REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Volumen 45, Número 2, 2004, Páginas 41–53

ON SOME FUNCTIONS OF THE LITTLEWOOD PALEY THEORY FOR γ_d AND A CHARACTERIZATION OF GAUSSIAN SOBOLEV SPACES OF INTEGER ORDER.[1]

IRIS A. LÓPEZ AND WILFREDO O. URBINA

ABSTRACT. We define an Area function of higher order with respect to the Gaussian measure S^k_γ and prove its $L^p(\gamma_d)$-continuity for $1 < p < \infty$. Also we consider an associated Littlewood Paley g type function $\widetilde{g}^k_\gamma$, of higher order with respect to the Gaussian measure and show its $L^p(\gamma_d)$-continuity for $1 < p < \infty$. Finally, we give a characterization of Gaussian Sobolev spaces $L^p_s(\gamma_d)$ of integer order by means of a related Littlewood Paley type function g_k, of higher order with respect to the Gaussian measure, defined by C. Gutierrez, C. Segovia and J. Torrea.

1. INTRODUCTION

Let us consider the Gaussian measure $\gamma_d(dx) = \frac{1}{\pi^{d/2}} e^{-|x|^2} dx$ in $\mathbb{R}^d$ and the Ornstein-Uhlenbeck differential operator

$$L = \frac{1}{2}\triangle_x - \langle x, \nabla_x \rangle .$$

Let $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$ be a multi-index, let $\alpha! = \prod_{i=1}^d \alpha_i!$, $|\alpha| = \sum_{i=1}^d \alpha_i$, $\partial_i = \frac{\partial}{\partial x_i}$, for each $1 \le i \le d$ and $\partial^\alpha = \partial_1^{\alpha_1} ... \partial_d^{\alpha_d}$.

Let us consider the (normalized) Hermite polynomials of degree $|\alpha|$, in d variables

$$h_\alpha(x) = \frac{1}{\left(2^{|\alpha|}\alpha!\right)^{1/2}} \prod_{i=1}^d (-1)^{\alpha_i} e^{x_i^2} \frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}} (e^{-x_i^2}).$$

Given a function $f \in L^1(\gamma_d)$ we consider its Fourier-Hermite coefficients defined by

$$c_\alpha = \int_{\mathbb{R}^d} f(x) h_\alpha(x) \gamma_d(dx).$$

Let us consider the Ornstein-Uhlenbeck semigroup $\{T_t\}_{t\ge 0}$ defined as

$$T_t(f)(x) = \frac{1}{\left(1-e^{-2t}\right)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{e^{-2t}(|x|^2+|y|^2)-2e^{-t}\langle x,y\rangle}{1-e^{-2t}}} f(y)\gamma_d(dy).$$

2000 Mathematics Subject Clasification: Primary 42C10, Secondary 42B25.

Key words and phrases: Gaussian Measure, Hermite expansions, Littlewood Paley functions, Sobolev spaces.

(1)Partially supported by Grant FONACIT #G-97000668 and by ECOS Nord/FONACIT Action V00M02/PI-200000000860.

By Bochner subordination principle, we define the Poisson-Hermite semigroup $\{P_t\}_{t\geq 0}$ associated to L (see [6]) as

$$\begin{aligned} P_t(f)(x) &= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_{t^2/4u}(f)(x)du \\ &= \frac{1}{2\pi^{(d+1)/2}} \int_{\mathbb{R}^d} \int_0^1 t \frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}} \frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}} \frac{dr}{r} f(y)dy. \end{aligned}$$

We will also consider the translated semigroups $\left\{T_t^k\right\}_{t\geq 0}$ and $\left\{P_t^k\right\}_{t\geq 0}$ defined by

$$T_t^k(f)(x) = e^{-kt}T_t(f)(x) \quad \text{and}$$

$$P_t^k(f)(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T^k_{t^2/4u}(f)(x)du \quad \text{for } k \geq 1.$$

In 1994 C. Gutierrez [3] introduced a Littlewood-Paley-Stein g_γ-function with respect to the Gaussian measure as

$$g_\gamma(f)(x) = \left(\int_0^\infty |t\nabla P_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2}, \tag{1.1}$$

where $\nabla = \left(\frac{\partial}{\partial t}, \frac{1}{\sqrt{2}}\nabla_x\right)$ is the (total) gradient and proved its $L^p(\gamma_d)$-continuity for $1 < p < \infty$.

The Area function and the Area function of higher order are important in the classical case of the Lebesgue measure, since for a harmonic function on $\mathbb{R}^{d+1}_+$ the existence of non-tangencial limits is equivalent to the finitess of S operator. Also they are very important in the characterization of Hardy spaces and they are related with atomic descomposition of Hardy spaces. Therefore the study of the analogous notions in the Gaussian case are important and interesting.

In 1994 E. Fabes and L. Forzani [1] introduced an Area function for the Gaussian measure γ_d as

$$S_\gamma(f)(x,\delta) = \left(\int_{\Gamma_\gamma(x,\delta)} |\nabla P_t(f)(y)|^2 \, t\left(t^{-d} \vee |x|^d \vee 1\right) dydt\right)^{1/2} \tag{1.2}$$

where $\Gamma_\gamma(x,\delta) = \left\{(y,t) \in \mathbb{R}^{d+1}_+ : |y-x| < \delta(t \wedge \frac{1}{|x|} \wedge 1)\right\}$ is what is called a Gaussian cone with opening with $\delta > 0$ and vertex $x \in \mathbb{R}^d$, and they proved its $L^p(\gamma_d)$ continuity for $1 < p < \infty$ of S_γ (see [1]).

