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SMOOTHNESS PROPERTIES OF VARIETIES OF BORELS

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ABSTRACT. Let G be a connected reductive group defined over an algebraically closed field k, T a fixed maximal torus in G, and B a fixed Borel subgroup containing T, W the Weyl group of G relative to T, and S the set of simple reflections in W defined by (T, B). We denote by X the projective variety of Borel subgroups of G. Let $O(w)$, $w \in W$, be the orbit of $(B, {}^wB) \in X \times X$ under the left action of G. Now we define $\overline{O}(s_1, \ldots, s_n)$, $s_i \in S$, to be the closed subvariety of X^{n+1} whose points are the sequences $(B_0, B_1, \ldots, B_n)$, $B_i \in X$, where $(B_{i-1}, B_i) \in \overline{O(s_i)}$ for $i = 1, \ldots, n$. In this paper, we prove that the canonical projections

$$\pi : \overline{O}(s_1, \ldots, s_i) \to \overline{O}(s_1, \ldots, s_{i-1})$$

are $\mathbf{P^1}$-bundles, which implies that the variety $\overline{O}(s_1, \ldots, s_n)$ is smooth over k.

All varieties or schemes and all morphisms are defined over a fixed algebraically closed field k. For the main part, the only points of a variety under consideration are the points rational over k. The context makes it clear when this is not the case. Let G be a connected reductive algebraic group over k, T a fixed maximal torus, and B a fixed Borel subgroup containing T, $W = N_G(T)/T$ the Weyl group of G relative to T, and S the set of simple reflections in W defined by (T, B). We denote by X the projective variety of Borel subgroups of G. Its structure of algebraic variety is defined by the canonical G-equivariant bijection ${}^gB = gBg^{-1} \mapsto gB$ of X onto G/B. Note that if $w \in W$, we can also define ${}^wB = wBw^{-1}$ and wB in the usual way, and they do not depend on the representative $\dot{w}$ of w chosen to define them. Let $O(w)$, $w \in W$, be the orbit of $(B, {}^wB) \in X \times X$ under the left action of G. The set $O(w)$, being locally closed ([Bor], 1.8, Proposition, p.53), defines a subvariety of $X \times X$. Now we define $\overline{O}(s_1, \ldots, s_n)$, $s_i \in S$, to be the closed subvariety (resp. $O(s_1, \ldots, s_n)$, $s_i \in S$, to be the subvariety) of X^{n+1} whose points are the sequences $(B_0, B_1, \ldots, B_n)$, $B_i \in X$, where $(B_{i-1}, B_i) \in \overline{O(s_i)}$ for $i = 1, \ldots, n$ (resp. $(B_{i-1}, B_i) \in O(s_i)$ for $i = 1, \ldots, n$). In particular, for $n = 0$, $\overline{O}() = O() = X$. Moreover, if G is defined over a finite field $\mathbf{F_q}$ and F is the Frobenius morphism of G, we define the subvariety $\overline{X}(s_1, \ldots, s_n)$ of $\overline{O}(s_1, \ldots, s_n)$ (resp. the subvariety $X(s_1, \ldots, s_n)$ of $O(s_1, \ldots, s_n)$) by the additional condition that $F(B_0) = B_n$. We call these subvarieties, which are defined over $\mathbf{F_q}$, *Deligne-Lusztig varieties.*

The varieties considered above were first introduced by Deligne and Lusztig as a technical tool in representation theory (See [D-L], 9.1, p.147). Lately they have

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appeared in other contexts (e.g. S. H. Hansen [Han1], [Han2]). In particular, the applications of Deligne-Lusztig varieties to the theory of error-correcting codes depend on two properties: smoothness and the existence of "many" rational points. The existence of "many" rational points is known only in very special cases (see J. P. Hansen [H], F. Rodier [R]). Nevertheless, it seems reasonable to conjecture such results in general. With respect to smoothness, the proofs are part of the folklore known to the experts, but complete proofs have never appeared. As preparation for further research, in this paper we prove a fundamental theorem that implies the main smoothness results.

