpha} \sup_{s>0} |(-s\frac{d}{ds})^\gamma f(s)|$ *defines a norm in* $\mathcal{M}_\infty^{(\alpha)}$ *which is equivalent to* $\|\cdot\|_{\infty,\alpha}$.

As said before, we want to prove the formula

$$\int_0^\infty f(s)\, h(s)\, \frac{ds}{s} = \int_0^\infty (-s\frac{d}{ds})^\alpha f(s)\, \mathcal{J}^\alpha h(s)\, \frac{ds}{s} \tag{3.2}$$

for $f \in \mathcal{M}_\infty^{(\alpha)} \cap C^{(n+1)}(\mathbf{R}^+)$. According to the lemma, and the fact that $h(s)$ and $\mathcal{J}^\alpha h(s)$ are $\mathrm{O}\big((\log s)^{-2}\big)$ as $s \to 0^+, \infty$, both integrals are defined.

Lemma 3.2. *Let* $\alpha = n + \delta$, $n = [\alpha]$, $f \in \mathcal{M}_\infty^{(\alpha)} \cap C^{(n+1)}(\mathbf{R}^+)$. *Then*

$$\int_0^\infty (-s\frac{d}{ds})^\alpha f(s)\, \mathcal{J}^\alpha h(s)\, \frac{ds}{s} = \int_0^\infty (-s\frac{d}{ds})^\delta f(s)\, \mathcal{J}^\delta h(s)\, \frac{ds}{s}.$$

Proof: Note that $\mathcal{J}^k h(0) = \mathcal{I}^k g(-\infty) = 0 = \mathcal{I}^k g(+\infty) = \mathcal{J}^k h(+\infty)$ in particular for every $k = \delta + 1, ..., \delta + n$. Also, $\sup_{s>0} |(s\frac{d}{ds})^k f(s)| < \infty$ for $k = \delta, \delta + 1, ..., \delta + n$ by Lemma 3.1. Now, it is sufficient to integrate by parts n times in the integral of the right hand member of the equality. ∎

Lemma 3.3. *For $\rho > 1$, set*

$$H_{1,\rho} := \frac{1}{\Gamma(-\delta)} \int_0^\infty \int_s^{\rho s} \frac{f(t) - f(s)}{(\log(t/s))^{1+\delta}} \, \frac{dt}{t} \, \mathcal{J}^\delta h(s) \, \frac{ds}{s}.$$

Then

$$\lim_{\rho \to 1} H_{1,\rho} \; = \; 0.$$

Proof: Let consider $H_{1,\rho}$ in the form

$$H_{1,\rho} = \frac{1}{\Gamma(-\delta)} \int_0^\infty \int_1^{\rho} \frac{f(ts) - f(s)}{(\log t)^{1+\delta}} \, \frac{dt}{t} \, \mathcal{J}^\delta h(s) \, \frac{ds}{s}.$$

Take ε such that $\delta < \varepsilon < 1$. By [GT, p. 256],

$$f(ts) - f(s) = \frac{1}{\Gamma(\varepsilon)} \int_0^\infty \{(u - ts)_+^{\varepsilon - 1} - (u - s)_+^{\varepsilon - 1}\} f^{(\varepsilon)}(u) \, du,$$

for every t such that $1 \le t \le \rho$. From this, it follows that $|f(ts) - f(s)| \le C_\varepsilon (t-1)^\varepsilon$ uniformly in $s > 0$ [GM]. Hence,

$$\begin{aligned} |H_{1,\rho}| &\le C_\varepsilon \int_0^\infty \int_1^\rho (t-1)^\varepsilon (\log t)^{-(1+\delta)} t^{-1} \, dt \, |\mathcal{J}^\delta h(s)| s^{-1} \, ds \\ &= C_\varepsilon \left(\int_1^\rho (t-1)^\varepsilon (\log t)^{-(1+\delta)} t^{-1} \, dt \right) \left(\int_0^\infty |\mathcal{J}^\delta h(s)| s^{-1} \, ds \right) \longrightarrow_{\rho \to 1^+} 0 \end{aligned}$$

since the function $(t-1)^\varepsilon (\log t)^{-(1+\delta)} t^{-1}$ is integrable on $(1, 2)$ and the function $\mathcal{J}^\delta h(s) \; s^{-1}$ is integrable on $(1, \infty)$. ∎

Lemma 3.4. *For $\rho > 1$, set*

$$H_{2,\rho} := \frac{1}{\Gamma(-\delta)} \int_0^\infty \int_{\rho s}^\infty \frac{f(t) - f(s)}{(\log(t/s))^{1+\delta}} \, \frac{dt}{t} \, \mathcal{J}^\delta h(s) \, \frac{ds}{s}.$$