Theorem 1.1. *Suppose $f \in L^p(\gamma_d)$. Then*

i) *There exist a constant $C_\delta > 0$ such that for every $x \in \mathbb{R}^d$*

$$g_\gamma(f)(x) \leq C_\delta S_\gamma(f)(x,\delta). \tag{1.3}$$

ii) *If $1 < p < \infty$, then exist a constant $C_p > 0$ such that*

$$\|S_\gamma(f)(\cdot,\delta)\|_{p,\gamma_d} \leq C_p \|f\|_{p,\gamma_d}. \tag{1.4}$$

Inspired by C. Segovia and R.Whedeen article [9], we define an Area function of order k with respect to γ_d as

$$S_{\gamma}^{k}(f)(x,\delta)=\left(\int_{\Gamma_{\gamma}(x,\delta)}\left(t\wedge 1\wedge|x|^{-1}\right)^{2k-1+d}\left|\nabla^{k}P_t(f)(y)\right|^{2}t^{-d}\left(t^{-d}\vee|x|^{d}\vee 1\right)dydt\right)^{1/2}, \tag{1.5}$$

for $k\geq 1$ and $\delta>0$ where

$$\left|\nabla^{k}P_t(f)(x)\right|^{2}=\sum_{j=0}^{d}\left|\nabla^{k-1}\frac{\partial}{\partial x_j}P_t(f)(x)\right| \quad \text{and } x_0=t.$$

We define an associated Littlewood Paley g-type function of order k as

$$\widetilde{g}_{\gamma}^{k}(f)(x)=\left(\int_0^{\infty}t^{-(d+1)}\left(t\wedge 1\wedge|x|^{-1}\right)^{2k+d}\left|\nabla^{k}P_t(f)(x)\right|^{2}dt\right)^{1/2}. \tag{1.6}$$

We will prove the $L^p(\gamma_d)$-continuity for $1<p<\infty$, of the functions S_{γ}^{k} and $\widetilde{g}_{\gamma}^{k}$. Formally

Theorem 1.2. *Suppose $f\in L^p(\gamma_d)$. Then*

i) *If $1<p<\infty$, then exist a constant $C_{p,\delta,k,d}>0$ such that*

$$\left\|S_{\gamma}^{k}(f)(\cdot,\delta)\right\|_{p,\gamma_d}\leq C_{p,\delta,k,d}\left\|f\right\|_{p,\gamma_d}. \tag{1.7}$$

ii) *There exist a constant $C_{k,\delta,d}>0$ such that for every $y\in\mathbb{R}^d$*

$$\widetilde{g}_{\gamma}^{k}(f)(y)\leq C_{k,\delta,d}S_{\gamma}^{k}(f)(y,\delta). \tag{1.8}$$

If $p=1$ R. Scotto has proved (see [8])

$$\gamma_d\left(\left\{x\in\mathbb{R}^d: g_{\gamma}(f)(x)>\lambda\right\}\right)\leq\frac{C}{\lambda}\left\|f\right\|_{1,\gamma_d}.$$

Using a similar argument to Scotto's proff it can be proved that

$$\gamma_d\left(\left\{x\in\mathbb{R}^d: S_{\gamma}^{k}(f)(x,\varepsilon)>\lambda\right\}\right)\leq\frac{C_{\varepsilon}}{\lambda}\left\|f\right\|_{1,\gamma_d}$$

for all $\lambda>0$ and $k\geq 1$. In consecuence for all $k\geq 1$ the function $\widetilde{g}_{\gamma}^{k}$ can be weak-type $(1,1)$ continuos.

In 1996 C. Gutierrez, C. Segovia and J. Torrea (see [4]) introduced Littlewood-Paley functions of higher order with respect to the γ_d as

$$g_{S,\gamma}^{k}(f)(x)=\left(\int_0^{\infty}\left|t^{k}\nabla_x^{k}P_t(f)(x)\right|^{2}\frac{dt}{t}\right)^{1/2} \tag{1.9}$$

where

$$\left|\nabla_x^{k}P_t(f)(x)\right|^{2}=\sum_{j=1}^{d}\left|\nabla_x^{k-1}\frac{\partial}{\partial x_j}P_t(f)(x)\right|$$

and

$$g_{t,\gamma}^{k}(f)(x)=\left(\int_0^{\infty}\left|t^{k}\frac{\partial^{k}}{\partial t^{k}}P_t(f)(x)\right|^{2}\frac{dt}{t}\right)^{1/2} \tag{1.10}$$

They proved (see [4]) the $L^p(\gamma_d)$-continuity, $1 < p < \infty$ of these functions, namely

Theorem 1.3. *Given $1 < p < \infty$ let $k \geq 1$. Then there exist positive constants $C_{p,k}$ and $A_{p,k}$, only depending on p and k such that for every polynomial f we have*

$$\left\| g^k_{S,\gamma}(f) \right\|_{p,\gamma_d} \leq C_{p,k} \left\| f \right\|_{p,\gamma_d}, \tag{1.11}$$

$$\left\| g^k_{t,\gamma}(f) \right\|_{p,\gamma_d} \leq A_{p,k} \left\| f \right\|_{p,\gamma_d}. \tag{1.12}$$

On the other hand, they proved the opposite inequality from $g^k_{t,\gamma}(f)(\cdot)$ (see [4]). Formally

Theorem 1.4. *Given $1 < p < \infty$, let $k \geq 1$. Then there exist a constant $B_{p,k} > 0$ such that for every polynomial f we have*

$$\left\| f \right\|_{p,\gamma_d} \leq B_{p,k} \left\| g^k_{t,\gamma}(f) \right\|_{p,\gamma_d}. \tag{1.13}$$

As an application of these functions $g^k_{S,\gamma}$ and $g^k_{t,\gamma}$, we will provide a characterization of Gaussian Sobolev spaces, $L^p_s(\gamma_d)$ for $1 < p < \infty$ and s a positive integer.

