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TRANSFERENCE RESULTS FOR MULTIPLIERS, MAXIMAL MULTIPLIERS AND TRANSPLANTATION OPERATORS ASSOCIATED WITH FOURIER-BESSEL EXPANSIONS AND HANKEL TRANSFORM

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Abstract. Our objective in this survey is to present some results concerning to transference of multipliers, maximal multipliers and transplantation operators between Fourier-Bessel series and Hankel integrals. Also we list some related problems that can be interesting and that have not been studied yet.

From August 31st to September 3rd, 2004, was held in Merlo (San Luis, Argentine) the congress "VII Encuentro de Analistas Alberto Calderón y I Encuentro Conjunto Hispano-Argentino de Análisis". This meeting was dedicated to Professor Roberto Macías in his 60th birthday. In these notes we include the main results that were commented in the talk presented by the author there. That talk was devoted to Professor Roberto Macías who has been an example for many Spanish and Argentine mathematicians and so is this note.

Our purpose is to present some results about transference of boundedness of multipliers, maximal operators associated to multipliers and transplantation operators relating mainly Hankel transforms and Fourier-Bessel expansions settings. The author knew these topics when he wrote the papers [10] and [11] jointly with Professor Krzysztof Stempak who introduced him in the questions of transference. The author would like to thank to Professor Stempak all the very fruitfull and nice discussions about these and other mathematical topics.

1. Transference of boundedness for multipliers.

Although all the following definitions and results related to Fourier integrals and series can be given in dimension $n \geq 1$, to simplify we will write them for $n = 1$. As it is wellknown the Fourier transform on $\mathbf{R}$ is defined by

$$\mathcal{F}(f)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iyx} dx, \;\; y \in \mathbf{R},$$

provided that $f \in L^1(\mathbf{R})$. Moreover Hausdorff-Young Theorem says that the Fourier transform can be extended to $L^p(\mathbf{R})$ as a bounded operator from $L^p(\mathbf{R})$ into $L^{p'}(\mathbf{R})$, for every $1 \leq p \leq 2$, where p' denotes the exponent conjugated to p, that is, $p' = \frac{p}{p-1}$.

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$\mathcal{F}$ is a surjective map (from $L^p(\mathbf{R})$ onto $L^{p'}(\mathbf{R})$) only when $p = 2$. An interesting characterization of the range of $\mathcal{F}$ on L^p-spaces can be encountered in [38].

Assume that $1 \leq p < \infty$. A bounded measurable function m defined on $\mathbf{R}$ is a Fourier p-multiplier when, for every $f \in L^p(\mathbf{R})$, there exists $g \in L^p(\mathbf{R})$ such that $\mathcal{F}(g) = m\mathcal{F}(f)$. In this case we represent by T_m the mapping defined by $T_m(f) = g = (\mathcal{F}^{-1}(m\mathcal{F}(f)))$, $f \in L^p(\mathbf{R})$. A simple argument using the closed graph theorem shows in this case that T_m is a bounded operator on $L^p(\mathbf{R})$. We denote by $M_p(\mathbf{R})$ the space of Fourier p-multipliers and by $\|m\|_{M_p(\mathbf{R})}$ the multiplier norm of m, that is, the norm of T_m as a bounded operator from $L^p(\mathbf{R})$ into itself. It is clear that T_m is bounded from $L^2(\mathbf{R})$ into itself. Many results have been proved establishing conditions of the function m in order that $m \in M_p(\mathbf{R})$ (Mihlin-Hormander theorem, Marcinkiewicz theorem, ...).

Analogous definitions can be made when Fourier series instead of Fourier integrals are considered. If $\{m_n\}_{n\in\mathbf{Z}}$ is a bounded sequence then it is a Fourier p-multiplier when, for every $f \in L^p(0, 2\pi)$, there exists $g \in L^p(0, 2\pi)$, such that $\hat{g}(n) = m_n\hat{f}(n)$, $n \in \mathbf{Z}$. Here $\hat{f}(n)$ denotes the n-th Fourier coefficient of f, for every $n \in \mathbf{Z}$. These multipliers can be seen as Fourier p-multipliers on the discrete subgroup $\mathbf{Z}$ of $\mathbf{R}$. By $M_p(\mathbf{Z})$ we represent the space of Fourier p-multipliers and by $\|m\|_{M_p(\mathbf{Z})}$ the mutiplier norm of m, that is, the norm of T_m as a bounded operator from $L^p(0, 2\pi)$ into itself.

Above definitions can be given also in a weak L^p-setting.

There are a number of theorems relating L^p multipliers on $\mathbf{R}$ and on $\mathbf{Z}$. One well-known of them due to de Leeuw ([31]) is the following. We call a bounded measurable function m defined on $\mathbf{R}$ regulated when

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} m(x+t)dt = f(x),$$

for every $x \in \mathbf{R}$.

