REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Volumen 45, Número 2, 2004, Páginas 1–10

BOUNDARY NONLINEARITIES FOR A ONE-DIMENSIONAL p-LAPLACIAN LIKE EQUATION

P. AMSTER, M.C. MARIANI AND P. DE NÁPOLI

ABSTRACT. We study a nonlinear ordinary second order vector equation of p-Laplacian type under nonlinear boundary conditions. Applying Leray-Schauder arguments we obtain solutions under appropriate conditions. Moreover, for the scalar case we prove the existence of at least one periodic solution of the problem applying the method of upper and lower solutions.

INTRODUCTION

We consider a nonlinear one-dimensional problem for a vector function $u : [0, T] \to \mathbb{R}^N$ satisfying

$$(\phi(u'))' = f(t, u, u') \qquad \text{in} \quad (0, T) \tag{1}$$

where $\phi : \mathbb{R}^N \to \mathbb{R}^N$ is a homeomorphism.

We study problem (1) under the following nonlinear boundary conditions:

$$u(0) = h_1(u(T), u'(T)), \qquad u'(0) = h_2(u(T), u'(T)) \tag{NBC}$$

with $h_i : \mathbb{R}^{2N} \longrightarrow \mathbb{R}^N$ continuous.

Nonlinear boundary conditions for systems of semilinear ODE's have been considered by different authors (see e.g. [BL], [E], [KL], [S], [AMP] and [BL2] for more references). Using coincidence degree, Gaines and Mawhin [GM] proved a continuation theorem for a general nonlinear condition which in the scalar second order case reads

$$\gamma(u) = 0$$

where $\gamma : C^1([0, T], \mathbb{R}) \to \mathbb{R}^2$ is continuous and takes bounded sets into bounded sets. The particular case

$$\gamma(u) = h(u(0), u(T), u'(0), u'(T)),$$

corresponds to the general nonlinear two-point condition. Explicit results are known for the special case

$$h(u) = \big(h_1(u(0), u'(0)), h_2(u(T), u'(T))\big)$$

which appears in different physical models, as a second order analogue for the axial deformation of a nonlinear elastic beam [RS]. On the other hand, problems involving a p-Laplacian have been widely studied for example in [DJM], [MM], [M].

SOME REMARKS ON THE GENERAL CASE

For $M = (M_1, M_2)$ define:

$$\begin{aligned} h_{1,M} &= \sup_{|u|\le M_1, |\widetilde{u}|\le M_2} |h_1(u,\widetilde{u})|, \\ h_{2,M} &= \sup_{|u|\le M_1, |\widetilde{u}|\le M_2} |\phi(h_2(u,\widetilde{u}))|, \\ m_\phi(M_2) &= \inf_{|\widetilde{u}|=M_2} |\phi(\widetilde{u})| \end{aligned}$$

and

$$\mathcal{R}_M = \{X \in C([0,T], \mathbb{R}^{2N}) : \|x_1\|_\infty \le M_1, \|x_2\|_\infty \le m_\phi(M_2)\}.$$

We shall assume that f is a Caratheodory function i.e. $f(\cdot, u, \widetilde{u})$ is measurable for any fixed $(u, \widetilde{u})$, and for any M there exists $\rho_M \in L^1(0,T)$ such that

$$|f(t,u,\widetilde{u})| \le \rho_M \quad a.e. \quad \text{for any} \quad |u| \le M_1, |\widetilde{u}| \le M_2.$$

Theorem 1. *Let $M_1, M_2 > 0$ and assume that*

$$h_{1,M} + TM_2 \le M_1, \qquad h_{2,M} + \|\rho_M\|_{L^1} \le m_\phi(M_2)$$

Then (1)-(NBC) admits at least one solution u with $(u, \phi(u')) \in \mathcal{R}_M$.