Following S. Watanabe (see [13]) Gaussian Sobolev spaces $L^p_s(\gamma_d)$ for $1 < p < \infty$ and $s > 0$ can be defined as follows:

Let f be a polynomial function and let

$$J_n(f) = \sum_{|\alpha|=n} c_\alpha h_\alpha$$

be its orthogonal projection onto the closed subspace generated by $\{h_\alpha : |\alpha| = n\}$, consider the Bessel potentials for the Gaussian measure (see [1]),

$$(I - L)^{-s/2}(f) = \sum_{n=0}^{\infty} (1+n)^{-s/2} J_n(f), \tag{1.14}$$

and define the norm

$$\left\| f \right\|_{p,s} := \left\| (I - L)^{s/2}(f) \right\|_{p,\gamma_d}. \tag{1.15}$$

Then Gaussian Sobolev spaces, $L^p_s(\gamma_d)$, is define as the completion of the polynomials functions with the norm $\left\| \cdot \right\|_{p,s}$.

In what follows the following notation will be needed. We will say that a multi-index β has order k if $\beta \in \mathbb{N}^k$, $\beta = (\beta_1, ..., \beta_k)$ with $1 \leq \beta_j \leq d$ and we denote $\partial_\beta = \partial_{\beta_1} ... \partial_{\beta_k}$. Let $\mathcal{M}_k$ be the set of all these multi-index of order k, then $Card(\mathcal{M}_k) = d^k$.

Observe that given a $\beta \in \mathcal{M}_k$, we can get a multi-index $\alpha \in \mathbb{N}^d$ such that $|\alpha| = k$, taking

$$\alpha_i = Card\left\{\beta_j : \beta_j = i\right\}.$$

Now given $\alpha \in \mathbb{N}^d$ with $|\alpha| = k$, we denote

$$\mathcal{U}_\alpha = \left\{\beta \in \mathcal{M}_k : \partial_\beta = \partial^\alpha\right\}, \quad \mathcal{J}_k = \left\{\alpha \in \mathbb{N}^d : |\alpha| = k\right\}$$

and thus $\mathcal{M}_k = \bigcup_{\mathcal{J}_k} \mathcal{U}_\alpha$.

We will need to define another Littlewood Paley g-type function of order k. Let $\alpha \in \mathbb{N}^d$ a multi-index and let $R_\alpha(f)(x)$ be the Riesz transform of order $|\alpha|$ (see [12]), with respect to the Gaussian measure. Given f a polynomial function, let us consider

$$f_\alpha(y,t) = P_t^{|\alpha|}(R_\alpha(f))(y), \quad f(y,t) = P_t(f)(y)$$

and

$$\overrightarrow{f}(y,t) = (f(y,t),\ f_\alpha(y,t)) \quad \text{with } \alpha \in \mathcal{J}_{k-s}.$$

Define

$$\left|\frac{\partial^k}{\partial t^k}\overrightarrow{f}(y,t)\right|^2 = \left|\frac{\partial^k}{\partial t^k} f(y,t)\right|^2 + \sum_{\mathcal{J}_{k-s}} \sum_{\mathcal{U}_\alpha} \left|\frac{\partial^k}{\partial t^k} f_\alpha(y,t)\right|^2$$

and the following Litllewood Paley type g-function

$$g_\gamma^{k,s}(\overrightarrow{f})(y) = \left(\int_0^\infty t^{2(k-s)-1} \left|\frac{\partial^k}{\partial t^k}\overrightarrow{f}(y,t)\right|^2 dt\right)^{1/2}. \tag{1.16}$$

There is an important relation between the function $g_\gamma^{k,s}(\overrightarrow{f})(\cdot)$ and the function $g_{S,\gamma}^{k-s}(f)(\cdot)$ previously defined, which is consequence of the following identity

$$\frac{\partial^{k-s}}{\partial t^{k-s}} P_t^{k-s}(R_\alpha(f))(x) = (-1)^{k-s}\partial^\alpha P_t(f)(x), \tag{1.17}$$

where $\alpha \in \mathbb{N}^d$ is a multi-index such that $|\alpha| = k-s$ (see [4]), and this will be one of key ingredients of the next result.

Theorem 1.5. *Let k, s be positive integers such that $0 < s < k$ and $1 < p < \infty$. Then there exist posistive constants $A_{p,k-s}$ and $B_{p,k-s}$, only depending on p and $k-s$, such that if $f \in L_s^p(\gamma_d)$, we have*

$$B_{p,k-s}\|f\|_{p,s} \le \left\|g_\gamma^{k,s}(\overrightarrow{f})\right\|_{p,\gamma_d} \le A_{p,k-s}\|f\|_{p,s}. \tag{1.18}$$

Therefore we have obtained a characterization of the Sobolev spaces of order $s \in \mathbb{N}$, $L_s^p(\gamma_d)$, by using esentially the $L^p(\gamma_d)$-continuity of the functions $g_{S,\gamma}^{k-s}$ and $g_{t,\gamma}^{k-s}$.

We thank Liliana Forzani for some helpful remarks.

2. Proofs

Proof of the Theorem 1.2:

i) In order to prove the $L^p(\gamma_d)$ continuity of S_γ^k we will need the following technical lemmas.