Theorem 1.1. *(*[31, Proposition 3.3]*). Let m be a bounded measurable function on $\mathbf{R}$. Suppose that m is regulated and in $M_p(\mathbf{R})$. Then the restriction of m to $\mathbf{Z}$ is in $M_p(\mathbf{Z})$.*

A converse to the theorem of de Leeuw is the following result proved by Igari.

Theorem 1.2. *(*[23]*). Let $1 < p < \infty$ and assume that m is a bounded measurable function on $\mathbf{R}$, continuous except on a set of Lebesgue measure zero. If $\{m(\varepsilon n)\}_{n\in\mathbf{Z}} \in M_p(\mathbf{Z})$ for all sufficiently small $\varepsilon > 0$ and $\liminf_{\varepsilon\to 0^+} \|\{m(\varepsilon n)\}_{n\in\mathbf{Z}}\|_{M_p(\mathbf{Z})} < \infty$, then $m \in M_p(\mathbf{R})$ and*

$$\|m\|_{M_p(\mathbf{R})} \leq \liminf_{\varepsilon\to 0^+} \|\{m(\varepsilon n)\}\|_{M_p(\mathbf{Z})}.$$

In [36, Problem 5] asked if a bounded measurable function m on $\mathbf{R}$ that is continuous at lattice points and determines a weak type $(1, 1)$ multiplier operator, then the restriction of m of these points determine a weak type $(1, 1)$ multiplier opeartor on $\mathbf{Z}$. That is, can [31, Proposition 3.3] be extended to the weak $(1, 1)$ case?. The positive answer to this question was given by Asmar, Berkson and Bourgain in [1]. Wozniakowski ([57]) got another proof of the conjecture of Pelczynski.

Saeki ([39]) extended the theorems of de Leeuw in the context of the general locally compact abelian groups. Also the results about transference of multipliers established in [1], [2], [3], [4] and [57] work in this abstract setting.

Notions of p-multipliers also can be defined in a similar way for other orthonormal systems or integral transforms. Motivated by the studies of de Leeuw, Igari ([24]) got a transference results for p-multipliers of Fourier series associated with Jacobi polynomials and p-multipliers for Hankel transforms.

Let $\alpha, \beta > -1$. For every $n \in \mathbf{N}$ the Jacobi polynomial of degree n is

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\bigg((1-x)^{n+\alpha}(1+x)^{n+\beta}\bigg), \ \ x \in (-1,1).$$

The system $\{P_n^{(\alpha,\beta)}\}_{n\in\mathbf{N}}$ is orthogonal on $(0, 2\pi)$ respect to the measure

$$d\mu_{\alpha,\beta}(\theta) = (sin(\theta/2))^{2\alpha+1}(cos(\theta/2))^{2\beta+1}d\theta.$$

If $f \in L^1((0, 2\pi), d\mu_{\alpha,\beta})$ and $n \in \mathbf{N}$, the n-th (α, β)-Fourier coefficient is

$$\hat{f}(n) = \int_0^{2\pi} f(\theta)P_n^{\alpha,\beta}(cos\theta)d\mu_{\alpha,\beta}(\theta),$$

and the normalizing number $h_n^{\alpha,\beta}$ is defined by

$$h_n^{\alpha,\beta} = \bigg(\int_0^{2\pi}(P_n^{\alpha,\beta}(cos\theta))^2 d\mu_{\alpha,\beta}(\theta)\bigg)^{-1}.$$

We say that a sequence $\{m_n\}_{n\in\mathbf{N}}$ of complex numbers is an (α, β)-Jacobi p-multiplier when the operator $T_{\{m_n\}}$ defined through

$$T_{\{m_n\}}(f)(\theta) = \sum_{n=0}^{\infty} m_n \hat{f}(n)h_n^{\alpha,\beta}P_n^{(\alpha,\beta)}(cos\theta)$$

is bounded form $L^p((0, 2\pi), d\mu_{\alpha,\beta})$ into itself.

The Hankel integral transform $h_\mu(f)$ of $f \in L^1((0,\infty), x^{2\mu+1})$ is given by

$$h_\mu(f)(y) = \int_0^\infty (xy)^{-\mu}J_\mu(xy)f(x)x^{2\mu+1}dx, \ \ y \in (0,\infty).$$

Here J_μ denotes the Bessel function of the first kind and order $\mu > -1$. The Fourier transform of a radial function in $\mathbf{R}^n$ reduces to a Hankel transform of order $\mu = (n-2)/2$.

A bounded measurable function m on $(0,\infty)$ is a μ-Hankel p-multiplier when the operator

$$T_m(f)(x) = \int_0^\infty m(y)h_\mu(f)(y)(xy)^{-\mu}J_\mu(xy)y^{2\mu+1}dy$$

is bounded from $L^p((0,\infty), x^{2\mu+1}dx)$ into itself.

The result of Igari relating (α, β)-Jacobi p-multiplier with α-Hankel p-multiplier can be written as follows.