Then

$$\lim_{\rho \to 1} H_{2,\rho} \; = \; \int_0^\infty f(t) \; h(t) \; \frac{dt}{t}.$$

Proof: Since $\mathcal{J}^\delta h(s) \; s^{-1}$ is integrable on $(1, \infty)$ we can apply Fubini Theorem to obtain

$$\int_0^\infty \int_{\rho s}^\infty \frac{f(t)}{(\log(t/s))^{1+\delta}} \frac{dt}{t} \mathcal{J}^\delta h(s) \frac{ds}{s} = \int_0^\infty \int_0^{t/\rho} \frac{\mathcal{J}^\delta h(s)}{(\log(t/s))^{1+\delta}} \frac{ds}{s} f(t) \frac{dt}{t}.$$

On the other hand,

$$\int_0^\infty \int_{\rho s}^\infty f(s) \frac{dt}{t(\log(t/s))^{1+\delta}} \mathcal{J}^\delta h(s) \frac{ds}{s} = \int_0^\infty \delta^{-1} (\log \rho)^{-\delta} f(s) \mathcal{J}^\delta h(s) \frac{ds}{s}$$

$$= \int_0^\infty \int_0^{s/\rho} \frac{\mathcal{J}^\delta h(s)}{(\log(s/u))^{1+\delta}} \frac{du}{u} f(s) \frac{ds}{s} \equiv \int_0^\infty \int_0^{t/\rho} \frac{\mathcal{J}^\delta h(t)}{(\log(t/s))^{1+\delta}} \frac{ds}{s}\ f(t) \frac{dt}{t}.$$

Hence

$$H_{2,\rho} = \frac{1}{\Gamma(-\delta)} \int_0^\infty \int_0^{t/\rho} \frac{\mathcal{J}^\delta h(s) - \mathcal{J}^\delta h(t)}{(\log(t/s))^{1+\delta}} \frac{ds}{s}\ f(t) \frac{dt}{t} = K_{2,\rho} + L_{2,\rho}$$

where

$$K_{2,\rho} = \frac{1}{\Gamma(-\delta)} \int_0^\infty \int_0^{t} \frac{\mathcal{J}^\delta h(s) - \mathcal{J}^\delta h(t)}{(\log(t/s))^{1+\delta}} \frac{ds}{s}\ f(t) \frac{dt}{t},$$

$$L_{2,\rho} = \frac{1}{\Gamma(-\delta)} \int_0^\infty \int_{t/\rho}^{t} \frac{\mathcal{J}^\delta h(s) - \mathcal{J}^\delta h(t)}{(\log(t/s))^{1+\delta}} \frac{ds}{s}\ f(t) \frac{dt}{t}.$$

After suitable change of variable in Lemma 2.3 we get that

$$h(t) = \frac{1}{\Gamma(-\delta)} \int_0^{t} \frac{\mathcal{J}^\delta h(s) - \mathcal{J}^\delta h(t)}{(\log(t/s))^{1+\delta}} \frac{ds}{s}$$

for $t > 0$, whence

$$K_{2,\rho} = \int_0^\infty f(t)\ h(t)\ \frac{dt}{t}.$$

So, to prove the lemma, it only remains to show that $\lim_{\rho\to 1} L_{2,\rho} = 0$.

First, let us write $L_{2,\rho}$ as

$$\begin{aligned} L_{2,\rho} &= \int_0^\infty \int_{1/\rho}^{1} \frac{\mathcal{J}^\delta h(ts) - \mathcal{J}^\delta h(t)}{(-\log s)^{1+\delta}} \frac{ds}{s}\ f(t) \frac{dt}{t} \\ &= \int_0^\infty \int_{1/\rho}^{1} \int_t^{st} (\mathcal{J}^\delta h)'(u) du\ \frac{ds}{s(-\log s)^{1+\delta}}\ \frac{dt}{t}. \end{aligned}$$

Recall that $\mathcal{J}^\alpha h(s) = O\left((\log s)^{-2}\right)$, as $s \to 0^+, \infty$. This automatically implies that $(\mathcal{J}^\alpha h)'(s) = O\left(s(\log s)^{-2}\right)$, as $s \to 0^+, \infty$. Thus, if $t > 0$ and $(1/\rho) < s < 1$,

$$\int_{st}^t |(\mathcal{J}^\delta h)'(u)| du \le C \int_{st}^t \frac{du}{u(\log u)^2} = C\{\frac{1}{\log(st)} - \frac{1}{\log t}\} = C \frac{|\log s|}{|\log t|\ |\log st|}.$$

Now, if $0 < t < (1/2)$ and $(1/\rho) < s < 1$ then $\log(st) < \log t < 0$ and therefore $|\log(st)|^{-1} < |\log t|^{-1}$. For $t > 2$ and $(1/\rho) < s < 1$, take $1 < \rho < (3/2)$. Then $0 < \log(2t/3) < \log(st)$ whence $|\log(st)|^{-1} < |\log(2t/3)|^{-1}$.