Lemma 2.1. *Let $0 < \delta < \varepsilon$, and $x \in \mathbb{R}^d$. Let $\Gamma_\gamma(x,\varepsilon)$, $\Gamma_\gamma(x,\delta)$ and $(y,\tau) \in Int(\Gamma_\gamma(x,\delta))$. We consider $c > 0$ and $r \le c\left(\tau \wedge 1 \wedge |x|^{-1}\right)$ such that $B_r(y,\tau) \subset \Gamma_\gamma(x,\varepsilon)$. Let $F \in C(\mathbb{R}_+^{d+1})$ and $G \in C(\mathbb{R}_+^{d+1})$ two positive functions such that*

$$F(y,\tau) \le Cr^{-2(k-1)-(d+1)} \int_{B_r(y,\tau)} G(v,t)\,dvdt.$$

Then there exist $c = c(\varepsilon,\delta)$ such that

$$\int_{\Gamma_\gamma(x,\delta)} \left(\tau \wedge 1 \wedge |x|^{-1}\right)^{2k+d-1} F(y,\tau)\tau^{-d}\left(\tau^{-d} \vee |x|^d \vee 1\right) dyd\tau$$

$$\le C_{k,\delta,d} \int_{\Gamma_\gamma(x,\varepsilon)} G(y,\tau)\tau\left(\tau^{-d} \vee |x|^d \vee 1\right) dyd\tau.$$

Proof. Let $k \geq 1$ and $0 < \delta < \varepsilon$ then $\Gamma_\gamma(x,\delta) \subset \Gamma_\gamma(x,\varepsilon)$. Fix $k \in \mathbb{N}$ and let $(y,\tau) \in Int(\Gamma_\gamma(x,\delta))$. If we choose $c = \left(\varepsilon - \delta \wedge \sin\left(\frac{\delta_1-\varepsilon_1}{2}\right) \wedge \frac{1}{2}\right)$, where $\delta = \tan(\delta_1)$ and $\varepsilon = \tan(\varepsilon_1)$, we can consider $B_r(y,\tau) \subset \Gamma_\gamma(x,\varepsilon)$ with $r = c\left(\tau \wedge 1 \wedge |x|^{-1}\right)$. Then we define

$$A_\tau = \left\{(v,t) : |x-v| < \varepsilon\left(t \wedge 1 \wedge |x|^{-1}\right);\ |t-\tau| \leq r\right\},$$

$$A_{\tau,\delta} = \left\{(v,t) : |x-v| < \varepsilon\left(t \wedge 1 \wedge |x|^{-1}\right);\ (1+\delta^2)^{-1/2}(1-c)\tau < t < (1+c)\tau\right\}$$

and

$$H_{\tau,\delta} = \int_{A_{\tau,\delta}} G(v,t)dvdt.$$

Since $B_r(y,\tau) \subset A_\tau$ we have

$$F(y,\tau) \leq Cr^{-2(k-1)-(d+1)} \int_{A_\tau} G(v,t)dvdt. \tag{2.1}$$

Now consider the following change of coordinates,

$$\begin{aligned} \tau &= \rho\cos(\theta) \\ x-y &= |x-y|\,\sigma = \rho\sin(\theta).\sigma \\ dyd\tau &= \rho^d d\rho d\sigma \end{aligned}$$

where $\sigma \in \mathbb{R}^d$ such that $|\sigma| = 1$ and $\rho > 0$.

Let us denote $\Omega = B_1(x,0) \cap \Gamma_\gamma(x,\delta)$. Since $\Gamma_\gamma(x,\delta) \subset \{(y,\tau) : |x-y| < \delta\tau\}$ and $A_{\rho\cos\theta} \subset A_{\rho,\delta}$, then

$$\int\int_{\Gamma_\gamma(x,\delta)} \left(\tau \wedge 1 \wedge |x|^{-1}\right)^{2k-1+d} F(y,\tau)\tau^{-d}\left(\tau^{-d} \vee |x|^d \vee 1\right) dyd\tau$$

$$\leq C\int_\Omega (\cos\theta)^{-2d}\left[\int_0^\infty H_{\rho,\delta}\left(\rho^{-d} \vee |x|^d \vee 1\right) d\rho\right] d\sigma.$$

Therefore in order to prove the lemma, it is enough to prove that

$$\int_0^\infty H_{\rho,\delta}\left(\rho^{-d} \vee |x|^d \vee 1\right) d\rho \leq C_{k,\delta}\int_{\Gamma_\gamma(x,\varepsilon)} G(y,\tau)\tau\left(\tau^{-d} \vee |x|^d \vee 1\right) dyd\tau \tag{2.2}$$

since $(1+\delta^2)^{-1/2} \leq \cos\theta \leq 1$.

Let us consider two cases: $|x| \leq 1$ and $|x| > 1$.

In first case, we have $\left(\rho^{-d} \vee |x|^d \vee 1\right) = \max\left(1, \rho^{-d}\right)$ and therefore

$$\int_0^\infty H_{\rho,\delta}\max(1,\rho^{-d})d\rho = \int_0^1 H_{\rho,\delta}\rho^{-d}d\rho + \int_1^\infty H_{\rho,\delta}d\rho. \tag{2.3}$$

In the second case, we have $\left(\rho^{-d} \vee |x|^d \vee 1\right) = \max\left(\rho^{-d}, |x|^d\right)$ thus

$$\int_0^\infty H_{\rho,\delta}\max(\rho^{-d},|x|^d)d\rho = \int_0^{|x|^{-1}} H_{\rho,\delta}\rho^{-d}d\rho + |x|^d\int_{|x|^{-1}}^\infty H_{\rho,\delta}d\rho. \tag{2.4}$$

Let us bound in (2.3) and (2.4) each of the integral in the right hand side. It is clear that

$$\int_0^1 H_{\rho,\delta}\rho^{-d}d\rho \leq \int_0^\infty H_{\rho,\delta}\rho^{-d}d\rho$$

and

$$\int_1^\infty H_{\rho,\delta}d\rho \leq \int_0^\infty H_{\rho,\delta}d\rho.$$

Then by Tonelli's Theorem,

$$\begin{aligned}\int_0^\infty H_{\rho,\delta}\rho^{-d}d\rho &= \int_0^\infty \tau^{-d}\left[\int_{\Gamma_\gamma(x,\varepsilon)} G(v,t)\chi_{A_{\tau,\delta}}(v,t)dvdt\right]d\tau \\ &\leq \int_{\Gamma_\gamma(x,\varepsilon)} G(v,t)\left(\int_{t/(1+c)}^{(1+\delta^2)^{1/2}t/(1-c)}\tau^{-d}d\tau\right)dvdt \\ &\leq C_{\delta,d}\int_{\Gamma_\gamma(x,\varepsilon)} G(v,t)t\max\left(t^{-d},1,|x|^d\right)dvdt.\end{aligned}$$

Wich allows us to bound (2.3).