Theorem 1.3. *([24, Theorem]) Let $1 \le p < \infty$ and $\alpha, \beta > -1$. Suppose that m is a bounded measurable function on $(0,\infty)$ that is continuous except at most on a set* *of Lebesgue measure zero. If for every* $\varepsilon > 0$ *(small) the sequence* $\{m(\varepsilon n)\}_{n\in\mathbf{N}}$ *is an* (α,β)*-Jacobi p-multiplier, and*

$$lim\,inf_{\varepsilon\to 0}\|\{m(\varepsilon n)\}_{n\in\mathbf{N}}\|_{(\alpha,\beta)-M_p((0,2\phi),d\mu_{\alpha,\beta})} < \infty,$$

then m *is an* α*-Hankel p-multiplier and*

$$\|m\|_{\alpha-M_p((0,\infty),x^{2\alpha+1}dx)} \leq lim\,inf_{\varepsilon\to 0}\|\{m(\varepsilon n)\}_{n\in\mathbf{N}}\|_{(\alpha,\beta)-M_p((0,2\phi),d\mu_{\alpha,\beta})}.$$

The relation between the Jacobi and Hankel contexts comes given by the asymptotic behaviour of Jacobi polynomials

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(cos\theta) = m\frac{\Gamma(n+\alpha+1)}{n^\alpha}\left(\frac{\theta}{sin\theta}\right)^{1/2} J_\alpha(m\theta) + g(\theta,n),$$

where

$$g(\theta,n) = \begin{cases} O(\theta^{1/2}n^{-3/2}), & \text{cuando } C/n \leq \theta \leq \pi-\varepsilon \\ O(\theta^{\alpha+2}n^\alpha), & \text{cuando } 0 < \theta \leq C/n \end{cases}$$

$m = n + \frac{\alpha+\beta+1}{2}$, $n \in \mathbf{N}$ and $\varepsilon, C > 0$ are fixed.

Later Connet and Schwartz ([14]) obtained the corresponding weak type results relating (α,β)-Jacobi p-multiplier with α-Hankel p-multiplier. It is quotable also the paper of Gasper and Trebels ([16]) about Jacobi and Hankel multipliers.

Stempak ([41]) established relations between multipliers for Laguerre expansions and Hankel multipliers. Let L_n^α denote the Laguerre polynomial of degree $n \in \mathbf{N}$ and order α. We define, for every $n \in \mathbf{N}$, the Laguerre function

$$\phi_n^\alpha(x) = \left(\frac{2n!}{\Gamma(n+\alpha+1)}\right)^{1/2} e^{-x^2/2}L_n^\alpha(x^2), \;\; x \in (0,\infty).$$

The system $\{\phi_n^\alpha\}_{n\in\mathbf{N}}$ is an orthonormal basis in $L^2((0,\infty),x^{2\alpha+1})$. As above, if $1 \leq p < \infty$, a sequence $\{m_n\}_{n\in\mathbf{N}}$ of complex numbers is be said an α-Laguerre p-multiplier when the operator $T_{\{m_n\}}$ defined through

$$T_{\{m_n\}}(f)(x) = \sum_{n=0}^{\infty} m_n\hat{f}(n)\phi_n^\alpha(x)$$

is bounded from $L^p((0,\infty),x^{2\alpha+1}dx)$ into itself. Here $\hat{f}(n)$ represents the n-th coefficient of f respect to the Laguerre system $\{\phi_n^\alpha\}_{n\in\mathbf{N}}$. A relation between Laguerre functions and Hankel transforms can be encountered in the context of the algebra $L^1_{rad}(\mathbf{C}^n)$ (see [41, p. 287-288]). Moreover, Hilb's asymptotic formula for Laguerre functions ([48, Theorem 8.22.4]) establishes a relation between Laguerre and Bessel functions that is fundamental to prove the following transference result.

Theorem 1.4. *(*[41, Theorem 1.1]*) Let* $1 \leq p < \infty$ *and* $\alpha > -1$*. Suppose that* m *is a bounded measurable function on* $(0,\infty)$ *that is continuous except at most on a set of Lebesgue measure zero. If for every* $\varepsilon > 0$ *(small) the sequence* $\{m(\varepsilon\sqrt{n})\}_{n\in\mathbf{N}}$ *is an* α*-Laguerre p-multiplier, and*

$$lim\,inf_{\varepsilon\to 0}\|\{m(\varepsilon\sqrt{n})\}_{n\in\mathbf{N}}\|_{\alpha(Laguerre)-M_p((0,\infty),x^{2\alpha+1}dx)} < \infty,$$

then m *is an* α*-Hankel p-multiplier and*

$$\|m\|_{\alpha(Hankel)-M_p((0,\infty),x^{2\alpha+1}dx)} \leq lim\,inf_{\varepsilon\to 0}\|\{m(\varepsilon n)\}_{n\in\mathbf{N}}\|_{\alpha(Laguerre)-M_p((0,\infty),x^{2\alpha+1}dx)}.$$

An interesting question is whether converses of Theorems 1.3 and 1.4 similar to de Leeuw's result are true.