Put $M := \sup_{(1/2)<u<2} |\mathcal{J}^\delta h)'(u)|$. We have

$$\begin{aligned} L_{2,\rho} \le C &\left(\int_{1/\rho}^1 \frac{ds}{s(-\log s)^\delta} \right) \left(\int_0^{1/2} \frac{dt}{t(\log t)^2} + \int_2^\infty \frac{dt}{t(\log t)(\log(2t/3)} \right) \\ &+ \int_{1/2}^2 M \int_{1/\rho}^1 (1-s) \frac{ds}{s(-\log s)^{1+\delta}} dt \\ &\equiv C_1 (\log \rho)^\delta + C_2 \int_{1/\rho}^1 (1-s) \frac{ds}{s(-\log s)^{1+\delta}} \longrightarrow 0 \end{aligned}$$

as $\rho \to 1^+$, as we wanted to show. ■

§4. Proof of the inclusion $\mathcal{M}_\infty^{(\alpha)} \hookrightarrow \Lambda_{\infty,1}^{\beta}(\mathbf{R}^+)$.

Let $\beta > 0$. Recall that $\Lambda_{\infty,1}^{\beta}(\mathbf{R}^+)$ is normed by $\|f\|_{\Lambda,\beta} = \sum_{k=-\infty}^{\infty} 2^{|k|\beta}\|f * \sigma_k\|_\infty$. For convenience, we express the convolution in a slightly different way. This is

$$(f * \sigma_k)(s) = \int_0^\infty f(s/t)\ \sigma_k(t)\ \frac{dt}{t} = \int_0^\infty f_s(u)\ h_k(u)\ \frac{du}{u}$$

where $f_s(u) := f(su)$ and $h_k(u) := \sigma_k(1/u)$ $(u, s > 0)$. Note that $h_k(u) := \check{\phi}_k(-\log u)$ or, alternatively, $h_k(u) = g_k(\log u)$ where $g_k(x) := \check{\phi}_k(-x)$ if $x \in \mathbf{R}$.

Let $\alpha > \beta$ with $n = [\alpha]$ and $\delta = \alpha - n$. Take f in $\mathcal{M}_\infty^{(\alpha)} \cap C^{n+1}(\mathbf{R}^+)$. Let $s > 0$. Using the Marchaud formula

$$f^{(\alpha)}(u) = \frac{1}{\Gamma(-\delta)}\frac{d^n}{du^n}\int_0^\infty \frac{f(t+u) - f(u)}{t^{1+\delta}}dt.$$

(see [GT, p. 256]) we have that $f_s^{(\alpha)}(u) = s^\alpha f^{(\alpha)}(su)$ for all $u > 0$. Hence $f_s \in \mathcal{M}_\infty^{(\alpha)}$ with $\|f_s\|_{\infty,\alpha} = \|f\|_{\infty,\alpha}$ for every $s > 0$.

Thus we can apply, and we do, the results of Section 3 to $f \equiv f_s$ and $h \equiv h_k$, with $s > 0$, $k \in \mathbf{Z}$.

By formula (3.2),

$$(f * \sigma_k)(s) = \int_0^\infty (-u\frac{d}{du})^\alpha f_s(u)\ \mathcal{J}^\alpha h_k(u)\ \frac{du}{u}$$

whence, from Lemma (3.1), we obtain that

$$|(f * \sigma_k)(s)| \le C\|f\|_{\infty,\alpha}\int_0^\infty |\mathcal{J}^\alpha h_k(u)|\ \frac{du}{u}$$

for all $s > 0$, $k \in \mathbf{Z}$. Now, writing $h_k(u) := \check{\phi}_k(-\log u) = \pm 2^{|k|-1}\check{\phi}_1(\mp 2^{|k|-1}\log u)$ in formula (3.1), or, alternatively, the corresponding expression of g_k in formula $\mathcal{I}^\alpha = \mathcal{I}^n\mathcal{I}^\delta$, it is straightforward to check that

$$\|\mathcal{J}^\alpha h_k\|_1 = C\|\mathcal{J}^\alpha h_1\|_1 = C_\alpha 2^{-|k|\alpha}.$$

Since $\alpha > \beta$ it implies that $\|f\|_{\Lambda,\beta} \le C\|f\|_{\infty,\alpha}$ as we wanted to show. ■

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