For the second integral of the right hand side of (2.4) we have

$$\begin{aligned}|x|^d\int_{|x|^{-1}}^\infty H_{\rho,\delta}d\rho &\leq |x|^d\int_0^\infty H_{\rho,\delta}d\rho \\ &= |x|^d\int_0^\infty\left(\int_{\Gamma_\gamma(x,\varepsilon)} G(v,t)\chi_{A_{\tau,\delta}}(v,t)d\tau\right)dvdt \\ &\leq \int_{\Gamma_\gamma(x,\varepsilon)} G(v,t)\left(|x|^d\int_{t/(1+c)}^{(1+\delta^2)^{1/2}t/(1-c)}d\tau\right)dvdt \\ &\leq C_\delta\int_{\Gamma_\gamma(x,\varepsilon)} G(v,t)t\max\left(t^{-d},1,|x|^d\right)dvdt.\end{aligned}$$

Then we conclude the proof of the lemma. ∎

Following the same type of argument of the proof of Lemma 4.1 of [1], we have,

Lemma 2.2. *Let us consider* $(y,\tau)\in\mathbb{R}_+^{d+1}$*, a ball* $B_r(y,\tau)$ *with center at* (y,τ) *and radius* $r\leq\left(\tau\wedge 1\wedge|y|^{-1}\right)$ *and let* $\xi=(\xi_1,...,\xi_k)$ *with* $0\leq\xi_j\leq d$*,* $j=1,...,k$*. If* $u(y,\tau)$ *is a solution of* $\frac{\partial^2 u}{\partial\tau^2}+2Lu=0$ *then,*

$$|\partial_\xi u(y,\tau)|\leq C_{k,d}r^{-k-\frac{(d+1)}{2}}\left(\int_{B_r(y,\tau)}|u(x,s)|^2dxds\right)^{1/2}. \tag{2.5}$$

Proof. For each $(y_0,\tau_0)\in\mathbb{R}_+^{d+1}$, $|y_0|>1$, set $B=B_{\frac{1}{|y_0|}}(y_0,\tau_0)$. Let us define on B the transformation

$$y=y_0+\frac{1}{|y_0|}y',$$
$$\tau=\tau_0+\frac{1}{|y_0|}\tau'.$$

Then $(y,\tau) \in B$ if and only if $(y^{'},\tau^{'}) \in B_1(0,0)$. Now, let us define the function

$$U(y^{'},\tau^{'}) = u\left(y_0 + \frac{1}{|y_0|}y^{'}, \tau_0 + \frac{1}{|y_0|}\tau^{'}\right) \text{ on } B_1(0,0).$$

The function U satisfies the equation

$$\Delta_{y^{'},\tau^{'}}U - 2\frac{1}{|y_0|}\left(y_0 + \frac{1}{|y_0|}y^{'}\right)\nabla_{y^{'}}U = 0.$$

Since $(y^{'},\tau^{'}) \in B_1(0,0)$ then $\frac{1}{|y_0|}\left(y_0 + \frac{1}{|y_0|}y^{'}\right)$ is bounded by a constant. Thus, U is a solution of a elliptic differential operator with bounded first order coefficients and then from it follows, from Theorem 8.13 of [2] and Sobolev immersion Theorem, that U satisfies the inequality,

$$|\partial_\xi U(0,0)| \leq C_{k,d}s^{-k-\frac{(d+1)}{2}}\left(\int_{B_s(0,0)}\left|U(y^{'},\tau^{'})\right|^2 dy^{'}d\tau^{'}\right)^{1/2},$$

for all $s \leq 1$.
Now, by the definition of U, the last inequality can be rewritten as

$$\begin{aligned}|\partial_\xi u(y_0,\tau_0)| &\leq C_{k,d}s^{-k-\frac{(d+1)}{2}}\left(\int_{B_s(0,0)}\left|u\left(x_0 + \frac{1}{|y_0|}y^{'}, \tau_0 + \frac{1}{|y_0|}\tau^{'}\right)\right|^2 dy^{'}d\tau^{'}\right)^{1/2}\\ &= C_{k,d}|y_0|^{\frac{(d+1)}{2}}s^{-k-\frac{(d+1)}{2}}\left(\int_{B_{\frac{s}{|y_0|}}(y_0,\tau_0)}|u(y,\tau)|^2 dy d\tau\right)^{1/2}.\end{aligned}$$

Hence, in order to obtain inequality (2.5), if $\tau_0 < \frac{1}{|y_0|}$, take $s = |y_0|\tau_0$ and if $\tau_0 > \frac{1}{|y_0|}$, $s = 1$.
If $|y_0| \leq 1$ we can apply to u the classical inequality, on the ball $B_1(y_0,\tau_0)$. ∎

Lemma 2.3. *Let $k \geq 1$ and $0 < \delta < \varepsilon$. Then exist $C_{\delta,k,d} > 0$ such that for each $x \in \mathbb{R}^d$*

$$S_\gamma^k(f)(x,\delta) \leq C_{\delta,k,d}S_\gamma(f)(x,\varepsilon). \tag{2.6}$$