In [10] Betancor and Stempak proved transference boundedness results relating multipliers for Hankel transforms and Fourier-Bessel series in a weighted L^p-setting. There they considered for the Hankel transform a different definition. Following for instance Zemanian [58] (see also [50]), in [10] the Hankel transform $\mathcal{H}_\mu f$ of $f \in L^1(0,\infty)$ is defined by

$$\mathcal{H}_\mu(f)(y) = \int_0^\infty \sqrt{xy} J_\mu(xy) f(x) dx, \;\; y \in (0,\infty).$$

This transform $\mathcal{H}_\mu$ can be extended to $L^2(0,\infty)$ as an isometry of $L^2(0,\infty)$ for every $\mu > -1$. In [10, Lemma 2.7] a proof of this property is presented using the corresponding one for Fourier-Bessel series. Thus [10, Lemma 2.7] can be seen a transference result. If $1 \leq p < \infty$ and $a \in \mathbf{R}$, we denote by $\|.\|_{p,a}$ the norm in the weighted Lebesgue space $L^p((0,\infty), x^a dx)$. A bounded measurable function m on $(0,\infty)$ is called a μ-Hankel (p,a)-multiplier provided that

$$\|\mathcal{H}_\mu(m \cdot \mathcal{H}_\mu f)\|_{p,a} \leq C\|f\|_{p,a}.$$

To simplify the multiplier norm of m and is denoted by $\|m\|_{(p,a)}$.

Given $\mu > -1$, let $\lambda_n = \lambda_{n,\mu}$, $n = 1, 2, \dots$, denote the sequence of succesive positive zeros of $J_\mu(x)$. Then the functions

$$\psi_n^\mu(x) = d_{n,\mu}(\lambda_n x)^{1/2} J_\mu(\lambda_n x), \qquad\qquad d_{n,\mu} = \sqrt{2}|\lambda_n^{1/2} J_{\mu+1}(\lambda_n)|^{-1},$$

$n = 1, 2, \dots$, form a complete orthonormal system in $L^2((0,1), dx)$ (for completeness, see [22]).

To every $f \in L^1(0,1)$ we associate its Fourier-Bessel series

$$f(x) \sim \sum_{n=1}^\infty \hat{f}(n)\psi_n^\mu(x), \qquad \hat{f}(n) = \int_0^1 f(x)\psi_n^\mu(x)dx.$$

A comprehensive study of Fourier–Bessel expansions is contained in Chapter XVII of Watson's monograph [54]. Also, in a serie of papers Benedek and Panzone ([7], [8] and [9]) investigated the convergence of series of Bessel functions.

Slightly abusing the notation we will use the symbol $\| \cdot \|_{p,a}$ in the same sense as before but now restricted to functions defined on $(0,1)$. A bounded sequence $\{m_n\}_{n=1}^\infty$ of complex numbers is called a μ-Fourier-Bessel (p,a)-multiplier provided

$$\|\sum_{n=1}^\infty m_n \hat{f}(n)\psi_n^\mu(x)\|_{p,a} \leq C\|f\|_{p,a}$$

The multiplier norm of $\{m_n\}_{n=1}^\infty$ is denoted by $\|\{m_n\}_{n\in\mathbf{N}}\|_{(p,a)}$.

In this context the following result is the corresponding version of the Igari's theorem.

Theorem 1.5. *(*[10, theorem 2.1]*) Let $1 < p < \infty$, $a \in \mathbf{R}$ and m be a bounded function on $(0,\infty)$ continuous except on a set of Lebesgue measure zero. If $\{m(\varepsilon\lambda_n)\}_{n\in\mathbf{N}}$ is a μ-Fourier-Bessel (p,a)-multiplier for all sufficiently small $\varepsilon > 0$ and we have*

$\liminf_{\varepsilon\to 0^+} \|\{m(\varepsilon\lambda_n)\}_{n\in\mathbf{N}}\|_{(p,a)}$ *is finite then* m *is a* μ*-Hankel transform* (p,a)*-multiplier and*

$$\|m\|_{(p,a)} \leq \liminf_{\varepsilon\to 0^+} \|\{m(\varepsilon\lambda_n)\}_{n\in\mathbf{N}}\|_{(p,a)}.$$

For $p = 1$ and $a = 0$ [10, Theorem 2.2] is a weak substitute of Theorem 1.5.

We now comment some applications and extensions of Theorem 1.5.

