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EXTREMAL SPACES RELATED TO SCHRÖDINGER OPERATORS WITH POTENTIALS SATISFYING A REVERSE HÖLDER INEQUALITY

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ABSTRACT. We describe some elements of the theory of semigroups generated by second order differential operators needed to study the Hardy-type space $H^1_{\mathcal{L}}$ related to the time independent Schrödinger operator $\mathcal{L} = -\Delta + V$, with $V \geq 0$ a potential satisfying a reverse Hölder inequality. Its dual space is a BMO-type space $BMO_{\mathcal{L}}$, that turns out to be the suitable one for the versions of some classical operators associated to $\mathcal{L}$ (Hardy-Littlewood, semigroup and Poisson maximal functions, square function, fractional integral operator). We also recall a characterization of $BMO_{\mathcal{L}}$ in terms of Carlesson measures.

1. Introduction

These notes are intended to give an overview of the results that appeared in the talk presented by the author in the congress "VII Encuentro de Analistas Alberto Calderón y I Encuentro Conjunto Hispano-Argentino de Análisis", that was held in Merlo (San Luis), from August 31st to September 3rd, 2004. This meeting had a double purpose. On one hand, we had the privilege of congratulating Professor Roberto Macías in his 60th birthday. On the other hand, it was a great opportunity to develop the connections between Argentine and Spanish analysts, following the example of the admirable labour of Professor Roberto Macías.

The talk mentioned above was mainly based in a joint work with J. Dziubański, G. Garrigós, J.L. Torrea and J. Zienkiewicz, [4]. The present paper does not contain any new result in the subject, and its purpose is to serve as a concise and expository summary of the topic, including some proofs not appearing in [4] that may contribute to clarify the reading of the original work. We will also refer to other works of J. Dziubański and J. Zienkiewicz, for the study of the H^p spaces associated to Schrödinger operators, and other authors that studied the properties of the associated semigroups.

Let us start by recalling some very well known facts. Typically, when one tries to study the boundedness of an operator (Calderón-Zygmund operators, for instance) in the spaces, L^p with $1 \leq p \leq \infty$, one obtains the boundedness from L^p into L^p in the case $1 < p < \infty$. In the extremes $p = 1$ and $p = \infty$, appropriate substituting spaces may be used. Some of those substituting spaces are $H^1(\mathbb{R}^d)$ and $BMO(\mathbb{R}^d)$, for the cases $p = 1$ and $p = \infty$, respectively. It is well known that the classical $H^1(\mathbb{R}^d)$ can be

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defined as the space of the functions in $L^1(\mathbb{R}^d)$ such that the heat semigroup maximal function is also in $L^1(\mathbb{R}^d)$. In this sense, both $H^1(\mathbb{R}^d)$ and its dual $BMO(\mathbb{R}^d)$ are spaces associated to the Laplacian operator, $\Delta = \sum_{k=1}^d \partial^2_{x_k}$. It is also known that this point of view allows to give a suitable definition of the analogous spaces associated to other differential operators. Concretely, we will be interested in the spaces H^1 and BMO associated to time independent Schrödinger operators with potential V:

$$\mathcal{L} = -\Delta + V. \tag{1.1}$$

Here, V is a fixed non-negative function on $\mathbb{R}^d$, $d \geq 3$, satisfying a *reverse Hölder inequality* $V \in RH_s(\mathbb{R}^d)$ for some $s > \frac{d}{2}$. That is, there exists $C = C(s, V) > 0$ such that

$$\left(\frac{1}{|B|}\int_B V(x)^s\,dx\right)^{\frac{1}{s}} \leq \frac{C}{|B|}\int_B V(x)\,dx, \tag{1.2}$$

for every ball $B \subset \mathbb{R}^d$. It is classical that under certain conditions on the potencial, the operator $\mathcal{L}$ generates a heat semigroup (see section 2):

$$\mathcal{T}_t f(x) = e^{-t\mathcal{L}} f(x) = \int_{\mathbb{R}^d} k_t(x,y) f(y)dy, \quad f \in L^2(\mathbb{R}^d), \quad t > 0. \tag{1.3}$$

In section 3 we see that a Hardy-type space related to $\mathcal{L}$ is naturally defined by:

$$H^1_{\mathcal{L}} = \left\{ f \in L^1(\mathbb{R}^d) : \ \mathcal{T}^* f(x) = \sup_{t>0} |\mathcal{T}_t f(x)| \in L^1(\mathbb{R}^d) \right\},$$

where $\mathcal{T}^* f(x) = \sup_{t>0} |\mathcal{T}_t f(x)|$ and

$$\|f\|_{H^1_{\mathcal{L}}} := \|\mathcal{T}^* f\|_{L^1(\mathbb{R}^d)}. \tag{1.4}$$

For the above class of potentials, it was shown in [5] that $H^1_{\mathcal{L}}$ admits a special atomic characterization, where cancellation conditions are only required for atoms with small supports. The following step is studying the properties of the dual space of $H^1_{\mathcal{L}}$, which we shall identify with a subclass of BMO functions, namely:

$$BMO_{\mathcal{L}} = \left\{ f \in BMO : \ \frac{1}{|B|}\int_B |f| \leq C, \text{ for all } B = B_R(x) \ : \ R > \rho(x) \right\}. \tag{1.5}$$

The precise definition of the norm in this space is given in section 4. The critical radii above are determined by the function $\rho(x; V) = \rho(x)$ which determines the behavior of both spaces $H^1_{\mathcal{L}}$ and $BMO_{\mathcal{L}}$ (see section 2 for a precise description of the function ρ). This $BMO_{\mathcal{L}}$ space turns out to be the suitable extreme point for $p = \infty$ concerning the boundedness of the classical operators associated to the operator $\mathcal{L}$. We shall use the following notations:

$$\begin{aligned}
Mf(x) &= \sup_{B\ni x}\frac{1}{|B|}\int_B |f(y)|\,dy, && (1.6)\\
\mathcal{T}^*f(x) &= \sup_{t>0}|\mathcal{T}_tf(x)f(x)|, \quad \mathcal{T}_tf(x)=e^{-t\mathcal{L}}f(x) && (1.7)\\
\mathcal{P}^*f(x) &= \sup_{t>0}\Big|\mathcal{P}_tf(x)\Big|, \text{ where } \mathcal{P}_t=e^{-t\sqrt{\mathcal{L}}}=\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\mathcal{T}_{t^2/4u}\,du, && (1.8)\\
s_{\mathcal{Q}}f(x) &= \Big(\int_0^\infty |\mathcal{Q}_tf(x)|^2\tfrac{dt}{t}\Big)^{\frac12}, \quad \mathcal{Q}_tf(x)=t^2\Big(\frac{d\mathcal{T}_sf}{ds}\Big|_{s=t^2}f\Big)(x) && (1.9)\\
\mathcal{I}_\alpha f(x) &= \mathcal{L}^{-\alpha/2}f(x)=\int_0^\infty e^{-t\mathcal{L}}f(x)\,t^{\alpha/2-1}\,dt \quad \text{for} \quad 0<\alpha<d. && (1.10)
\end{aligned}$$

These notations correspond, respectively, to the Hardy-Littlewood maximal function, the semigroup and Poisson-semigroup maximal functions, the $\mathcal{L}$-square function and the $\mathcal{L}$-fractional integral operator. We observe that in the classical case (i.e. $V\equiv 0$) these operators fail to be bounded in BMO, in fact they may be identically infinity for functions with certain growth (see section 5). However in our case it turns out that they behave correctly in $BMO_{\mathcal{L}}$, as it is shown in section 5. Finally, we also show a characterization of $BMO_{\mathcal{L}}$ in terms of Carleson measures, parallel to the classical one (see section 4).

In order to introduce the concepts in a natural order, the paper is organized as follows. In section 2 we discuss the fundamental properties of the semigroup generated by $\mathcal{L}$. In section 3 we consider the Hardy space $H^1_{\mathcal{L}}$, and in section 4, its dual space $BMO_{\mathcal{L}}$ is studied. Finally, section 5 is devoted to the study of the boundedness of the operators mentioned above, defined in the context of Schrödinger operators, in the space $BMO_{\mathcal{L}}$.