Proof. Let f be a bounded, smooth and with bounded first and second order derivates. We know that if $u(y,\tau) = P_\tau(f)(y)$, then

$$\frac{\partial^2}{\partial\tau^2}u(y,\tau) + 2Lu(y,\tau) = 0.$$

Now if $\xi = (\xi_1,...,\xi_k)$ with $0 \leq \xi_j \leq d$, then $\partial_\xi u(y,\tau) = \partial_\mu\partial_i u(y,\tau)$ if $\mu = (\mu_0,..,\mu_{k-1})$ and $i = 0,...,d$. Also, u is a bounded function, with bounded first and second derivates on $\overline{B_r(y,\tau)}$ (see [3]). Then using (2.5) of lemma 2.2 we get

$$|\partial_\xi P_\tau(f)(y)|^2 \leq C_{k-1,d}r^{-2(k-1)-(d+1)}\int_{B_r(y,\tau)}\left|\frac{\partial}{\partial v_i}P_t(f)(v)\right|^2 dvdt,$$

and

$$\left|\nabla^k P_\tau(f)(y)\right|^2 \leq C_{k-1,d}r^{-2(k-1)-(d+1)}\int_{B_r(y,\tau)}|\nabla P_t(f)(v)|^2 dvdt.$$

Applying the lemma 2.1 with $F(y,\tau) = \left|\nabla^k P_\tau(f)(y)\right|^2$and $G(v,t) = |\nabla P_t(f)(v)|^2$ we obtain the result of this lemma. ∎

The proof of i) the Theorem 1.2 is now inmediate using (2.6), and the $L^p(\gamma_d)$-continuity of the Area function S_γ (see [1]) when f is a smooth function. For general $f \in L^p(\gamma_d)$, we need only approximate in norm by a sequence of indefinitely differentiable functions with compact support.

ii) Fix $\delta > 0$, $(y,t) \in \Gamma_\gamma(y,\delta)$ and $r = c\left(t \wedge |y|^{-1} \wedge 1\right)$ such that $B_r(y,t) \subset \Gamma_\gamma(y,\delta)$. The Mean Valued inequality (see [1]) implies

$$\left|\nabla^k P_t(f)(y)\right|^2 \le C_{\delta,d}\left(t^{-(d+1)} \vee |y|^{d+1} \vee 1\right)\int_{B_r(y,t)} \left|\nabla^k P_s(f)(x)\right|^2 dxds.$$

Then by Schwarz's inequality and Tonelli's Theorem we obtain

$$\left[\widetilde{g}^k_\gamma(f)(y)\right]^2 = \int_0^\infty t^{-(d+1)}\left(t \wedge 1 \wedge |y|^{-1}\right)^{2k+d}\left|\nabla^k P_t(f)(y)\right|^2 dt$$

$$\le C_{\delta,d}\int_0^\infty t^{-(d+1)}\left(t \wedge 1 \wedge |y|^{-1}\right)^{2k+d}\left(t^{-(d+1)} \vee 1 \vee |y|^{d+1}\right)$$
$$[\int_{B_r(y,t)} \left|\nabla^k P_s(f)(x)\right|^2 dxds]dt$$

$$\le C_{\delta,d}\int_{\Gamma_\gamma(y,\delta)} \left|\nabla^k P_s(f)(x)\right|^2$$
$$\left(\int_{s-\delta\left(s\wedge 1\wedge|y|^{-1}\right)}^{s+\delta\left(s\wedge 1\wedge|y|^{-1}\right)} t^{-(d+1)}\left(t \wedge 1 \wedge |y|^{-1}\right)^{2k+d}\left(t^{-(d+1)} \vee |y|^{d+1} \vee 1\right)dt\right)dxds$$

Therefore, in order to proof the lemma, it is enough to proof

$$\int_{s-\delta\left(s\wedge 1\wedge|y|^{-1}\right)}^{s+\delta\left(s\wedge 1\wedge|y|^{-1}\right)} t^{-(d+1)}\left(t \wedge 1 \wedge |y|^{-1}\right)^{2k+d}\left(t^{-(d+1)} \vee |y|^{d+1} \vee 1\right)dt \tag{2.7}$$
$$\le C_{\delta,k,d}s^{-d}\left(s \wedge 1 \wedge |y|^{-1}\right)^{2k-1+d}\left(s^{-d} \vee |y|^d \vee 1\right).$$

We will prove (2.7) considering two cases as in the proof of the Lemma 2.1: $|y| \le 1$ and $|y| > 1$.

In the first case we have

$$\left(t^{-(d+1)} \vee |y|^d \vee 1\right) = \max\left(t^{-(d+1)}, 1\right) \quad \text{and} \quad \left(t \wedge 1 \wedge |y|^{-1}\right) = \min(t,1)$$

then

$$\int_{s-\delta\left(s\wedge 1\wedge|y|^{-1}\right)}^{s+\delta\left(s\wedge 1\wedge|y|^{-1}\right)} t^{-(d+1)}\left(t \wedge 1 \wedge |y|^{-1}\right)^{2k+d}\left(t^{-(d+1)} \vee |y|^{d+1} \vee 1\right)dt$$
$$\le \int_{s(1-\delta)}^{s(1+\delta)} t^{2k-d-2}dt \le C_{\delta,k,d}s^{2k-1}s^{-d}$$

if $t \in (0,1)$. On the other hand

$$\int_{s-\delta\left(s\wedge 1\wedge |y|^{-1}\right)}^{s+\delta\left(s\wedge 1\wedge |y|^{-1}\right)} t^{-(d+1)}\left(t\wedge 1\wedge |y|^{-1}\right)^{2k+d}\left(t^{-(d+1)}\vee |y|^{d+1}\vee 1\right)dt$$
$$\leq \int_{s(1-\delta)}^{s(1+\delta)} t^{-(d+1)}dt \leq C_{\delta,d}s^{-d}$$

if $t \in (1,\infty)$.