Wing [55] proved that the partial sum operators

$$S_N^{\psi} f(x) = \sum_{n=1}^{N} \hat{f}(n)\psi_n^{\mu}(x)$$

for the $\{\psi_n^{\mu}\}$-expansions, $\mu \geq -1/2$, are uniformly bounded in any $L^p((0,1),dx)$, $1 < p < \infty$. Benedek and Panzone [8] then extended this result to $-1 < \mu < -1/2$ and the p-range $2/(2\mu+3) < p < -2/(2\mu+1)$. Theorem 1.5 thus gives (with $m = \chi_{(0,1)}$ and $a = 0$).

Corollary 1.6. *(*[10, Corollary 4.4]*) Let either* $\mu \geq -1/2$ *and* $1 < p < \infty$ *or* $-1 < \mu < -1/2$ *and* $2/(2\mu+3) < p < -2/(2\mu+1)$*. Then the Hankel transform partial sum operators*

$$S_R^{\mathcal{H}} f(x) = \mathcal{H}_\mu(\chi_{(0,R)} \cdot \mathcal{H}_\mu f)(x), \qquad R > 0,$$

are uniformly bounded in $L^p((0,\infty),dx)$*.*

Observe that uniform boundedness of $S_R^{\mathcal{H}} f$ in the case $\mu \geq -1/2$ is known as Wing's theorem [56]. For the result when $-1 < \mu < -1/2$ can see also [53].

In [11] Betancor and Stempak established a vector valued analogue of Theorem 1.5.

Now $\underline{\mu}$ denotes an arbitrary fixed sequence of indices, $\underline{\mu} = (\mu_0, \mu_1, \dots)$, each μ_k is bigger than -1, $k \in \mathbf{N}$. By $L^{p,a}(\ell^q)$, $1 \leq p < \infty$, $1 \leq q < \infty$, $a \in \mathbb{R}$, we denote the vector valued (weighted) Lebesgue space of those sequences of functions $\{g_k\}_{k=0}^{\infty}$ for which the quantity

$$\|\{g_k\}\|_{p,a;q} = \left(\int_I \left(\sum |g_k(x)|^q\right)^{p/q} x^a dx\right)^{1/p}$$

is finite where, depending on the context, either $I = (0,\infty)$ or $I = (0,1)$.

A sequence $\underline{m} = (m_0, m_1, \dots)$ of jointly bounded, $\sup_{k\geq 0} \|m_k\|_\infty < \infty$, measurable functions m_k on $(0,\infty)$ is called an $L^{p,a}(\ell^q)$ $\underline{\mu}$-Hankel multiplier provided

$$\|\{\mathcal{H}_{\mu_k}(m_k \cdot \mathcal{H}_{\mu_k} g_k)\}\|_{p,a;q} \leq C\|\{g_k\}\|_{p,a;q}$$

with a constant C independent of $\{g_k\}$, $g_k \in C_c^\infty(0,\infty)$. The least constant for which the above inequality holds is called the multiplier norm of the multiplier operator generated by $\underline{m}$ and is denoted by $\|\underline{m}\|_{(p,a;q)}$.

A sequence $\underline{m} = (m_0, m_1, \dots)$ of jointly bounded sequences $m_k = (m_k(1), m_k(2), \dots)$, $k = 0, 1, \dots$, is called an $L^{p,a}(\ell^q)$ $\underline{\mu}$-Fourier-Bessel multiplier provided

$$\|\{\sum_{n=1}^{\infty} m_k(n) c_n^{\mu_k}(g_k) \psi_n^{\mu_k}(x)\}_{k=0}^{\infty}\|_{p,a;q} \leq C\|\{g_k\}\|_{p,a;q}$$

with a constant C independent of $\{g_k\}$, $g_k \in C_c^\infty(0,1)$. The multiplier norm of the multiplier operator generated by $\underline{m}$ is denoted by $\|\underline{m}\|_{(p,a;q)}$.

A vector valued version of Theorem 1.5 is the following.

Theorem 1.7. *Assume $\underline{\mu}$, $1<p<\infty$, $1<q<\infty$, $a\in\mathbb{R}$ are given. Let $\underline{m}=(m_0,m_1,\dots)$ be a sequence of bounded functions on $(0,\infty)$ such that $\sup_{k\ge 0}\|m_k\|_\infty<\infty$ and each m_k is continuous except on a set of Lebesgue measure zero. If, for all sufficiently small $\varepsilon>0$, the sequence $\underline{m}^\varepsilon=(m_0^\varepsilon,m_1^\varepsilon,\dots)$ of sequences*

$$m_k^\varepsilon=(m_k(\varepsilon\lambda_{1,\mu_k}),m_k(\varepsilon\lambda_{2,\mu_k}),\dots),\qquad k=0,1,\dots,$$

is an $L^{p,a}(\ell^q)$ $\underline{\mu}$-Fourier–Bessel multiplier and, moreover,

$$\liminf_{\varepsilon\to 0^+}\|\underline{m}^\varepsilon\|_{(p,a;q)}<\infty$$

then $\underline{m}$ is also an $L^{p,a}(\ell^q)$ $\underline{\mu}$-Hankel multiplier and

$$\|\underline{m}\|_{(p,a;q)}\le\liminf_{\varepsilon\to 0^+}\|\underline{m}^\varepsilon\|_{(p,a;q)}.$$