Proof. Let us consider the equivalent system for $X = (u, \phi(u'))$:

$$\begin{cases} X' = F(t,X) & \text{in} \quad (0,T) \\ X(0) = H(X(T)) \end{cases}$$

where

$$F(t, x_1, x_2) = \left(\phi^{-1}(x_2), f(t, x_1, \phi^{-1}(x_2))\right)$$

and

$$H(x_1, x_2) = \left(h_1(x_1, \phi^{-1}(x_2)), \phi[h_2(x_1, \phi^{-1}(x_2))]\right).$$

Next, define the operator $\mathcal{N} : C([0,T], \mathbb{R}^{2N}) \to C([0,T], \mathbb{R}^{2N})$ given by

$$\mathcal{N}X(t) = H(X(T)) + \int_0^t F(s, X).$$

By Arzelá-Ascoli Theorem, $\mathcal{N}$ is compact. Moreover, for $X \in \mathcal{R}_M$ we have:

$$\|(\mathcal{N}X)_1\|_\infty \le |h_1\left(x_1(T), \phi^{-1}(x_2(T))\right)| + T \sup_{t\in[0,T]} |\phi^{-1}(x_2(t))|$$

$$\|(\mathcal{N}X)_2\|_\infty \leq |\phi\big(h_2\left(x_1(T), \phi^{-1}(x_2(T))\right)\big)| + \int_0^T |f(\cdot, x_1, \phi^{-1}(x_2))|.$$

As $\|x_2\|_\infty \leq m_\phi(M_2)$, it is clear that $|\phi^{-1}(x_2)| \leq M_2$, and then

$$\|(\mathcal{N}X)_1\|_\infty \leq h_{1,M} + TM_2, \qquad \|(\mathcal{N}X)_2\|_\infty \leq h_{2,M} + \|\rho_M\|_{L^1}.$$

Hence, $\mathcal{N}(\mathcal{R}_M) \subset \mathcal{R}_M$ and by Schauder's Theorem $\mathcal{N}$ has a fixed point X. Then $u = x_1$ is a solution of (1)-(NBC).

Theorem 2. *Under the hypothesis of the previous theorem, assume also the Lipschitz-type conditions for $(u, \widetilde{u}), (v, \widetilde{v}) \in \mathcal{R}_M$:*

(H1) $$|h_1(u, \phi^{-1}(\widetilde{u})) - h_1(v, \phi^{-1}(\widetilde{v}))| \leq k_1|(u - v, \widetilde{u} - \widetilde{v})|$$

(H2) $$|\phi(h_2(u, \phi^{-1}(\widetilde{u}))) - \phi(h_2(v, \phi^{-1}(\widetilde{v})))| \leq k_2|(u - v, \widetilde{u} - \widetilde{v})|$$

(F) $$|f(t, u, \phi^{-1}(\widetilde{u})) - f(t, v, \phi^{-1}(\widetilde{v}))| \leq k_f|(u - v, \widetilde{u} - \widetilde{v})|$$

(Φ) $$|\phi^{-1}(\widetilde{u}) - \phi^{-1}(\widetilde{v})| \leq k_\phi|(\widetilde{u} - \widetilde{v})|$$

with

$$k_1 + Tk_\phi < 1, \qquad k_2 + Tk_f < 1.$$

Then (1)-(NBC) admits a unique solution in $\mathcal{R}_M$.

Proof. In the situation of Theorem 1, using conditions (H1)-(H2)-(F)-(Φ) it is immediate to prove that $\mathcal{N}$ is a contraction.

As a simple corollary, we deduce the existence of solutions under appropriate growth conditions on f and h_i:

Corollary 3. *Let us assume that*

$$|h_1(u, \widetilde{u})| \leq \psi_{1,1}(|u|) + \psi_{1,2}(|\widetilde{u}|),$$

$$|\phi(h_2(u, \widetilde{u}))| \leq \psi_{2,1}(|u|) + \psi_{2,2}(|\widetilde{u}|)$$

and

$$\|\rho_M\|_{L^1} \leq \psi_{f,1}(M_1) + \psi_{f,2}(M_2)$$

for any $t \in [0, T]$, with $\psi_{i,j}$ positive and nondecreasing. Further, assume that the following conditions hold:

$$\limsup_{r \to \infty} \frac{\psi_{1,1}(r)}{r} = A < 1$$

and

$$[\psi_{2,1}+\psi_{f,1}]\left(\frac{\psi_{1,2}(r)+rT}{1-A}\right)+[\psi_{2,2}+\psi_{f,2}](r)<m_\phi(r)$$

for r large. Then (1)-(NBC) admits at least one solution.