2. The heat semigroup associated to $\mathcal{L}$

Let us start by recalling some basic facts about semigroups (see [2], [11], [20] and [26] for a general account on semigroup theory). A $\mathcal{C}_0$-semigroup is a collection of linear operators $\{\mathcal{T}_t\}_{t\geq 0}$ defined on a Banach space $\mathcal{X}$ (the example to have in mind is $\mathcal{X}=L^p(\Omega,d\mu)$ for some $p\in[1,\infty]$ and measure space Ω), satisfying the following properties

$$\mathcal{T}_0=\text{ Id}, \quad \mathcal{T}_t\mathcal{T}_s=\mathcal{T}_{t+s}, \quad t\to\mathcal{T}_tf \text{ is continuous in } X \text{ for every } f\in X. \qquad (2.1)$$

The *infinitesimal generator* of any semigroup $\mathcal{T}_t$ acting on some space of functions $\mathcal{X}$, is the operator A, defined as

$$\lim_{t\to 0}\frac{\mathcal{T}_tf-f}{t}=Af$$

for f in a suitable dense class of functions in $\mathcal{X}$ called the *domain* of A (we could think, for example, in the infinitely differentiable functions with compact support). If the operator A is bounded in $\mathcal{X}$, the associated semigroup is given by the formal series $\mathcal{T}_t=e^{tA}=\sum_{n=0}^\infty \frac{t^nA^n}{n!}$, which in fact converges in norm. Nevertheless, in most cases we are interested in unbounded operators A (such as differential operators). The general theory on semigroups states that under certain conditions, for example if A is closed, densely defined and $\|(\lambda-A)^{-1}\|\leq 1/\lambda$ for every $\lambda>0$ (Yosida's generating theorem), we still have that A is the infinitesimal generator of a semigroup. Formally, we will denote this semigroup also as $\mathcal{T}_t = e^{tA}$. Also formally, from this formula it holds that $\partial_t \mathcal{T}_t f = A\mathcal{T}_t f$, $\mathcal{T}_0 f = f$. It turns out that this calculation can be made rigorous for f satisfying certain conditions. For this reason, $\{\mathcal{T}_t\}$ is called the *heat semigroup* of A.

Let us recall some typical examples of heat semigroups:

Example 2.1. The most classical example of a $\mathcal{C}_0$-semigroup is the one generated by the Laplacian in $\mathbb{R}^d$, $d \geq 1$, with the Lebesgue's measure. It is well known that the corresponding heat semigroup is given by the heat kernel,

$$T_t f(x) = e^{t\Delta} f(x) = \int_{\mathbb{R}^n} h_t(x-y) f(y)\, dy, \qquad h_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{\frac{|x|^2}{4t}}. \tag{2.2}$$

Observe that the kernel of T_t is a Gaussian density. It is very well known the connection between semigroups and Markov processes, and in fact what the expression of T_t says is that, under certain conditions on f, the solution $u(t,x)$ to the equation $\partial_t u(t,x) = \Delta u(t,x)$, $u(0,x) = f$ is $u(t,x) = T_t f(x) = E^x(f(B_{2t}))$, where $\{B_t\}$ is a Brownian motion and the expectation E^x is taken with respect to the law of the Brownian motion started at x. We will not comment further on this very interesting connection between Probability and PDE's and we suggest [8] for a detailed treatment of this topic.

Example 2.2. Another operator generating a $\mathcal{C}_0$-semigroup (see [20]) is the Ornstein-Uhlenbeck operator, $A = \frac{1}{2}\Delta - x\cdot\nabla$, in $(\mathbb{R}^d, d\gamma(x))$, $d \geq 1$, where $d\gamma(x) = \pi^{-d/2} e^{-|x|^2}\, dx$ is the Gaussian measure. The action of this semigroup is most commonly expressed as

$$e^{tA} f(x) = M_r f(x) = \frac{1}{\pi^{d/2}} \int_{\mathbb{R}^n} K_r(x,y) f(y)\, dy,$$

where $r = e^{-t}$ and

$$K_r(x,y) = \frac{1}{(1-r^2)^{n/2}} \exp\left(- \frac{|y - rx|^2}{1-r^2} \right), \qquad 0 < r < 1$$

is called the *Mehler kernel*. The semigroups in this example and example 2.1 show specially good semigroups, called symmetric diffusion semigroups (see [20] for the details).

Example 2.3. An example of an operator that generates a semigroup, but not a symmetric diffusion one (because it is not Markovian, that is, it does not map constants into constants) is the harmonic oscillator (also called Hermite operator), $A = \frac{1}{2}\Delta - \frac{1}{2}|x|^2$, in $(\mathbb{R}^d, dx)$.

From the heat semigroup associated to an operator A, we can define a number of semigroups as *subordinated semigroups*. There exist several ways of creating subordinated semigroups. We will be interested in the *Poisson subordinated semigroup*, which is defined, by using spectral techniques, as

$$\mathcal{P}_t f = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \mathcal{T}_{t^2/4u} f\, du = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-t^2/4u}}{u^{3/2}} \mathcal{T}_u f\, du. \tag{2.3}$$

It is easy to see that if $\{\mathcal{T}_t\}$ is a $\mathcal{C}_0$-semigroup, so is $\{\mathcal{P}_t\}$, and with some more effort (see, for example [2]), it can be seen that if $\mathcal{T}_t = e^{tA}$, then $\mathcal{P}_t = e^{-t\sqrt{-A}}$. At least formally, this last formula can be understood by using a well known formula for the Gamma function (see the book by Folland, Introduction to Partial Differential Equations for a proof of it)

$$\mathcal{P}_t = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{\frac{t^2}{4u}A}\, du = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\beta^2/4u}\, du = e^{-\beta}, \tag{2.4}$$

with $\beta^2 = t^2(-A)$, which is "positive". Also formally, we can differentiate twice in the formula for $\mathcal{P}_t$ (this calculation can be also made rigourously for f satisfying certain properties), and see that it satisfies $\partial_t^2 \mathcal{P}_t f + A\mathcal{P}_t f = 0$, $\mathcal{P}_0 f = f$, that is, the Laplace equation for A.

Example 2.4. In particular, in the situation of Example 2.1, when A is the Laplace operator on $\mathbb{R}^d$ with Lebesgue's measure, $\mathcal{P}_t f(x)$ should be the harmonic extension to $(0,\infty)\times\mathbb{R}^d$ of f. In fact, if we substitute in (2.4) T_t by its expression in terms of the integral against the gaussian kernel, and change the order of integration, we obtain that

$$P_t f(x) = e^{-t\sqrt{-\Delta}} f(x) = C_d \int_{\mathbb{R}^d} \frac{t}{(t^2+|x-y|^2)^{(d+1)/2}} f(y)\, dy = P_t * f(x)$$

where the kernel

$$P_t(x) = \frac{C_d\; t}{(t^2+|x|^2)^{(d+1)/2}} = \frac{1}{t^d} P\left(\frac{x}{t}\right), \qquad P(x) = P_1(x) = \frac{C_d}{(1+|x|^2)^{(d+1)/2}}$$

is the *Poisson kernel for the upper half space.* As in the semigroup of Example 2.1, this kernel is a density, and the process associated to these densities is the Cauchy process, which is called the subordinated process to Brownian motion.