Córdoba [15, Theorem] (see also [33]), proving that the Fourier disc multiplier is bounded in the mixed norm space $L^p_{rad}(L^2_{ang})(\mathbb{R}^d)$ for $2d/(d+1)<p<2d/(d-1)$, $d\ge 2$, reduced the problem to showing the following vector valued inequality ([15, p. 23, (A)])

$$\Big(\int_0^\infty\Big(\sum_0^\infty|\mathcal{H}_{\mu(k,\alpha)}(\chi_{(0,1)}\cdot\mathcal{H}_{\mu(k,\alpha)}g_k)(x)|^2\Big)^{p/2}x^a\,dx\Big)^{1/p}\le C\Big(\int_0^\infty\Big(\sum_0^\infty|g_k(x)|^2\Big)^{p/2}x^a\,dx\Big)^{1/p}$$

where $\alpha=(d-2)/2$, $\mu(k,\alpha)=k+\alpha$, $k=0,1,\dots$, and $a=(2\alpha+1)(1-p/2)$. In our terminology this means that the constant sequence $\{\chi_{(0,1)}\}_{k=0}^\infty$ is an $L^{p,a}(\ell^2)$ $\underline{\mu}$-Hankel multiplier for the index sequence $\underline{\mu}=\{\mu(k,\alpha)\}_{k=0}^\infty$ and the value of a specified above.

In [6] Balodis and Córdoba considered the convergence of multidimensional Fourier-Bessel series in the mixed norm space $L^p_{rad}(L^2_{ang})(\mathbb{B}^d)$, $\mathbb{B}^d$ denotes the unit ball in $\mathbb{R}^d$. More precisely, they proved that the partial sums operators S_{NM} are uniformly bounded on $L^p_{rad}(L^2_{ang})(\mathbb{B}^d)$ provided $2d/(d+1)<p<2d/(d-1)$. This was done under the additional condition $N\ge AM+1$ where $A=A(d)$ was an absolute constant. By the partial sum operator $S_{N,M}$ the following is meant: take a suitable function f on $\mathbb{B}^d$ and consider the expansion

$$f(\overline{x})=f(rx')\sim\sum_{m=0}^\infty\sum_{s=1}^{d(m)}f_{ms}(r)Y_{ms}(x');$$

$\{Y_{ms}\}_{s=1}^{d(m)}$ is an orthonormal basis of spherical harmonics of degree m, $r=\|\overline{x}\|$, $\|x'\|=1$. Then expand each f_{ms} with respect to the complete orthonormal system $\{\phi^d_{m,n}\}_{n=1}^\infty$ on $L^2((0,1),r^{d-1}dr)$, $\phi^d_{m,n}(r)=\psi_n^{m+(d-2)/2}(r)r^{-(d-1)/2}$:

$$f_{ms}(r)\sim\sum_{n=1}^\infty\langle f_{ms},\phi^d_{m,n}\rangle\phi^d_{m,n}(r),$$

and by $S_{N,M}f$ we mean

$$S_{N,M}f(\overline{x})=\sum_{m=0}^M\sum_{s=1}^{d(m)}\sum_{n=1}^N\langle f_{ms},\phi^d_{m,n}\rangle\phi^d_{m,n}(r)Y_{ms}(x').$$

As application of Theorem 1.7 in [11] it is shown that the result established in [15, Theorem] is implied by the one proved in [6, Theorem 3]. Moreover, by arguing as in in the proof of [6, Theorem 3] we can extend this result by replacing the exponent 2 in the angular part by other exponent $\frac{4}{3} < q < 4$. Then, again by using Theorem 1.7, we deduce [37, Theorem 1].

In this moment it is not known if a converse of Theorem 1.5 is true. If we have proved this converse result then, we can deduce, for instance, the result proved in [6] about n-dimensional Fourier series from the one concerning to Hankel integrals established in [15]. In a similar line of this question is the following: to transfer Hausdorff-Young inequalities from Hankel integrals to Fourier-Bessel series when vector valued functions are considered. The usual arguments in the Fourier setting developed for instance in [29] and [30] do not work in this case, because there not exists a group structure in the Hankel setting (see [21] about convolutions and translations for Hankel transforms). I think that is an interesting question to obtain characterizations for Hilbert spaces via Hankel transforms and Fourier-Bessel series similar to the known ones in terms of Fourier integrals and series.

Recently Stempak [45] has obtained an abstract version of Igari's theorem that applies to get transference results for Fourier multipliers in Lorentz spaces.