More precisely, we prove that the canonical projections

$$\pi : \overline{O}(s_1, \ldots, s_i) \to \overline{O}(s_1, \ldots, s_{i-1}) \tag{1}$$

are $\mathbf{P^1}$-bundles, which implies that the variety $\overline{O}(s_1, \ldots, s_n)$ is smooth over k and that the variety $\overline{X}(s_1, \ldots, s_n)$ is smooth over $\mathbf{F_q}$. In a special case, this result seems to be implicit in [D-L], 9.2, p.148. As a corollary , we obtain desingularizations of $\overline{O(w)}$ and of the Schubert varieties considered by Demazure ([Dem]). Finally, with the hope of mitigating the stylistic dryness of this paper, we apply our results to construct a new class of error-correcting codes.

We call a $\mathbf{P^n}$-*bundle* a morphism of schemes $f : E \to Y$ such that for an open covering U_α of Y, $E|U_\alpha = f^{-1}(U_\alpha)$ is U_α-isomorphic to $U_\alpha \times \mathbf{P^n}$, i.e. there is an isomorphism of $E|U_\alpha$ onto $U_\alpha \times \mathbf{P^n}$ compatible with the projections of these two schemes onto U_α. As indicated at the beginning, all schemes and morphisms are defined over k. This notion coincides with that of a locally trivial fiber space with fiber $\mathbf{P^n}$ and structure group $\mathbf{PGL_{n+1}}$, the full group of automorphisms of $\mathbf{P^n}$ ([Gro], ch.IV, 4.7).

Before proceeding, note that the projections π in (1) are G-equivariant with respect to the left action of G.

Lemma 1. *The morphism $\pi : \overline{O(s)} \to X$, $s \in S$, is a $\mathbf{P^1}$-bundle.*

To avoid introducing additional notation, in the course of this proof we will view X as G/B with the pertinent changes for $\overline{O(s)}$ and π. Since π is G-equivariant, it will be enough to find a nonempty open subset V of G/B such that $\overline{O(s)}|V$ is V-isomorphic to $V \times \mathbf{P^1}$. The family $\{gV \mid g \in G\}$ gives the required covering of G/B because each $g \in G$ induces isomorphisms $V \longrightarrow gV$, $\overline{O(s)}|V \longrightarrow g\overline{O(s)}|gV$, and $V \times \mathbf{P^1} \longrightarrow gV \times \mathbf{P^1}$ that allow us to transport the given V-isomorphism $\overline{O(s)}|V \longrightarrow V \times \mathbf{P^1}$ to a gV-isomorphism $g\overline{O(s)}|gV \longrightarrow gV \times \mathbf{P^1}$. Let U be the unipotent part of B, B' the Borel subgroup opposite to B, and U' its unipotent part. We have $U \cap B' = U' \cap B = \{e\}$.

1. The set $U'B$, the big cell, is open in G ([Hum], 28.5, Proposition, p.174); since it is also saturated with respect to the equivalence relation defined by B (i.e. it is a union of B-equivalence classes), the quotient $U'B/B$ is open in G/B. We set $V = U'B/B$ and observe that the canonical bijective quotient morphism

$$\tau : U' \to V = U'B/B$$

is an isomorphism ([Bor], 6.1, Corollary, p.95).

2. Let $P_s = BsB \cup B$ be the minimal parabolic generated by B and s ([Hum], 29.3, Lemma B, p.178). Since $\dim(P_s/B) = 1$, it follows ([Hum], 25.3, Theorem p.154, or [Bor], 13.13, Proposition, p.171) that $P_s/B \cong \mathbf{P^1}$.

3. We need to show that

$$\pi^{-1}(V) = \{(uB, upB) \mid u \in U', p \in P_s\}.$$

The inclusion of the second set into the first is clear. To prove that the first set is included in the second, pick an element (gB, gsB), $gB = uB$, $g \in G$ and $u \in U'$. Then $g = ub$, and, setting $p = b\dot{s}$ with $\dot{s}$ a representative of s, $bsB = ubsB = upB$.