We are interested in the heat semigroup associated to a Schrödinger operator of the form given in (1.1). Example 2.3 shows a particular case of these class of operators, that have a deep physical significance. Consider a non relativistic, spinless quantum mechanical particle of mass m that moves in $\mathbb{R}^d$ under the influence of a potencial $V:\mathbb{R}^d\to\mathbb{R}$. Then the wave function of the particle is given by the solution $u(t,x)$ of the equation

$$\begin{cases} \dfrac{1}{i}\dfrac{\partial u}{\partial t} = \dfrac{1}{2m}\Delta u - V(x)u \\ u(0,x) = f(x), \qquad x\in\mathbb{R}^d. \end{cases}$$

Under the additional condition $\int_{\mathbb{R}^d} |f(x)|^2\, dx = 1$, the function $|u(t,x)|^2$ can be understood (among other interpretations) as the position density of the particle in time t. That is, for any measurable set $A\in\mathbb{R}^d$, the probability that at time t the particle is in the set A is given by $\int_A |u(t,x)|^2\, dx$. In this sense, the most significant dimension is $d=3$. On the other hand, the equation

$$\frac{\partial u}{\partial t} = \Delta u - V(x)u$$

describes a heat flow with cooling, and the temperature $u(t,x)$ represents the temperature in x at time t when there does not exist a perfect conduction of heat, but the heat dissipates at rate V. Also, the Schrödinger operator is useful in the study of certain sub-elliptic operators. For instance, if in the operator $-\Delta_x - V(x)\partial_t^2$ we take Fourier transform in the t variable, we obtain the operator $-\Delta_x + V(x)\xi^2$.

We will consider Schrödinger operators with potentials that satisfy a reverse Hölder inequality (1.2), $V \in RH_s(\mathbb{R}^d)$, for some $s > d/2$ and $d \geq 3$. The reverse Hölder classes RH_s are defined for any $s > 1$ and they satisfy the following properties (see [16] and [5], and the references therein):

- they are decreasing: $RH_s \subset RH_t$ for $s \geq t$.
- If $V \in RH_s$ then there exists $\varepsilon > 0$ such that $V \in RH_{s+\varepsilon}$ (thus, it is equivalent to consider $s > d/2$ or $s \geq d/2$ in our hypothesis).
- If $V \in RH_s$ for any $s > 1$, then $d\mu(x) = V(x)\,dx$ is a doubling measure and V is a Muckenhoupt A_∞ weight.
- If V is a polynomial, then $V \in RH_s$ for any $s > 1$.

We impose the hypothesis $s \geq d/2$ in order to guarantee that the critical radius ρ defined in (2.9) is well defined (see [16]).

Let us point out that the operator $\mathcal{L}$ is defined in all dimensions, but we will consider only dimension three or higher. The results that we show in this paper rely on previous estimates of the densities of the operators of the heat semigroup generated by $-\mathcal{L}$ and also in properties of the critical radius ρ. These estimates are strongly connected with the expression of the fundamental solution for $\mathcal{L}$, and its relationship with the fundamental solution for $-\Delta$, that for $d \geq 3$ has the common expression $\Gamma(x, y) = C_d/|x-y|^{d-2}$.

Under the conditions we have imposed on the potential V, it is well known that $-\mathcal{L}$ generates a $\mathcal{C}_0$-semigroup $\mathcal{T}_t = e^{-t\mathcal{L}}$ (see [17] and the references therein for an account on these semigroups). The Feynman-Kac formula states that

$$\mathcal{T}_t f(x) = e^{-t\mathcal{L}} f(x) = E_x\bigg(\exp\bigg(-\int_0^t V(B(s))\,ds\bigg) f(B(t))\bigg), \tag{2.5}$$

where E_x denotes the expectation of a Brownian motion $\{B(t)\}_{t\geq 0}$ started at $x \in \mathbb{R}^d$. This process is an $\mathbb{R}^d$-valued Gaussian process (its finite dimensional distributions are gaussian random vectors) with

$$E_x(B_j(t)) = x_j, \qquad E_x\big((B_j(t) - x_j)(B_k(s) - x_k)\big) = \delta_{j,k}\min\{s, t\}.$$

It is also well known that $e^{t\frac{1}{2}\Delta} f(x) = E_x(f(B(t)))$ (and the reason of the 1/2 in this definition, different from the one in example 2.1, comes from the usual normalization of the density function of $B(t)$). The operators $\mathcal{T}_t$ are given by symmetric (in x and y), jointly continuous (in t, x and y), uniformly bounded integral kernels,

$$\mathcal{T}_t f(x) = \int_{\mathbb{R}^d} k_t(x, y) f(y)\,dy$$

whose concrete expression is not known. Nevertheless, from the Feynman-Kac formula and (2.2), it follows that

$$0 \leq k_t(x, y) \leq h_t(x - y) = (4\pi t)^{-\frac{d}{2}}\ \exp(-\tfrac{|x-y|^2}{4t}). \tag{2.6}$$

It is possible to prove better estimates, as the next result shows.

Proposition 2.7. (see [6], [12]) *For every N, there is a constant C_N such that*

$$0 \le k_t(x,y) \le C_N\, t^{-\frac{d}{2}} \exp(-\tfrac{|x-y|^2}{5t}) \left(1 + \tfrac{\sqrt{t}}{\rho(x)} + \tfrac{\sqrt{t}}{\rho(y)}\right)^{-N}. \tag{2.8}$$

Here, $\rho(x) := \rho(x,V)$ is the critical radius associated to the potencial V. It is defined in the following way:

$$\rho(x,V) := \sup\left\{ r > 0 : \quad \frac{1}{r^{d-2}} \int_{B(x,r)} V(y)\, dy \le 1 \right\}. \tag{2.9}$$

The condition $V \in RH_s$, $s > d/2$ guarantees (together of course with $V \not\equiv 0$, see [16]) that $0 < \rho(x) < \infty$ for any $x \in \mathbb{R}^d$, since in that case

$$\lim_{r\to 0} \frac{1}{r^{d-2}} \int_{B(x,r)} V(y)\, dy = 0, \qquad \lim_{r\to\infty} \frac{1}{r^{d-2}} \int_{B(x,r)} V(y)\, dy = \infty.$$

The function $m(x,V) := 1/\rho(x)$ was introduced in [15] to study the Neumann problem above a Lipschit graph, with data in L^p for the Schrödinger operators $-\Delta + V$ for general $V \in RH_\infty$. Similar conditions as the one used to define RH_s were used in [9] to study the eigenvalues of Schrödinger operators as (1.1): it holds that, roughly speaking, in dimension $d \ge 3$, (1.1) has a number of negative eigenvalues equivalent to N if there exists a collection of N pairwise disjoint cubes where $1/|Q_j| \int_{Q_j} |V| \ge C(\operatorname{diam} Q_j)^{-2}$ (see Theorem 6 in [9]).

In [18] operators $\mathcal{L}$ were considered in the case that $V = \sum_{\beta\le\alpha} a_\beta x^\beta$, α, $\beta \in \mathbb{N}^d$ multi-index, is a positive polynomial. In this work it is proved that if $K(x)$ is the only tempered distribution such that $\mathcal{L}K(x) = \delta_x$, and $a(\xi) = \hat{K}(\xi)$, then the derivatives of the smooth function a are controlled by a constant times a certain power of $|\xi| + M_N(0)$, where $M_N(x) = \left(\sum_{\beta\le\alpha} \left|D^\beta V(x)\right|^{N/(|\beta|+2)} \right)^{1/N} \sim M_1(x)$. In the case that V is a positive polynomial, it can be seen (see [15], [27])

$$m(x,V) \sim M_1(x) = \sum_{\beta\le\alpha} \left|D^\beta V(x)\right|^{1/(|\beta|+2)}.$$

This function is a fundamental tool in order to obtain suitable estimates for the kernel of the operators $\mathcal{T}_t$, as it is shown in [5], [6], [7], [4], [12], [15], [16].