2. Transference of boundedness for maximal operators associated with multipliers.

If m is a bounded measurable function on $\mathbf{R}$, we define the maximal operator T_m^* associated to the multiplier m respect to the Fourier integral on $\mathbf{R}$ as follows

$$T_m^*(f) = \sup_{\varepsilon>0} |\mathcal{F}^{-1}(m(\varepsilon y)\mathcal{F}(f)(y))|.$$

We say m is p-maximal on $\mathbf{R}$ (or weak-p maximal) when T_m^* is bounded from $L^p(\mathbf{R})$ into itself (or from $L^p(\mathbf{R})$ into $L^{p,\infty}(\mathbf{R})$). The corresponding maximal multiplier operator associated with the Fourier series is defined in a similar way. That is,

$$\tilde{T}_m^*(f)(x) = \sup_{\varepsilon>0} |\sum_{n\in\mathbf{Z}} m(\varepsilon n)\hat{f}(n)e^{inx}|.$$

We refer to the boundedness of the maximal operator $\tilde{T}_m^*$ saying that m is p-maximal or weak p-maximal on $(0, 2\pi)$.

Kenig and Tomas ([28]) established the following important result that shows that the boundedness of the operator T_m^* is equivalent to the boundedness of the operator $\tilde{T}_m^*$.

Theorem 2.1. *(*[28, Theorem 1]*) Let $1 < p < \infty$ and let m be a regulated bounded measurable function on $\mathbf{R}$. Then, m is p-maximal (or weak p-maximal) on $(0, 2\pi)$ if and only if m is p-maximal (or weak p-maximal on $\mathbf{R}$).*

As it is wellknown by taking $m = 1$ that $\tilde{T}_m^*$ is weak p-maximal, $1 \leq p \leq 2$, is equivalent that the almost everywhere convergence of Fourier series of functions in $L^p(0, 2\pi)$. In this context, transference theorems as the one due to Kenig and Tomas plays a relevant role (see for instance [28, Section 3]).

Extensions of Theorem 2.1 to abstract settings including the end case $p = 1$ can be found in [1], [2], [3] and [4], amogst others.

Analogous definitions can be given for maximal multipliers operators when Fourier integral and Fourier series are replaced by other integral transforms (say Hankel transforms) or other orthogonal series expansions (say Jacobi, Laguerre or Bessel expansions).

Kanjin [25] showed a relation similar to the one stated in Theorem 2.1 between maximal operators defined by Jacobi series and maximal operators defined by Hankel integral multipliers. That relation was used for proving almost everywhere convergence of spherical means for radial functions on $\mathbf{R}^n$ by taking into account known L^p estimates for certain maximal multipliers associated with Jacobi expansions.

By understanding the maximal multipliers operators T_m^* and $\tilde{T}_m^*$ in the Hankel and Fourier-Bessel setting, respectively, Betancor and Stempak established in [10] the following result involving weighted L^p-spaces.

Theorem 2.2. *(*[10, Theorem 2.3]*) Let $1 < p < \infty$, $a \in \mathbf{R}$ and m be a bounded measurable function on $(0,\infty)$ continuous except at most in a set of Lebesgue measure zero. If*

$$\|\tilde{T}_m^* f\|_{p,a} \le C\|f\|_{p,a}$$

with a constant $C > 0$ independent of f in $C_c^\infty(0,1)$ then

$$\|T_m^* f\|_{p,a} \le C\|f\|_{p,a}$$

independently of f in $C_c^\infty(0,\infty)$ (with the same constant C).

It is not known if the converse of Theorem 2.2 holds true.

L^p-estimates of maximal operators for the partial sums of Fourier-Bessel expansions had be proved by Gilbert [17] (see also [19] for a weighted version). Theorem 2.2 allows us to obtain L^p-estimates for maximal operators associated with partial Hankel integrals (that appear when we evaluate spherical partial Fourier integrals of radial functions in $\mathbf{R}^n$).

Stempak ([42]) established a result similar to Kanjin's one relating maximal operators defined for multipliers for Laguerre expansions and Hankel transforms.

3. Transference of boundedness for transplantation operators.

Guy [20] showed that the size of the Hankel transform of any suitable function, when measured in the (weighted) L^p-norm, remains the same whatever the order of the Hankel transform is. More precisely, given $\nu, \mu \ge -1/2$, $1 < p < \infty$ and $-1 < a < p-1$ there is a constant $C = C(\nu,\mu,p,a)$ such that for every appropriate function f

$$C^{-1}\|\mathcal{H}_\mu f\|_{p,a} \le \|\mathcal{H}_\nu f\|_{p,a} \le C\|\mathcal{H}_\mu f\|_{p,a}.$$

In another way, this can be expressed as

$$\|(\mathcal{H}_\nu \circ \mathcal{H}_\mu) f\|_{p,a} \le C\|f\|_{p,a}.$$

Another proof of Guy's transplantation theorem was given by Schindler [40] who found a explicit expression for the transplantation kernel $\mathcal{H}_\mu\mathcal{H}_\nu$. Recently Nowak and Stempak [35] have obtained a weighted version of the transplantation for Hankel transforms. They have seen that the transplantation kernel is in a local part ("near" the diagonal) a Calderón-Zygmund kernel, while in the global part he is something like Hardy kernels. They have introduced a class of weights wider than the usual Muckenhoupt A_p weights on $(0,\infty)$ suitable for Hankel transplantation operators.