Proof. With the previous notations, set $M_1=\frac{\psi_{1,2}(M_2)+TM_2}{1-A}+\varepsilon$ and then, for M_2 large and $\varepsilon>0$ sufficiently small conditions of Theorem 1 are fulfilled.

A PARTICULAR CASE

In this section we study the following particular case of (1)-(NBC):

$$\left\{\begin{array}{ll}(\phi(u'))'+a(t)\phi(u')+g(t,u)=f(t) & \text{in}\quad (0,T)\\ u(0)=h_1(u(T)), & u'(0)=h_2(u'(T))\end{array}\right. \tag{2}$$

with g bounded and $a\in C([0,T],\mathbb{R}^{n\times n})$. For example, when $n=1$ we may consider the forced pendulum equation with friction for a p-Laplacian:

$$\left\{\begin{array}{ll}(|u'|^{p-2}u')'+|u'|^{p-2}u'+b(t)\sin_p(u)=f(t) & \text{in}\quad (0,T)\\ u(0)=h_1(u(T)), & u'(0)=h_2(u'(T))\end{array}\right.$$

We recall that $\sin_p$ is defined as the solution of the initial value problem

$$\left(|u'|^{p-2}u'\right)'+(p-1)|u|^{p-2}u=0,\qquad u(0)=0,\qquad u'(0)=1,$$

or implicitly by the formula

$$t=\int_0^{\sin_p t}\frac{ds}{(1-s^p)^{1/p}}$$

for $t\in[0,\frac{\pi_p}{2}]$ and extended to a $2\pi_p$-periodic function, where π_p is the constant given by

$$\pi_p=2\int_0^1\frac{ds}{(1-s^p)^{1/p}}.$$

For simplicity we introduce the following notation: for any $\sigma\in[0,1]$ let U_σ be the unique solution of the matricial equation

$$U'=-\sigma aU,\qquad U(0)=Id\in\mathbb{R}^{n\times n}.$$

Moreover, we define

$$L(\sigma)=\limsup_{|x|\to\infty}\frac{\sigma\widehat{h}_2(U_\sigma(T)x)x}{|x|^2}$$

where $\widehat{h}_2=\phi^{-1}h_2\phi$. Then we have:

Theorem 4. *Assume that*

$$\limsup_{|x|\to\infty} \frac{h_1(x)x}{|x|^2} < 1$$

and that

$$\sup_{0\le\sigma\le 1} L(\sigma) < 1.$$

Then (2) admits at least one solution for any bounded g.

Proof. As before, let us consider $X = (u, \phi(u'))$ and the compact operator $\mathcal{N}X(t) = H(X(T)) + \int_0^t F(s, X)$. Then, if $X = \sigma\mathcal{N}X$ for $0 < \sigma \le 1$, we obtain:

$$x_1' = \sigma\phi^{-1}(x_2), \qquad x_1(0) = \sigma h_1(x(T))$$

$$x_2' = \sigma(f - ax_2 - g(\cdot, x_1)), \qquad x_2(0) = \sigma\widehat{h}_2(x(T))$$