Let us think in the case $V(x) = |x|^2$. Clearly we have $\alpha = (2,\dots,2)$ and since the crossed derivatives are null, we get that $m(x,V) \sim C + |x| + C\sum_{j=1}^d |x|^{1/3}$. Thus $m(x,V)$ behaves as a constant for small x, and as $|x|$ for big x. That is,

$$\text{for } V(x) = |x|^2, \qquad \rho(x) \sim \frac{1}{1+|x|}.$$

This critical radius appears in the study of the operators related to the Ornstein-Uhlenbeck semigroup (see, for example, [10] and the references therein). This is not a casual fact and it is due to the connection existing between the Hermite operator $\mathcal{H} = \frac{1}{2}\Delta - \frac{1}{2}|x|^2$ and the Ornstein-Uhlenbeck operator $\mathcal{O} = \frac{1}{2}\Delta - x\cdot\nabla$ (see examples 2.2 and 2.3). For both operators there exist orthogonal complete bases of eigenvectors, in the corresponding $L^2(\mathbb{R}^d, d\mu)$ for the suitable measure $d\mu$, that are connected by a change of scale. This connection is summarized in the following table

Operator	Measure space for L^2	Orthogonal basis of eigenvectors
Hermite $\mathcal{H} = \frac{1}{2}\Delta - \frac{1}{2}\|x\|^2$	$(\mathbb{R}^d, dx)$	Hermite polynomials $\{H_n\}_{n\geq 0}$
Ornstein-Uhlenbeck $\mathcal{O} = \frac{1}{2}\Delta - x\cdot\nabla$	$(\mathbb{R}^d, e^{-\|x\|^2}\,dx)$	Hermite functions $\mathcal{H}_n(x) = e^{\|x\|^2/2}H_n(x)$

Concretely, for $f \in \mathcal{C}^2$ one has

$$\mathcal{H}(f\, e^{-|x|^2/2})(x)e^{|x|^2/2} = \mathcal{O}f(x) - \frac{d}{2}f(x).$$

An important feature of the critical radius ρ is that for similar points, the corresponding critical radii are similar.

Proposition 2.10. [15] *There exist $c>0$ and $k_0 \geq 1$ so that, for all $x, y \in \mathbb{R}^d$*

$$c^{-1}\,\rho(x)\Big(1+\tfrac{|x-y|}{\rho(x)}\Big)^{-k_0} \;\leq\; \rho(y) \;\leq\; c\,\rho(x)\,\Big(1+\tfrac{|x-y|}{\rho(x)}\Big)^{\frac{k_0}{k_0+1}}. \tag{2.11}$$

In particular, $\rho(x)\sim\rho(y)$ when $y \in B_r(x)$ and $r \leq C\rho(x)$.

3. The Hardy space $H^1_{\mathcal{L}}(\mathbb{R}^d)$

Let us recall that in the classical clase, the space $H^1(\mathbb{R}^d)$ admits several equivalent definitions (see, for instance [22] and the references therein). Our interest will be focused in two of these definitions, namely the atomic and the maximal definitions.

A function a is an atom if its support is contained in a ball B, such that $|a(x)| \leq |B|^{-1}$ almost everywhere and it has null mean, that is $\int_B a(x)\,dx = 0$. Thus, $f \in H^1(\mathbb{R}^d)$ if and only if f admits an atomic decomposition, $f = \sum_{j=1}^\infty \lambda_j a_j(x)$ where a_j are atoms and $\sum_{j=1}^\infty |\lambda_j| < \infty$. The norm in the space $H^1(\mathbb{R}^d)$ is given as $\|f\|_{H^1} = \inf \sum_{j=1}^\infty |\lambda_j|$ where the infimum runs over all such possible atomic decompositions. In particular, this implies that $\int_{\mathbb{R}^d} f(x)\,dx = 0$.

Given $\Phi \in \mathcal{S}$, being $\mathcal{S}$ the Schwartz class in $\mathbb{R}^d$, we define the maximal function associated to Φ as

$$\mathcal{M}_\Phi f(x) = \sup_{t>0}|f*\Phi_t(x)|, \qquad \Phi_t(x) = \frac{1}{t^n}\Phi\left(\frac{x}{t}\right)$$

and for P_t as in example 2.4, the Poisson kernel in the upper half space, the non tangential maximal function is

$$u^*(x) = \sup_{|x-y|<t}|u(y,t)|, \qquad \text{where} \quad u(x,t) = f*P_t(x).$$

Observe that, with the notation of example 2.4, this non tangential maximal function can be expressed in terms of the Poisson semigroup of the Laplacian, namely $u^*(x) = P^*_{nt}f(x) = \sup_{|x-y|<t}|P_tf(y)|$. Then, the following sentences are equivalent

- f belongs to $H^1(\mathbb{R}^d)$.

- There exists $\Phi \in \mathcal{S}$ with $\int \Phi \neq 0$ such that $\mathcal{M}_\Phi f \in L^1(\mathbb{R}^d)$.
- The distribution f is bounded and $u^* \in L^1(\mathbb{R}^d)$.

The most typical example of function in $\mathcal{S}$ having non compact support and non zero mean is the gaussian function, $\Phi(x) = (4\pi)^{-d/2} e^{-|x|^2/4}$. Observe that $f * \Phi_{\sqrt{t}}(x) = T_t f(x)$ (see example 2.1). By the former result, if $T^* f(x) = \sup_{t\geq 0} |T_t f(x)| = \mathcal{M}_\Phi f(x)$ belongs to $L^1(\mathbb{R}^d)$, then f is in $H^1(\mathbb{R}^d)$. The converse is also true (since $T^* f(x)$ is the norm in ℓ^∞ of a vector-valued Calderón-Zygmund operator, see [14] for the details). In this sense, the space $H^1(\mathbb{R}^d)$ appears naturally associated to the Laplacian in $\mathbb{R}^d$. Thus, it is a natural extension of this notion to define

$$H^1_{\mathcal{L}} = \left\{ f \in L^1(\mathbb{R}^d) : \quad \mathcal{T}^* f(x) = \sup_{t>0} |\mathcal{T}_t f(x)| \in L^1(\mathbb{R}^d) \right\}, \tag{3.1}$$

where $\|f\|_{H^1_{\mathcal{L}}} := \|\mathcal{T}^* f\|_{L^1(\mathbb{R}^d)}$ to be the Hardy space with $p = 1$ associated to the Schrödinger operator (1.1). This space shares with the classical $H^1(\mathbb{R}^d)$ several features. One of the most important ones is that its elements admit atomic decompositions.

The atomic decomposition of the space $H^1_{\mathcal{L}}(\mathbb{R}^d)$ was proved in [5]. Define the sets

$$\mathcal{B}_n = \left\{ x : \quad 2^{-(n+1)/2} < \rho(x) \leq 2^{-n/2} \right\}.$$

A function a is an atom in $H^1_{\mathcal{L}}$ associated to a ball $B = B(x_0, r)$ if

i) the support of a is contained in B and $|a(x)| \leq |B|^{-1}$ almost everywhere,
ii) if $x_0 \in \mathcal{B}_n$, then $r \leq 2\, 2^{-n/2}$,
iii) if $x_0 \in \mathcal{B}_n$ and $r \leq \frac{1}{2} 2^{-n/2}$, then $\int_B a(x)\, dx = 0$.

Let us call $\mathcal{A}$ to the family of these atoms. Observe that the support and size conditions are the same than for the classical atoms in $H^1(\mathbb{R}^d)$. The difference comes because we only consider atoms with small support (those such that the radius of the ball B is smaller than twice the critical radius of the center), and that we only require the cancellation condition for atoms supported in a ball of radius smaller than half the critical radius. Clearly we have $H^1(\mathbb{R}^d) \subset H^1_{\mathcal{L}}(\mathbb{R}^d)$. The concrete result is the following

Theorem 3.2. [5, Theorem 1.5] *Let $V \not\equiv 0$ be a non-negative potencial such that $V \in RH_{d/2}$. Then, there exists a constant $C > 0$ such that*

$$C^{-1} \|f\|_{H^1_{\mathcal{L}}} \leq \|f\|_{H^1_{at,\mathcal{L}}} \leq C \|f\|_{H^1_{\mathcal{L}}}$$

where $\|f\|_{H^1_{at,\mathcal{L}}} = \inf \sum_{j=1}^\infty |\lambda_j|$. Here, the infimum runs over all possible atomic decompositions $f = \sum_{j=1}^\infty \lambda_j a_j(x)$ where a_j are $H^1_{\mathcal{L}}$-atoms and $\sum_{j=1}^\infty |\lambda_j| < \infty$.