With Guy's result a series of transplantion theorems involving continuous and discrete orthonormal expansions was initiated. Kanjin [26] established transplantation theorems for Laguerre expansions. A modified version of Kanjin's result has been given by Thangavelu [49]. This result can be seen a special case of a more general transplantation result shown by Stempak and Trebels [47].

Stempak [43] exhibited a connection between Laguerre expansions and Hankel transforms on the lebel of transplantation. He considered, for every $n \in \mathbf{N}$, the Laguerre function $\mathcal{L}_n^\alpha$, $\alpha > -1$, defined by

$$\mathcal{L}_n^\alpha(x) = \left(\frac{2n!}{\Gamma(n+\alpha+1)}\right)^{1/2} x^{\alpha+1/2} e^{-x^2/2} L_n^\alpha(x^2), \;\; x \in (0,\infty),$$

and established the following transplantation result.

Theorem 3.1. *([43, Theorem 1.1]). Let $1 < p < \infty$, $a \in \mathbf{R}$ and $\alpha, \gamma > -1$. If the Laguerre transplantation inequality*

$$\left\| \sum_{n=0}^{\infty} < f, \mathcal{L}_n^\gamma > \mathcal{L}_n^\alpha \right\|_{p,a} \leq C\|f\|_{p,a},$$

holds true, where, for every $n \in \mathbf{N}$,

$$< f, \mathcal{L}_n^\gamma >= \int_0^\infty f(x)\mathcal{L}_n^\gamma(x)dx,$$

and $\|.\|_{p,a}$ represents (as above) the norm in the space $L^p((0,\infty), x^a dx)$, the the Hankel transplantation inequality

$$\|(\mathcal{H}_\nu \circ \mathcal{H}_\mu)f\|_{p,a} \leq C\|f\|_{p,a},$$

ia also satisfied (with the same constant $C > 0$).

A result like Theorem 3.1 where Laguerre functions were replaced by Jacobi functions were established in [44, Theorem 2.1].

The corresponding property when Bessel functions are considered was proved in [10].

Theorem 3.2. *Let $1 < p < \infty$, $a \in \mathbf{R}$ and $\nu, \mu > -1$. If the Fourier–Bessel transplantation inequality*

$$\| \sum_{n=1}^{\infty} < f, \psi_n^\mu > \psi_n^\nu(x)\|_{p,a} \leq C\|f\|_{p,a},$$

holds true, where, for every $n \in \mathbf{N}$,

$$< f, \psi_n^\mu >= \int_0^\infty f(x)\psi_n^\mu(x)dx,$$

then the Hankel transplantation inequality

$$\|(\mathcal{H}_\nu \circ \mathcal{H}_\mu)f\|_{p,a} \leq C\|f\|_{p,a},$$

is also satisfied (with the same constant C).

Gilbert's result [17, Theorem 1], and its weighted extension [19, Theorem 1] give a weighted transplantation theorem for Fourier–Bessel expansions (the unweighted case is stated as Theorem A and Theorem B in [17]). Consequently, Theorem 3.2 gives the corresponding weighted transplantation inequality for the Hankel transform.

4. Some other questions.

In this section we present some questions related to the topics that we have discussed in the previous sections and that can be interesting for some readers.

a) Guadalupe and Kolyada [18] investigated transplantation theorems for ultraspherical expansions on Lorentz spaces. They completed, in some sense, previous results of Askey and Wainger [5]. We think that is an interesting question to obtain boundedness in Lorentz spaces for transplantation operators in the Bessel series setting. Then a transference result could allow us to prove the boundedness for transplantation Hankel operators.

b) In the celebrated paper [34], B. Muckenhoupt presented an exhaustive and deeper study about transplantation theorems for Jacobi series. Recently, Ciaurri and Stempak [12], inspired in the above quoted paper, have obtained transplantation results for Bessel series. An extension of Muckenhoupt's properties to Hardy spaces have been investigated by Miyachi [32]. In this moment Ciaurri and Stempak ([13]) are studying about Bessel series versions of Miyachi's results. Also Kanjin [27] have analized the transplantation problem in the Hankel setting on Hardy spaces. A transference theorem for the boundedness of the transplantion operators between Bessel series and Hankel transforms on Hardy spaces have not proved yet.

c) An open question is to obtain transference results relating discrete Jacobi and infinite Jacobi setting, that is, Jacobi expansions with Jacobi transforms. In this context the procedures used in the Bessel setting in [10] do not work because the homogeneity is missed (see [45]). A good understanding of the Jacobi case could allow to analyze more general situations like Chebli-Trimeche transforms ([51] and [52]).

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