Hence,

$$x_2(t) = U_\sigma(t)c(t),$$

with

$$c(t) = c_0 + \sigma\int_0^t U_\sigma^{-1}(s)[f(s) - g(s, x_1)]ds$$

Then

$$c_0 = x_2(0) = \sigma\widehat{h}_2(U_\sigma(T)[c_0 + R_\sigma])$$

where $R_\sigma = \sigma\int_0^T U_\sigma^{-1}(s)[f(s) - g(s, x_1)]ds$ is bounded. It follows that

$$|c_0|^2 + c_0R_\sigma = \sigma\widehat{h}_2(U_\sigma(T)[c_0 + R_\sigma])(c_0 + R_\sigma) \le (1 - \delta)(c_0 + R_\sigma)^2$$

for some $\delta > 0$ if $|c_0|$ is large enough. Thus c_0 (and hence x_2) is bounded. On the other hand, from the equality

$$x_1(t) = c_1 + \sigma\int_0^t \phi^{-1}(x_2), \qquad x_1(0) = \sigma h_1(x_1(T))$$

we obtain that $c_1 = \sigma h_1(c_1 + S_\sigma)$ where $S_\sigma = \sigma\int_0^T \phi^{-1}(x_2)$ is bounded. As before,

$$|c_1|^2 + c_1S_\sigma \le (1 - \delta)(c_1 + S_\sigma)^2$$

and we conclude that c_1 (and hence x_1) is bounded. Thus, the result follows from Leray-Schauder Theorem.

Remark . *In particular, for the scalar case it suffices to assume that*

$$\limsup_{|x|\to\infty} \frac{h_1(x)}{x} < 1$$

and

$$\limsup_{|x|\to\infty} \frac{\widehat{h}_2(x)}{x} = L < \begin{cases} e\int_0^T a & \textit{if } \int_0^T a \le 1 \\ e^{\int_0^T a} & \textit{otherwise.} \end{cases}$$

Indeed, it suffices to observe that in this case $U_\sigma(t) = e^{-\sigma\int_0^t a}$*, and letting* $y = e^{-\sigma\int_0^T a}x$ *we obtain:*

$$\frac{\sigma\widehat{h}_2\big(e^{-\sigma\int_0^T a}x\big)}{x} = \sigma e^{-\sigma\int_0^T a}\frac{\widehat{h}_2(y)}{y}$$

A simple computation shows that

$$\max_{0\le\sigma\le 1} \sigma e^{-\sigma\int_0^T a} = \begin{cases} \frac{1}{e\int_0^T a} & \textit{if } \int_0^T a \le 1 \\ e^{-\int_0^T a} & \textit{otherwise} \end{cases}$$

and the proof follows.

SCALAR CASE: UPPER AND LOWER SOLUTIONS FOR THE PERIODIC PROBLEM

In this section we apply the method of upper and lower solutions in order to obtain solutions for the scalar case of (2) when $a = 0$, $h_1 = h_2 = id$, and g is not necessarily bounded. We shall need the following auxiliary lemmas:

Lemma 5. *Assume that* ϕ *is nondecreasing and let* $\lambda > 0$*. Then for any* $\theta \in C([0,T])$ *the equation*

$$(\phi(u'))' - \lambda u = \theta(t)$$

admits a unique T*-periodic solution. Furthermore, the mapping* $\mathcal{T} : C([0,T]) \to C([0,T])$ *given by* $\mathcal{T}(\theta) = u$ *is compact.*

Proof. Let us first note that for any $\xi \in C([0,T])$ there exists a unique $c = c(\xi) \in \mathbb{R}$ such that

$$\int_0^T \phi^{-1}\big(c + \xi(t)\big)dt = 0$$

Furthermore, the mapping $\xi \to c(\xi)$ is compact. Indeed, existence and uniqueness follow immediately from the fact that ϕ is strictly monotone and that $\phi(x) \to \pm\infty$ as $x \to \pm\infty$. On the other hand, if $M > \|\xi\|_\infty$ then

$$\int_0^T \phi^{-1}\big(\phi(0) + M + \xi\big) > 0 > \int_0^T \phi^{-1}\big(\phi(0) - M + \xi\big)$$