As in the classical case, this atomic characterization of the space $H^1_{\mathcal{L}}(\mathbb{R}^d)$ provides a useful tool to study its dual space. Similar atomic characterizations (with appropriate cancellation conditions for atoms in iii)) hold also for the spaces $H^p_{\mathcal{L}}$ with $0 < p < 1$ for V a polynomial (see [3]), and for p in a certain range when V is a general non-negative potencial in a reverse Hölder class (see [6] and [7]). Let us mention that in [5] it is proved a characterization of $H^1_{\mathcal{L}}(\mathbb{R}^d)$ in terms of the Riesz transforms associated to $\mathcal{L}$, parallel to the well known one for the classical $H^1(\mathbb{R}^d)$. Indeed, define for $j = 1, \ldots, d$

the j-th Riesz transform as

$$R_j f(x) = \frac{\partial}{\partial x_j} \mathcal{L}^{-1/2} f(x).$$

Theorem 3.3. [5, Theorem 1.5] *Let $V \not\equiv 0$ be a non-negative potencial such that $V \in RH_{d/2}$. Then, there exists a constant $C > 0$ such that*

$$C^{-1} \|f\|_{H^1_{\mathcal{L}}} \leq \|f\|_{L^1} + \sum_{j=1}^{d} \|R_j f\|_{L^1} \leq C \|f\|_{H^1_{\mathcal{L}}}.$$

An important tool in the proofs of the results showed above are the following lemma and the corollaries bellow.

Lemma 3.4. [5, Lemma 2.3] *There exists a collection of balls $B_{n,k} = B(x_{n,k}, 2^{-n/2})$ for $n \in \mathbb{Z}$, $k \in \mathbb{N}$, such that $x_{n,k} \in \mathcal{B}_{n,k}$, $\mathcal{B}_n \subset \cup_k B_{n,k}$ and the balls have the finite intersection property, which means that there exists a constant $\kappa > 0$ such that*

$$\sharp\{n', k' : \ RB_{n',k'} \cup RB_{n,k} \neq \emptyset\} \leq R^{\kappa}$$

for every $R \geq 2$, where $RB_{n,k} = B(x_{n,k}, R2^{-n/2})$

From now on, B^* will be the ball with the same center than B and twice the radius.

Corollary 3.5. *For every $\alpha > 0$, there exists a constant $C = C(\alpha, \rho)$ such that for $B = B(x_0, R)$ with $R > \alpha\rho(x_0)$, we have*

$$|B| \leq \sum_{\substack{n,k: \\ B^*_{n,k} \cap B \neq \emptyset}} |B_{n,k}| \leq C\,|B|.$$

Proof. Since the balls $B_{n,k}$ cover $\mathbb{R}^d$, the first inequality is trivial. For the second one, let us observe that

$$\sum_{\substack{n,k: \\ B^*_{n,k} \cap B \neq \emptyset}} |B_{n,k}| = \int_{\mathbb{R}^d} \sum_{\substack{n,k: \\ B^*_{n,k} \cap B \neq \emptyset}} \chi_{|B_{n,k}|}(y)\, dy.$$

From (2.11), we have that with any $z \in B^*_{n,k} \cap B$

$$\begin{aligned} \rho(x_{n,k}) &\leq C\,\rho(z)\left(1 + \frac{|z - x_{n,k}|}{\rho(x_{n,k})}\right)^{k_0/(k_0+1)} \leq C\,\rho(z) \\ &\leq C\,\rho(x_0)\left(1 + \frac{|z - x_0|}{\rho(x_0)}\right)^{k_0/(k_0+1)} \\ &= C\,\rho(x_0)\left(1 + \frac{|z - x_0|}{\rho(x_0)}\right)\left(1 + \frac{|z - x_0|}{\rho(x_0)}\right)^{-1/(k_0+1)} \\ &\leq C\,(\rho(x_0) + |z - x_0|) \leq C\,R \end{aligned}$$

Therefore, for $y \in B_{n,k}$ such that $B^*_{n,k} \cap B \neq \emptyset$,

$$|y - x_0| \leq |y - z| + |z - x_0| \leq 4\,2^{n/2} + R \leq C\rho(x_{n,k}) + R \leq C_1 R.$$

With this, by the finite intersection property, we get

$$\sum_{\substack{n,k:\\ B^*_{n,k}\cap B\neq\emptyset}} |B_{n,k}| = \int_{C_1 B} \sum_{\substack{n,k:\\ B^*_{n,k}\cap B\neq\emptyset}} \chi_{|B_{n,k}|}(y)\, dy \leq 2^\kappa |C_1 B| = C\,|B|.$$

□

A proof of the following corollary of Lemma 3.4 can be found in [26].

Corollary 3.6. *There exist functions* $\psi_{n,k}$ *such that* $0 \leq \psi_{n,k} \leq 1$*,* $\psi_{n,k} \in \mathcal{C}_0^\infty(B_{n,k})$ *and* $\sum_{n,k} \psi_{n,k}(x) \equiv 1$

Remark 3.7. It is equivalent to consider atoms in $\mathcal{A}$ than atoms in the more general class $\tilde{\mathcal{A}}$ formed by atoms satisfying the same conditions that the atoms in $\mathcal{A}$, but erasing the condition ii). That is, in $\tilde{\mathcal{A}}$ there are atoms with supports of any size, and we only require cancellation conditions in the smallest ones. We will use this definition of the atoms in $H^1_{\mathcal{L}}$ in the sequel.

The proof of the equivalence of $\mathcal{A}$ and $\tilde{\mathcal{A}}$ for the atomic decomposition of $H^1_{\mathcal{L}}$ is as follows: clearly, $\mathcal{A} \subset \tilde{\mathcal{A}}$. For the converse it is enough to see that there exists a constant C such that for any $\tilde{a} \in \tilde{\mathcal{A}}$ with support in a ball $B = B(x_0, R)$ with $x_0 \in \mathcal{B}_n$ and $R > 2\; 2^{-n/2}$, we have

$$\tilde{a}(x) = \sum_j c_j a_j, \quad a_j \in \mathcal{A}, \quad \sum_j |c_j| \leq C.$$

Consider the functions $a_{n,k} = |B|\,|B_{n,k}|^{-1}\tilde{a}(x)\psi_{n,k}(x)$ and the coefficients $c_{n,k} = |B|^{-1}|B_{n,k}|$ for indexes n,k such that $B^*_{n,k} \cap B \neq \emptyset$, and zero otherwise. Thus, $a_{n,k}$ by Corollary 3.6, we get that

$$\tilde{a}(x) = \sum_{\substack{n,k:\\ B^*_{n,k}\cap B\neq\emptyset}} c_{n,k} a_{n,k}(x), \qquad \text{supp}\; a_{n,k} \subset B_{n,k}, \qquad \|a_{n,k}\|_\infty \leq |B_{n,k}|^{-1}.$$

The functions $a_{n,k}$ are atoms of $\mathcal{A}$, since $B_{n,k}$ are balls with radius $2^{-n/2} \geq \frac{1}{2}2^{-n/2}$ and we do not require cancellation conditions form them. On the other hand, by Corollary 3.5, we get ve that

$$\sum_{\substack{n,k:\\ B^*_{n,k}\cap B\neq\emptyset}} c_{n,k} = |B|^{-1} \sum_{\substack{n,k:\\ B^*_{n,k}\cap B\neq\emptyset}} |B_{n,k}| \leq C.$$

4. The space $BMO_{\mathcal{L}}(\mathbb{R}^d)$

Since the atomic definition of $H^1_{\mathcal{L}}$ is the same than for the classical one, except that we only require cancellation conditions for atoms with "small" support, and atoms with bigger support are allowed without any further requirement, the functions in the space $BMO_{\mathcal{L}}$ should satisfy the same conditions than the classical ones for "small" balls, and something stronger for bigger balls. Here and subsequently, $f_B = |B|^{-1}\int_B f(x)\,dx$.