In the second case,

$$\left(t^{-(d+1)}\vee |y|^{d}\vee 1\right) = \max\left(t^{-(d+1)}, |y|^{d}\right) \quad \text{and} \quad \left(t\wedge 1\wedge |y|^{-1}\right) = \min(t, |y|^{-1})$$

thus

$$\int_{s-\delta\left(s\wedge 1\wedge |y|^{-1}\right)}^{s+\delta\left(s\wedge 1\wedge |y|^{-1}\right)} t^{-(d+1)}\left(t\wedge 1\wedge |y|^{-1}\right)^{2k+d}\left(t^{-(d+1)}\vee |y|^{d+1}\vee 1\right)dt$$
$$\leq |y|^{d+1}\left(|y|^{-1}\right)^{2k+d}\int_{s(1-\delta)}^{s(1+\delta)} t^{-(d+1)}dt$$
$$= C_{\delta,k,d}\,|y|^{d}\left(|y|^{-1}\right)^{2k-1+d}s^{-d}$$

if $t \in (|y|^{-1},\infty)$. In the case that $t \in (0,|y|^{-1})$, we have that $(0,|y|^{-1}) \subset (0,1)$. ∎

Proof of the Theorem 1.5:

Let f be a function such can be written as

$$f = \sum_{n\geq 0}\left(\frac{1}{1+n}\right)^{s/2} J_n(\psi),$$

where ψ is a polinomial.

We define

$$\phi = \sum_{n\geq 0}\left(\frac{n}{1+n}\right)^{s/2} J_n(\psi),$$

then by using Meyer's Multiplier Theorem twice (see [13]) and the definition of the $\|\ \|_{p,s}$ norm, we have that there exist two positive constants A_p and B_p such that

$$A_p\left\|f\right\|_{p,s} \leq \left\|\phi\right\|_{p} \leq B_p\left\|f\right\|_{p,s}, \tag{2.8}$$

since $\|f\|_{p,s} := \|\psi\|_{p,\gamma_d}$. Let $\alpha \in \mathbb{N}^d$ be a multi-index and let us consider the functions,

$$f(y,t) = P_t(f)(y), \quad \phi(y,t) = P_t(\phi)(y),$$
$$f_\alpha(y,t) = P_t^{|\alpha|}(R_\alpha f)(y) \quad \text{and} \quad \phi_\alpha(y,t) = P_t^{|\alpha|}(R_\alpha \phi)(y).$$

Thus

$$f(y,t) = \sum_{n\geq 0}\frac{1}{(1+n)^{s/2}}e^{-t\sqrt{n}}J_n(\psi)(y), \tag{2.9}$$

$$f_\alpha(y,t) = \sum_{n\geq 1}\tfrac{1}{(1+n)^{s/2}}\left\{\sum_{|\eta|=n} c_\eta^{\psi} e^{-t\sqrt{n}}\left(\tfrac{2^{|\alpha|}}{|\eta|^{|\alpha|}}\right)^{1/2}\left(\prod_{i=1}^{d}\eta_i...(\eta_i-\alpha_i+1)\right)^{1/2} h_{\eta-\alpha}(y)\right\}, \tag{2.10}$$

and

$$\phi(y,t) = \sum_{n\geq 0} \frac{n^{s/2}}{(1+n)^{s/2}} e^{-t\sqrt{n}} J_n(\psi)(y), \tag{2.11}$$

$$\phi_\alpha(y,t) = \sum_{n\geq 1} \frac{n^{s/2}}{(1+n)^{s/2}} \left\{ \sum_{|\eta|=n} c_\eta^\psi e^{-t\sqrt{n}} \left(\frac{2^{|\alpha|}}{|\eta|^{|\alpha|}}\right)^{1/2} \left(\prod_{i=1}^d \eta_i...(\eta_i-\alpha_i+1)\right)^{1/2} h_{\eta-\alpha}(y) \right\}. \tag{2.12}$$

By using integration by parts $(s-1)$ times and using (2.9) and (2.11) we have

$$\int_0^\infty u^{s-1} \frac{\partial^k}{\partial u^k}\phi(y,t+u)du = (s-1)!\frac{\partial^k}{\partial t^k} f(y,t),$$

and using (2.10) and (2.12) we obtain

$$\int_0^\infty u^{s-1} \frac{\partial^k}{\partial u^k}\phi_\alpha(y,t+u)du = (s-1)!\frac{\partial^k}{\partial t^k} f_\alpha(y,t).$$

Also using integration by parts $(s-1)$ times and the fact that

$$\phi(y,t)\to 0, \quad \text{and} \quad \phi_\alpha(y,t)\to 0$$

as $t\to\infty$, we get

$$\int_0^\infty u^{s-1} \frac{\partial^k}{\partial u^k}\phi(y,t+u)du = (-1)^{s-2}(s-1)!\frac{\partial^{k-s}}{\partial t^{k-s}}\phi(y,t)$$

and

$$\int_0^\infty u^{s-1} \frac{\partial^k}{\partial u^k}\phi_\alpha(y,t+u)du = (-1)^{s-2}(s-1)!\frac{\partial^{k-s}}{\partial t^{k-s}}\phi_\alpha(y,t),$$

for $0<s<k$. Consequently

$$\left|\frac{\partial^k}{\partial t^k} f(y,t)\right| = \left|\frac{\partial^{k-s}}{\partial t^{k-s}}\phi(y,t)\right|$$