and hence $c(\xi) \in [\phi(0) - \|\xi\|_\infty, \phi(0) + \|\xi\|_\infty]$. Finally, note that if $\xi_n \to \xi$ uniformly, taking a subsequence we may assume that $c(\xi_n) \to c$ for some c. Thus,

$$0 = \int_0^T \phi^{-1}\big(c(\xi_n) + \xi_n(t)\big)dt \to \int_0^T \phi^{-1}\big(c + \xi(t)\big)dt,$$

and by uniqueness it follows that $c(\xi) = c$.
For $u \in C([0,T])$ let $\xi_u(t) = \int_0^t (\lambda u + \theta)$ and define, for fixed $s \in \mathbb{R}$,

$$\mathcal{T}u(t) = s + \int_0^t \phi^{-1}(c(\xi_u) + \xi_u(\tau))d\tau.$$

By Arzelá-Ascoli Theorem, it is easy to see that $\mathcal{T}$ is compact. Moreover, if $u = \sigma\mathcal{T}u$ for some $\sigma \in (0,1]$ we obtain for $v = \frac{u}{\sigma}$:

$$(\phi(v'))' - \sigma\lambda v = \theta, \qquad v(0) = v(T) = s$$

Let φ be defined by

$$\varphi(t) = s + \int_0^t \phi^{-1}\big(c(\xi_0) + \xi_0\big)$$

then a simple computation shows that

$$\|v - \varphi\|_{L^2} \leq \|\varphi\|_{L^2}$$

and hence $\|(\phi(v'))'\|_{L^2} \leq C$ for some constant C. Take t_0 such that $v'(t_0) = 0$, then

$$|\phi(v'(t))| \leq |\phi(0)| + \int_{t_0}^t |(\phi(v'))'| \leq |\phi(0)| + T^{1/2}C$$

and hence $\|v\|_\infty \leq \tilde{C}$ for some constant $\tilde{C}$. By Leray-Schauder Theorem, there exists u_s such that

$$(\phi(u_s'))' - \lambda u_s = \theta, \qquad u_s(0) = u_s(T) = s.$$

By monotonicity of ϕ it is immediate to prove that u_s is unique. Moreover, a simple computation proves that the mapping $s \to u_s$ is continuous for the $C([0,T])$-norm. For $s > 0$ large enough, note that if $u_s(t_0) > s$ is maximum, we have that u_s' is strictly nondecreasing in a neighborhood of t_0, a contradiction. Thus, $u_s(T) - u_s(0) \geq 0$. In the same way it follows that $u_s(T) - u_s(0) \leq 0$ for $s << 0$, and the existence of a T-periodic solution follows. Uniqueness follows immediately from the monotonicity of ϕ. Furthermore, if u_1 and u_2 are T-periodic with

$$(\phi(u_i'))' - \lambda u_i = \theta_i(t),$$

then

$$-\int_0^T (\theta_1 - \theta_2)(u_1 - u_2) = \int_0^T [\phi(u_1') - \phi(u_2')](u_1' - u_2') + \lambda \int_0^T (u_1 - u_2)^2.$$

It follows that

$$\|u_1 - u_2\|_{L^2} \leq \frac{1}{\lambda}\|\theta_1 - \theta_2\|_{L^2}$$

and hence

$$\|(\phi(u_1'))' - (\phi(u_2'))'\|_{L^2} \leq 2\|\theta_1 - \theta_2\|_{L^2}.$$

Thus,

$$\|\phi(u_1') - \phi(u_2')\|_\infty \le c\|\theta_1 - \theta_2\|_{L^2}$$

for some constant c and the compactness of the mapping $\theta \to u$ follows.