Definition 4.1. *We shall say that a locally integrable function f belongs to $BMO_{\mathcal{L}}$ whenever there is a constant $C \geq 0$ so that*

$$\frac{1}{|B_s|}\int_{B_s}|f - f_{B_s}| \leq C \quad \textit{and} \quad \frac{1}{|B_r|}\int_{B_r}|f| \leq C, \tag{4.1}$$

for all balls $B_s = B_s(x)$, $B_r = B_r(x)$ such that if $x \in \mathcal{B}_n$ $s \leq \frac{1}{2}2^{-n/2} \leq r$. We let $\|f\|_{BMO_{\mathcal{L}}}$ denote the smallest C in (4.1).

We observe that $\|f\|_{BMO_{\mathcal{L}}}$ is actually a norm (and not only a seminorm) making $BMO_{\mathcal{L}}$ a Banach space. Moreover, $\|f\|_{BMO} \leq 2\|f\|_{BMO_{\mathcal{L}}}$, $L^\infty \subset BMO_{\mathcal{L}} \subset BMO$, $f \in BMO_{\mathcal{L}}$ implies $|f| \in BMO_{\mathcal{L}}$ and it can be proved a John-Nirenberg's inequality: for any $p \in [1,\infty)$ there exists $c = c(p,\rho) > 0$ such that for every $f \in BMO_{\mathcal{L}}$

$$\Big(\tfrac{1}{|B|}\int_B |f - f_B|^p\Big)^{\frac{1}{p}} \leq c\,\|f\|_{BMO_{\mathcal{L}}}, \quad \text{for every } B,$$

$$\Big(\tfrac{1}{|B|}\int_B |f|^p\Big)^{\frac{1}{p}} \leq c\,\|f\|_{BMO_{\mathcal{L}}}, \quad \text{for } B = B_r(x) \text{ with } x \in \mathcal{B}_n,\ r \geq 2^{-1-n/2}.$$

By using the atomic decomposition of $H^1_{\mathcal{L}}$ and Remark 3.7, the proof of the duality of these spaces is similar to the one in the classical case, see section 3 in [4]. It turns out that to see that a function belongs to $BMO_{\mathcal{L}}$ it is enough to consider the critical balls $B_{n,k}$.

Lemma 4.2. *If there exists a constant $C > 0$ such that for every fixed $B_{n,k}$ (see Lemma 3.4), we have*

i) $\displaystyle\frac{1}{|B_{n,k}|}\int_{B_{n,k}}|f(x)|\,dx \leq K_1.$

ii) *For $\alpha > 1$, $\|f\|_{BMO(\alpha B_{n,k})} \leq K_2(\alpha)$,*

then $f \in BMO_{\mathcal{L}}$ and $\|f\|_{BMO_{\mathcal{L}}} \leq C\,K$, where K depends on K_1 and K_2.

Proof. From ii), in order to prove that for $B_s = B(x_0, s)$ with $x_0 \in \mathcal{B}_{n_0}$, $s \leq \frac{1}{2}2^{-n_0/2}$, $|B_s|^{-1}\int_{B_s}|f - f_{B_s}| \leq C$, it is enough to see that there exists n,k such that $B_s \subset \alpha B_{n,k}$ for certain (fixed) α. Let n,k be such that $x_0 \in B_{n,k}$. Thus, for any $y \in B_s$, by Proposition 2.10

$$\begin{aligned}
|y - x_{n,k}| &\leq |y - x_0| + |x_0 - x_{n,k}| \leq \frac{1}{2}2^{-n_0/2} + 2^{-n/2} \leq \frac{1}{\sqrt{2}}\rho(x_0) + \sqrt{2}\rho(x_{n,k}) \\
&\leq \frac{1}{\sqrt{2}}C\Big(1 + \frac{|x_0 - x_{n,k}|}{\rho(x_{n,k})}\Big)^{k_0/(k_0+1)}\rho(x_{n,k}) + \sqrt{2}\rho(x_{n,k}) \\
&\leq \alpha\rho(x_{n,k}) \leq \alpha 2^{-n/2}
\end{aligned}$$

where α is an absolute constant.

Now take $B_r = B(x_0, r)$ with $x_0 \in \mathcal{B}_{n_0}$, $r \geq \frac{1}{2}2^{-n_0/2} \geq \frac{1}{2}\rho(x_0)$. By Corollaries 3.5 and 3.6 and the hypothesis i), we have

$$\begin{aligned}\frac{1}{|B_r|}\int_{B_r} |g(x)|\,dx &= \frac{1}{|B_r|}\sum_{\substack{n,k:\\ B_{n,k}\cap B_r\neq\emptyset}} \int_{B_r} |g(x)|\psi_{n,k}(x)\,dx \\ &\leq \frac{1}{|B_r|}\sum_{\substack{n,k:\\ B^*_{n,k}\cap B_r\neq\emptyset}} \frac{|B_{n,k}|}{|B_{n,k}|}\int_{B_{n,k}} |g(x)|\psi_{n,k}(x)\,dx \\ &\leq \frac{1}{|B_r|}\sum_{\substack{n,k:\\ B^*_{n,k}\cap B_r\neq\emptyset}} |B_{n,k}|\,K \leq C\,K.\end{aligned}$$

□

It is also possible to prove a characterization of $BMO_{\mathcal{L}}$ in terms of Carlesson measures, parallel to the one existing in the classical case (see [22]). The concrete result is as follows

Theorem 4.3. [4, Theorem 2] *Let $V \not\equiv 0$ be a non-negative potential in $RH_s(\mathbb{R}^d)$ for some $s > \frac{d}{2}$, $\rho(x)$ be the critical radius* (2.9) *and $\mathcal{Q}_t$ be as in* (1.9)*.*

1. *If $f \in BMO_{\mathcal{L}}$, then $d\mu_f(x,t) := |\mathcal{Q}_t f(x)|^2\,dx\,dt/t$ is a Carleson measure, and*

$$\|\mu_f\|_{\mathcal{C}} := \sup_{x\in\mathbb{R}^d,\ r>0} \frac{\mu_f(B_r(x)\times(0,r))}{|B_r(x)|} < C\,\|f\|^2_{BMO_{\mathcal{L}}}.$$

2. *Conversely, if $f \in L^1((1+|x|)^{-(d+1)}dx)$ and $d\mu_f(x,t)$ is a Carleson measure, then $f \in BMO_{\mathcal{L}}$.*

Moreover, in either case, there exists $C > 0$ such that

$$\frac{1}{C}\,\|f\|^2_{BMO_{\mathcal{L}}} \leq \|d\mu_f\|_{\mathcal{C}} \leq C\,\|f\|^2_{BMO_{\mathcal{L}}}.$$

5. Boundedness of classical operators in the space $BMO_{\mathcal{L}}(\mathbb{R}^d)$

Once we have identified the dual space of $H^1_{\mathcal{L}}$, the next question we are interested in is investigating the boundedness of some classical operators in the space $BMO_{\mathcal{L}}$. Classical operators are often not well behaved in the space BMO. For instance, Bennet, DeVore, Sharpley [1] proved that for a function $f \in BMO$, its Hardy-Littlewood maximal function Mf (see (1.6)) is either identically infinite or $Mf \in BMO$ with $\|Mf\|_{BMO} \leq C\,\|f\|_{BMO}$. Other operators share this kind of behavior. The maximal operator of the heat and Poisson semigroups associated to the Laplacian present this dichotomy. By the subordination formula (2.3), one has

$$P^*f(x) \leq C\,T^*f(x) \leq C\,M(|f|)(x) :\ L^\infty \longrightarrow L^\infty,$$

but easy counterexamples, as $f(x) = |\log|x|| \in BMO$, show that $P^*f(x)$ may be infinity almost everywhere for functions in BMO. Fractional integrals

$$I_\alpha f(x) = \frac{1}{\gamma_\alpha}\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy, \qquad \gamma_\alpha = \frac{\pi^{n/2}2^\alpha\Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}, \qquad 0<\alpha<n$$

are other example of the dichotomy. For $f \in L^{n/\alpha}(\mathbb{R}^d)$, $I_\alpha f(x)$ is either identically infinite or $I_\alpha f \in BMO(\mathbb{R}^d)$ with $\|I_\alpha f\|_{BMO} \leq C\,\|f\|_{L^{d/\alpha}(\mathbb{R}^d)}$. For example, with $f(x) = (|x|^\alpha \log|x|)^{-1}$, $I_\alpha f = \infty$ almost everywhere.