and

$$\left|\frac{\partial^k}{\partial t^k} f_\alpha(y,t)\right| = \left|\frac{\partial^{k-s}}{\partial t^{k-s}}\phi_\alpha(y,t)\right|,$$

for any $\alpha\in\mathbb{N}^d$. Now let us consider $\alpha\in\mathbb{N}^d$ such that $|\alpha| = k-s$, then by using (1.17) we have

$$\frac{\partial^{k-s}}{\partial t^{k-s}} P_t^{k-s}(R_\alpha\phi)(y) = \partial^\alpha P_t(\phi)(y),$$

which implies that

$$\begin{aligned} \left|\frac{\partial^k}{\partial t^k}\overrightarrow{f}(y,t)\right|^2 &= \left|\frac{\partial^k}{\partial t^k} f(y,t)\right|^2 + \sum_{\mathcal{J}_{k-s}}\sum_{\mathcal{U}_\alpha}\left|\frac{\partial^k}{\partial t^k} f_\alpha(y,t)\right|^2 \\ &= \left|\frac{\partial^{k-s}}{\partial t^{k-s}}\phi(y,t)\right|^2 + \sum_{\mathcal{M}_{k-s}} \left|\partial_\beta\phi(y,t)\right|^2, \end{aligned}$$

therefore

$$g_\gamma^{k,s}(\overrightarrow{f})(y) \leq g_{t,\gamma}^{k-s}(\phi)(y) + g_{S,\gamma}^{k-s}(\phi)(y).$$

Using the fact that ϕ is a polynomial function, by the Theorem 1.3 and (2.8), we have

$$\left\|g_{\gamma}^{k,s}(\overrightarrow{f})\right\|_{p,\gamma_d} \le \left\|g_{t,\gamma}^{k-s}(\phi)\right\|_{p,\gamma_d} + \left\|g_{S,\gamma}^{k-s}(\phi)\right\|_{p,\gamma_d} \le A_{p,k-s}\left\|f\right\|_{p,s}.$$

On the other hand, we have

$$\left|\frac{\partial^k}{\partial t^k}\overrightarrow{f}(y,t)\right|^2 \ge \left|\frac{\partial^k}{\partial t^k}f(y,t)\right|^2 = \left|\frac{\partial^{k-s}}{\partial t^{k-s}}\phi(y,t)\right|^2,$$

therefore

$$g_{\gamma}^{k,s}(\overrightarrow{f})(y) \ge g_{t,\gamma}^{k-s}(\phi)(y)$$

and

$$\left\|g_{\gamma}^{k,s}(\overrightarrow{f})\right\|_{p,\gamma_d} \ge \left\|g_{t,\gamma}^{k-s}(\phi)\right\|_{p,\gamma_d}.$$

Using the Theorem 1.4 and (2.8) again, we get

$$\left\|f\right\|_{p,s} \le B_{p,k-s}\left\|g_{t,\gamma}^{k-s}(\phi)\right\|_{p,\gamma_d} \le B_{p,k-s}\left\|g_{\gamma}^{k,s}(\overrightarrow{f})\right\|_{p,\gamma_d}.$$

In the general case, we can use the density of the polynomial functions in $L^p(\gamma_d)$. ∎

References

[1] L. Forzani, R. Scotto, W. Urbina, *Riesz and Bessel Potentials, the g^k functions and an Area Function for the Gaussian Measure* γ, Revista UMA, Vol 42 (1) (2000), 17-37.

[2] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, (1983).

[3] C. Gutierrez, *On the Riesz transforms for the Gaussian measure*, J. Fourier Anal. Appl. 120 (1) (1994), 107-134.

[4] C. Gutierrez, C. Segovia, j. L. Torrea, *On higher Riesz transforms for Gaussian measure,* J. Fourier Anal. Appl. Vol 2 (1996), 583-596.

[5] I. López, *Teoría de Littlewood Paley para la medida Gaussiana. Tesis de Maestría,* Facultad de Ciencias, UCV (1998).

[6] P. A. Meyer, *Quelques resultats analytiques sur le semigroup d'Ornstein Uhlenbeck en dimension infinite.* Lectures Notes in Contr. and Inform. Sci 49 Springer-Verlag (1983), 201-214.

[7] B. Muckenhoupt, *Poisson Integrals for Hermite and Laguerre expansions.* Trans. Amer. Math. Soc. 139 (1969), 231-242.

[8] R. Scotto, *Weak Type estimates for Singular Integral operators associated with the Ornstein Uhlenbeck process.* PhD Thesis. University of Minnesota (1993).

[9] C. Segovia, R. Whedeen, *On certain fractional area integrals,* Journal of Math. and Mech. Vol 19, (1969), 247-262.

[10] E. Stein, *Topics in Harmonic Analysis related to the Littlewood Paley theory.* Princenton University Press, Princenton N.J. (1970).

[11] S. Thangavelu, *Lectures on Hermite and Laguerre expansions,* Princenton University Press, Princenton N.J.(1993).

[12] W. Urbina, *Análisis Armónico Gaussiano: una visión panorámica.* Trabajo de Ascenso, Facultad de Ciencias, UCV (1998).

[13] S. Watanabe, *Stochastic Differential Equations and Mallavin Calculus.* Tata Institute of Fundamental Research, Springer Verlag (1984).

Iris A. López – Wilfredo O. Urbina
Universidad Central de Venezuela
Facultad de Ciencias.
Escuela de Matemática
Av. Los Ilustres. Los Chaguaramos
Apt. 20513
Caracas 1041-A VENEZUELA.
iathamai@euler.ciens.ucv.ve, cathar@cantv.net - wurbina@euler.ciens.ucv.ve

Recibido: 11 de marzo de 2004
Revisado: 11 de marzo de 2005