Lemma 6. *Assume that ϕ is nondecreasing and let $\lambda > 0$. Let u, v be T-periodic functions such that $(\phi(u'))' - \lambda u \ge (\phi(v'))' - \lambda v$. Then $u \le v$.*

Proof. Let t_0 be the absolute maximum of $u - v$, and suppose that $u(t_0) > v(t_0)$. Then $u'(t_0) = v'(t_0)$, and $\varphi(t) = \phi(u'(t)) - \phi(v'(t))$ is strictly nondecreasing in a neighborhood U of t_0 in $[0, T]$. As $\phi(u'(t_0)) - \phi(v'(t_0)) = 0$, we obtain that

$$\phi(u'(t)) < \phi(v'(t)) \text{ for } t \in U,\ t < t_0$$

and

$$\phi(u'(t)) > \phi(v'(t)) \text{ for } t \in U,\ t > t_0.$$

This implies that $u - v$ has a local minimum in t_0, a contradiction.

Theorem 7. *Let ϕ be nondecreasing and assume there exist T-periodic functions $\alpha \le \beta$ such that*

$$(\phi(\alpha'))' + g(t, \alpha) \ge f(t) \ge (\phi(\beta'))' + g(t, \beta).$$

Further, assume there exists a constant $R > 0$ such that

$$\frac{g(t,u) - g(t,v)}{u - v} \ge -R$$

for any u, v such that $\inf_t \alpha(t) \le v < u \le \sup_t \beta$. Then the problem

$$\text{(Per)} \qquad \begin{cases} (\phi(u'))' + g(t,u) = f(t) & in \quad (0,T) \\ u(0) = u(T), \qquad u'(0) = u'(T) \end{cases}$$

admits at least one solution u such that $\alpha \le u \le \beta$.

Proof. Let $E = \{u \in C([0,T]) : \alpha \le u \le \beta\}$ and fix a constant $\lambda > R$. For fixed $\overline{u} \in E$ define $\mathcal{T}\overline{u} = u$ the unique periodic solution of the equation

$$(\phi(u'))' - \lambda u = f - g(\cdot, \overline{u}) - \lambda\overline{u}.$$

By Lemma 5, $\mathcal{T} : E \to C([0,T])$ is well defined and compact. Moreover, as $\overline{u} \le \beta$ we have that

$$(\phi(u'))' - \lambda u = f - (g(\cdot, \overline{u}) + \lambda\overline{u}) \ge f - (g(\cdot, \beta) + \lambda\beta).$$

Hence, $(\phi(u'))' - (\phi(\beta'))' - \lambda(u - \beta) \ge 0$, and by Lemma 6 it follows that $u \le \beta$. In the same way we conclude that $u \ge \alpha$ and the result follows by Schauder Theorem.

Example: the forced pendulum equation for a p-Laplacian

$$(|u'|^{p-2}u')' + \sin_p(u) = f(t)$$

admits a T-periodic solution for any forcing term f such that $-1 \leq f \leq 1$. Indeed, it suffices to take $\alpha = \frac{\pi_p}{2}$ and $\beta = \frac{3}{2}\pi_p$.

As a simple corollary we have:

Corollary 8. *Under the conditions of the previous theorem, let $\lambda > R$ and define the sequences $\{u_n^\pm\}$ given by*

$$u_0^- = \alpha, \qquad u_0^+ = \beta$$

and $u_{n+1}^\pm$ the unique T-periodic function such that

$$(\phi(u_{n+1}^{\pm}{}'))' - \lambda u_{n+1}^\pm = f - g(\cdot, u_n^\pm) - \lambda u_n^\pm.$$

Then $\{u_n^-\}$ (resp. $\{u_n^+\}$) is a nondecreasing (nonincreasing) sequence of subsolutions (supersolutions) that converges pointwise to a solution of (Per). Moreover, $u_n^- \leq u_n^+$ for every n.

Proof. From the proof of the previous theorem we have that $\alpha \leq u_1^+ \leq \beta$. Assume as inductive hypothesis that $\alpha \leq u_n^+ \leq u_{n-1}^+$, then $u_{n+1}^+ \geq \alpha$. Moreover,

$$(\phi(u_{n+1}^{+}{}'))' - \lambda u_{n+1}^+ = f - (g(\cdot, u_n^+) + \lambda u_n^+) \geq f - (g(\cdot, u_{n-1}^+) + \lambda u_{n-1}^+) =$$
$$= (\phi(u_n^{+}{}'))' - \lambda u_n^+$$

and it follows that $u_{n+1}^+ \leq u_n^+$.