Remark 5.1. It is an easy exercise to show that the kernel of I_α appears after a change of variables in the formula that defines the negative powers of the Laplacian

$$I_\alpha f(x) = (-\Delta)^{-\alpha/2} f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{t\Delta} f(x)\, t^{\alpha/2-1}\, dt,$$

by using the calculus formula that gives for positive s and a

$$s^{-a} = \frac{1}{\Gamma(a)} \int_0^\infty e^{-ts} t^a \frac{dt}{t}.$$

Square functions as

$$sf(x) = \left(\int_0^\infty \left| t\nabla_{t,x} P_t * f(x) \right|^2 \frac{dt}{t} \right)^{1/2}$$

also present the mentioned dichotomy. S. Wang [25] showed that for $f \in BMO$, either $sf = \infty$ almost everywhere or $sf \in BMO$, with $\|sf\|_{BMO} \leq C\,\|f\|_{BMO}$. Later, Kurtz [13] extended this result to the case of the Lusin area function S and to g^*_λ, where

$$Sf(x) = \left(\int\int_{\Gamma(x)} t^{1-n} \left|\nabla_{t,z} P_t * f(z)\right|^2 dt\, dz \right)^{1/2},$$

$$g^*_\lambda f(x) = \left(\int\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-z|} \right)^{\lambda n} t^{1-n} \left|\nabla_{t,z} P_t * f(z)\right|^2 dt\, dz \right)^{1/2}.$$

We can even find more pathological behavior. In a discrete setting, Torrea and de la Torre [24] showed that the one-sided discrete square function

$$Sf(x) = \left(\sum_{n=-\infty}^{\infty} |A_n f(x) - A_{n-1} f(x)|^2 \right)^{1/2}, \quad A_n f(x) = \frac{1}{2^n} \int_x^{x+2^n} f(y)\, dy,$$

fails to be bounded in L^∞. For $f \in L^\infty$, either $Sf = \infty$ a.e., for instance for $f(x) = \sum_{i=0}^\infty \chi_{[2^{2i}, 2^{2i+1}]}$, or $Sf \in BMO$, with $\|Sf\|_{BMO} \leq C\,\|f\|_{L^\infty}$.

In [4] it is shown that $BMO_{\mathcal{L}}$ is a better space for the operators associated to the operator $\mathcal{L}$, defined in (1.6)–(1.9), than the classical BMO is for the classical versions (associated to the Laplacian) of those operators. The gain is in the sense that the dichotomy of the classical case does not appear any more.

Theorem 5.2. *Let $V \not\equiv 0$ be a non-negative potential in $RH_s(\mathbb{R}^d)$ for some $s > \frac{d}{2}$. The operators M, $\mathcal{T}^*$, $\mathcal{P}^*$ and s_Q are well-defined and bounded in $BMO_{\mathcal{L}}$. For all $0 < \alpha < d$, the operator $\mathcal{I}_\alpha$ is bounded from $L^{d/\alpha}(\mathbb{R}^d)$ into $BMO_{\mathcal{L}}$.*

The complete proof of this result can be found in [4]. Let us just give a sketch of it here. We start with the Hardy-Littlewood maximal operator. The first step is showing that for $f \in BMO_{\mathcal{L}}$, $Mf < \infty$ almost everywhere. For that, it is enough to see that for any $x_0 \in \mathcal{B}_{n_0}$ and $C_0 \geq 1$, $Mf(x) < \infty$ for almost every $x \in B_0 = B(x_0, C_0/2\, 2^{-n_0/2})$. Split f as $f = f_1 + f_2$, $f_1(x) = f(x)\chi_{B_0^*}$. Thus, $Mf_1(x) < \infty$ since any function in $BMO_{\mathcal{L}}$ is locally integrable. Since supp $f_2 \subset (B_0^*)^c$, we only use balls $B \ni x$ with $B \cap (B_0^*)^c \neq \emptyset$ to calculate $Mf_2(x)$ for $x \in B_0$. In this case, $2r \geq C_0/2\, 2^{-n_0/2}$ and $B \subset B_{4r}(x_0) = \tilde{B}$. But then $r(\tilde{B}) = 4r > C_0 2^{-n_0/2}$, and

$$\frac{1}{|B|}\int_B |f_2(y)|\,dy \leq \frac{4^d}{|B_{4r}(x_0)|}\int_{B_{4r}(x_0)} |f(y)|\,dy \leq c\,\|f\|_{BMO_{\mathcal{L}}}.$$

Next, we see that Mf is bounded $BMO_{\mathcal{L}}$. By the results in [1], and the definition of $BMO_{\mathcal{L}}$, it is enough to see that for $B = B(x_0, r)$ with $x_0 \in \mathcal{B}_{n_0}$ and $r \geq \frac{1}{2}2^{-n_0/2}$

$$\frac{1}{|B|}\int_B |Mf(y)|\,dy \leq C\|f\|_{BMO_{\mathcal{L}}}.$$

Split now f as $f = f_1 + f_2$, $f_1(x) = f(x)\chi_{B^*}$. By the previous argument, $Mf_2(x) \leq c\|f\|_{BMO_{\mathcal{L}}}$ for every $x \in B$. For the other term, we use the boundedness of M in L^2:

$$\begin{aligned}\frac{1}{|B|}\int_B |Mf_1(y)|\,dy &\leq \Big(\frac{1}{|B|}\int_B |Mf_1(y)|^2\,dy\Big)^{1/2} \\ &\leq C\Big(\frac{1}{|B|}\int_{B^*} |f(y)|^2\,dy\Big)^{1/2} \lesssim \|f\|_{BMO_{\mathcal{L}}}.\end{aligned}$$

Observe that in this proof we have not used any property of the potential, except the ones that give rise to the definition of $BMO_{\mathcal{L}}$. The proof for the operators (1.7)–(1.9) needs indeed deeper properties of the potential and the kernels of the semigroups. Let us sketch the proof for the boundedness of $\mathcal{T}^*$. A first step is reducing matters to critical balls: by Lemma 4.2, it is enough to see that for any n, k

$$\frac{1}{|B_{n,k}|}\int_{B_{n,k}} |\mathcal{T}^* f(x)|\,dx \leq C\,\|f\|_{BMO_{\mathcal{L}}}, \quad \|\mathcal{T}^* f\|_{BMO(\alpha B_{n,k})} \leq C\,\|f\|_{BMO_{\mathcal{L}}}. \tag{5.3}$$

Clearly, the first inequality shows that for $f \in BMO_{\mathcal{L}}$, $\mathcal{T}^* f$ is finite almost everywhere. This is a byproduct of the proof already seen for Mf, since from (2.6), we have $\mathcal{T}^* f(x) \leq \sup_{t>0} |f| * h_t(x) \leq C\,M|f|(x)$, and therefore

$$\frac{1}{|B_{n,k}|}\int_{B_{n,k}} |\mathcal{T}^* f(x)|\,dx \leq \frac{C}{|B_{n,k}|}\int_{B_{n,k}} M|f|(x)\,dx \leq C\,\|f\|_{BMO_{\mathcal{L}}}.$$

For the second inequality in (5.3), we may use the fact that if f satisfies

$$h - g_1 \leq f \leq h + g_2 \quad \text{a.e.}$$

given $h \in BMO(\alpha B_{n,k})$, g_1 and g_2 functions in L^∞, then $f \in BMO(\alpha B_{n,k})$ and $\|f\|_{BMO(\alpha B_{n,k})} \leq \|h\|_{BMO(\alpha B_{n,k})} + \max\{\|g_1\|_{L^\infty}, \|g_2\|_{L^\infty}\}$. For K to be fixed later, one trivially has

$$\begin{aligned}&\sup_{t\leq K\rho(x_k)^2} |T_t f(x)| - \sup_{t\leq K\rho(x_k)^2} |(\mathcal{T}_t - T_t)f(x)| \leq \mathcal{T}^* f(x) \\ &\quad\leq \sup_{t\leq K\rho(x_k)^2} |T_t f(x)| + \sup_{t\leq K\rho(x_k)^2} |(\mathcal{T}_t - T_t)f(x)| + \sup_{t\geq K\rho(x_k)^2} |\mathcal{T}_t f(x)|\end{aligned}$$

it will suffice to see that

$$\Big\| \sup_{t \geq K\rho(x_k)^2} |\mathcal{T}_t f(x)| \Big\|_{L^\infty(\alpha B_{n,k})} \leq C \, \|f\|_{BMO_{\mathcal{L}}}, \tag{5.4}$$

$$\Big\| \sup_{t \leq K\rho(x_k)^2} |(\mathcal{T}_t - T_t) f(x)| \Big\|_{L^\infty(\alpha B_{n,k})} \leq C \, \|f\|_{BMO_{\mathcal{L}}}, \tag{5.5}$$

$$\Big\| \sup_{t \leq K\rho(x_k)^2} |T_t f(x)| \Big\|_{BMO(\alpha B_{n,k})} \leq C \, \|f\|_{BMO_{\mathcal{L}}}. \tag{5.6}$$

This seems to be the usual behavior of the operators (1.7)–(1.9): the critical radius "splits" the operators in a correct way (see [4] for the details). The first inequality $\Big\| \sup_{t \geq \rho(x_k)^2} |\mathcal{T}_t f(x)| \Big\|_{L^\infty(\alpha B_{n,k})} \leq C \, \|f\|_{BMO_{\mathcal{L}}}$ follows from the kernel decay (2.6):

$$\begin{aligned} |\mathcal{T}_t f(x)| &\leq C \int_{\mathbb{R}^d} |f(y)| \, t^{-d/2} (1 + |x-y|/\sqrt{t})^{-N} \, dy \\ &\leq C \, \frac{1}{t^{d/2}} \int_{|x-y| \leq \sqrt{t}} |f(y)| \, dy + \sum_{j=1}^{\infty} \frac{1}{2^{jN}} \frac{1}{t^{d/2}} \int_{|x-y| \sim 2^j \sqrt{t}} |f(y)| \, dy. \end{aligned}$$

Let us observe that for $x \in \alpha B_{n,k}$, there exist constants $C_1(\alpha)$ and $C_2(\alpha)$ such that $C_1(\alpha)\rho(x) \leq \rho(x_{n,k}) \leq C_2(\alpha)\rho(x)$. Choose K such that $C_1(\alpha)K/\sqrt{2} \geq 1/2$. Thus, for $j \geq 0$, one has $2^j \sqrt{t} \geq K\rho(x_k) \geq \frac{1}{2} \, 2^{-n_0/2}$ when $x \in \alpha B_{n,k}$, thus

$$\frac{1}{t^{d/2}} \int_{|x-y| \sim 2^j \sqrt{t}} |f(y)| \, dy \lesssim \frac{2^{jd}}{|B_{2^j \sqrt{t}}(x)|} \int_{B_{2^j \sqrt{t}}(x)} |f(y)| \, dy \, \leq \, 2^{jd} \, \|f\|_{BMO_{\mathcal{L}}},$$

which gives the result. For the second inequality (5.5), we use the following result that compares the kernels of the heat operators for the Laplacian and $\mathcal{L}$.

Lemma 5.7. [6, Proposition 2.16] *There exists a nonnegative Schwartz class function w in $\mathbb{R}^d$ so that*

$$|h_t(x-y) - k_t(x,y)| \leq \begin{cases} \left(\frac{\sqrt{t}}{\rho(x)} \right)^{\delta} w_t(x-y), & \text{for } \sqrt{t} \leq \rho(x) \\ \left(\frac{\sqrt{t}}{\rho(y)} \right)^{\delta} w_t(x-y), & \text{for } \sqrt{t} \leq \rho(y) \\ w_t(x-y), & \text{elsewhere,} \end{cases} \tag{5.8}$$

where $w_t(x-y) = t^{-d/2} w((x-y)/\sqrt{t})$.

Also, if $x \in \alpha B_{n,k}$, $\rho(x) \sim \rho(x_{n,k})$ and $\sqrt{t} \le K\rho(x_{n,k})$, hence

$$
\begin{aligned}
|(\mathcal{T}_t - T_t)f(x)| &\le \int_{\mathbb{R}^d} \left(\tfrac{\sqrt{t}}{\rho(x)}\right)^{\delta} w_t(x-y)|f(y)|\,dy \\
&\le C\left(\tfrac{\sqrt{t}}{\rho(x_{n,k})}\right)^{\delta} \sum_{j=0}^{\infty} 2^{-j(N-d)} \tfrac{1}{|B_{2^j\sqrt{t}}(x)|} \int_{B_{2^j\sqrt{t}}(x)} |f(y)|\,dy \\
&= C\left(\tfrac{\sqrt{t}}{\rho(x_{n,k})}\right)^{\delta} \Bigg(\sum_{1\le 2^j \le \frac{2^{-n_0/2}}{2\sqrt{t}}} 2^{-j(N-d)} \tfrac{1}{|B_{2^j\sqrt{t}}(x)|} \int_{B_{2^j\sqrt{t}}(x)} |f(y)|\,dy \\
&\qquad + \sum_{2^j > \frac{2^{-n_0/2}}{2\sqrt{t}}} 2^{-j(N-d)} \|f\|_{BMO_{\mathcal{L}}} \Bigg).
\end{aligned}
$$

To bound the first term, let us observe that for balls of radius smaller than a constant times the critical radius of its center, it is not difficult to see that there exists $C = C(\beta) > 0$ so that, for all $f \in BMO_{\mathcal{L}}$ and $B = B_r(x)$ with $r < \beta\rho(x)$, then

$$|f_B| \le C\,(1 + \log \tfrac{\rho(x)}{r})\,\|f\|_{BMO_{\mathcal{L}}}.$$

Thus, for j such that $1 \le 2^j \le \frac{2^{-n_0/2}}{2\sqrt{t}}$,

$$\tfrac{1}{|B_{2^j\sqrt{t}}(x)|} \int_{B_{2^j\sqrt{t}}(x)} |f(y)|\,dy \le C\left(1 + \log \tfrac{\rho(x)}{2^j\sqrt{t}}\right) \le C\left(1 + \log \tfrac{\rho(x_{n,k})}{\sqrt{t}}\right),$$

and therefore

$$
\begin{aligned}
|(\mathcal{T}_t - T_t)f(x)| &\le C\left(\tfrac{\sqrt{t}}{\rho(x_{n,k})}\right)^{\delta} \left(1 + \log \tfrac{\rho(x_{n,k})}{\sqrt{t}}\right) \|f\|_{BMO_{\mathcal{L}}} \sum_{j=0}^{\infty} 2^{-j(N-d)} \\
&\le C\,\|f\|_{BMO_{\mathcal{L}}}.
\end{aligned}
$$

Finally, the inequality (5.6) is a result coming from the vector-valued singular integrals theory (see [4] for the details).

Let us observe that the same arguments as the ones showed above give the boundedness of the non-tangential maximal function

$$\mathcal{T}^{**}f(x) = \sup_{|x-y|<t} |\mathcal{T}_t f(y)|, \quad x \in \mathbb{R}^d.$$

Also, the proofs of the boundedness in $BMO_{\mathcal{L}}$ for $P^{\mathcal{L},*}$ and $\mathcal{I}_\alpha$ is similar, while for the square function $s_{\mathcal{Q}}$ an extra ingredient, a “perturbation formula”is needed.

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