As u_n^+ is nonincreasing and bounded, it converges pointwise to a function u^+. Furthermore, from the proof of Lemma 5 $\{u_n^+\}$ is bounded for the H^1-norm, and from the compactness of the imbedding $H^1(0,T) \hookrightarrow C([0,T])$ it is easy to see that $u_n^+ \to u^+$ in $C([0,T])$. Using the definition of $\{u_n^+\}$ it is immediate that u^+ is a solution of the problem. The proof is analogous for $\{u_n^-\}$. Furthermore,

$$(\phi(u_{n+1}^{\pm}{}'))' + g(\cdot, u_{n+1}^\pm) = f + g(\cdot, u_{n+1}^\pm) - g(\cdot, u_n^\pm) + \lambda(u_{n+1}^\pm - u_n^\pm).$$

Thus, it is easy to prove by induction that u_n^+ is a supersolution and u_n^- is a subsolution, with $u_n^- \leq u_n^+$.

References

[AMP] Amster P., Mariani M. C. , Pinasco D.: Nonlinear periodic-type conditions for a second order ODE. Nonlinear Studies, Vol. 8, 2, 2001.

[BL] Bernfeld S., Lakshmikantham V.: Existence of solutions of ordinary differential equations with generalized boundary conditions, Trans.Am.Math.Soc 184 (1973), 261-274.

[BL2] Bernfeld S., Lakshmikantham V.: An introduction to nonlinear boundary value problems, Acad.Press, NY 1974.

[DJM] Mawhin J., Dinca G., Jebelean P.: Variational and topological methods for Dirichlet problems with p-Laplacian. Port. Math. (N.S.) 58, No.3, 339-378 (2001).

[E] Erbe L.: Nonlinear boundary value problems for second-order differential equations, J. of Diff.Eq. 7 (1970), 459-472.

[FM] Fournier G., Mawhin J.: On periodic solutions of forced pendulum-like equations. Journal of Differential Equations 60 (1985) 381-395.

[GM] Gaines R.E., Mawhin J.: Ordinary differential equations with nonlinear boundary conditions, J. of Diff. Eq. 26, 200-222 (1977).

[KL] Krasnosel'skii M., Lifshitz E.: Duality principles for boundary value problems for nonlinear systems, Soviet Math.Dokl. 8 (1967), 1236-39.

[M] Mawhin J.: Periodic Solutions of Systems with p-Laplacian-like Operators. Nonlinear Analysis and its Applications to Differential Equations (series: Progress in Nonlinear Differential Equations and Their Applications, 43) pp. 37–64, Birkhäuser (2001).

[MM] Manásevich R., Mawhin J.: Periodic Solutions for Nonlinear Systems with p-Laplacian Like Operators . J. Differential Equations vol 145, No. 2, May 20 (1998).

[RS] Rebelo C. and Sanchez L.: Existence and multiplicity for an O.D.E. with nonlinear boundary conditions. Differential Equations and Dynamical Systems, Vol 3, Number 4, October 1995, pp.383-396.

[S] Stampacchia G.: Sopra una generalizzazione dei problemi ai limiti per i sistemi di equazioni differenziali ordinarie, Ric.Mar. 3 (1954), 76-94.

P.Amster, M. C. Mariani* and P.De Nápoli*
Dpto. de Matemática,
Fac. de Cs. Exactas y Naturales,
Universidad de Buenos Aires,
Pab. I, Ciudad Universitaria,
(1428) Buenos Aires, Argentina.
* CONICET
pamster@dm.uba.ar - mcmarian@dm.uba.ar - pdenapo@dm.uba.ar

Recibido: 18 de febrero de 2003
Revisado: 7 de setiembre de 2004