4. Finally, the map

$$\pi^{-1}(V) \to V \times (P_s/B)$$

$$(uB, upB) \mapsto (uB, pB)$$

is well-defined as a set-theoretic map given that the $u \in U'$ is uniquely determined by uB, and pB is obtained by multiplying upB by u^{-1} on the right. To prove that this map is a morphism, we first notice that

$$\pi^{-1}(V) \subset V \times (G/B).$$

Call pr_1 and pr_2 the projections of $V \times G/B$ onto V and G/B, $\mathrm{inv} : U' \to U'$ the inverse, i.e. $\mathrm{inv}(u) = u^{-1}$, and recall that $\tau : U' \to V$ was the canonical isomorphism. With these notations, our map becomes

$$(\mathrm{pr}_1, (\mathrm{inv} \circ \tau^{-1} \circ \mathrm{pr}_1) \cdot \mathrm{pr}_2)$$

with the domain restricted to $\pi^{-1}(V)$ and the codomain restricted to $V \times (G/B)$ if necessary. This proves that our map is a morphism. We leave to the reader to define the inverse in the obvious way and to show that it is a morphism. Clearly both morphisms are V-morphisms. $\square$

Remark 2. We mention that $\overline{O(s)}$ is also a $\mathbf{P^1}$-bundle with respect to the second projection. This can be seen easily by starting with the new pair of maximal torus ${}^{s^{-1}}T = T$ and Borel ${}^{s^{-1}}B$ instead of T and B.

In the future we want to identify

$$\overline{O}(s_1, \dots, s_i) = \overline{O}(s_1, \dots, s_{i-1}) \times_X \overline{O}(s_i)$$

as schemes, where the morphisms into X defining the right hand side are respectively the last and the first projections. The reader can verify that the canonical map $(B_0, \dots, B_i) \mapsto ((B_0, \dots, B_{i-1}), (B_{i-1}, B_i))$ is an isomorphism on the level of varieties (=reduced schemes), and the only technical point left to check is that $\overline{O}(s_1, \dots, s_{i-1}) \times_X \overline{O}(s_i)$ is reduced. This is true because this fiber product is a $\mathbf{P^1}$-bundle over $\overline{O}(s_1, \dots, s_{i-1})$, being obtained from the $\mathbf{P^1}$-bundle $\overline{O}(s_i) = \overline{O(s_i)}$ by extending the base from X to $\overline{O}(s_1, \dots, s_{i-1})$. By induction, we can also get the identification

$$\overline{O}(s_1, \dots, s_i) = \overline{O}(s_1) \times_X \cdots \times_X \overline{O}(s_i)$$

with the appropriate projections. Now, combining these identifications with 1, we get the following theorem.

Theorem 3. *The morphism*

$$\pi : \overline{O}(s_1, \ldots, s_n) \to \overline{O}(s_1, \ldots, s_{n-1}),$$

$s_i \in S$*, is a* $\mathbf{P^1}$*-bundle.*

We call the sequence

$$\overline{O}(s_1, \ldots, s_n) \to \overline{O}(s_1, \ldots, s_{n-1}) \to \cdots \to \overline{O}(s_1) \to \overline{O}() = X$$

the *iterated* $\mathbf{P^1}$*-bundle* over X corresponding to $(s_1, \ldots, s_n)$.

Corollary 4. *The variety* $\overline{O}(s_1, \ldots, s_n)$*,* $s_i \in S$*, is irreducible and smooth of dimension* $\dim(X)\ + n$ *over* k*, and* $O(s_1, \ldots, s_n)$ *is dense open in* $\overline{O}(s_1, \ldots, s_n)$*.*

We recall that, if $E \longrightarrow Y$ is a $\mathbf{P^1}$-bundle, and Y is smooth over k (resp. irreducible over k), then E is smooth over k (resp. irreducible over k). For the irreducibility, see, for instance, [EGA], IV, 2.3.5 (iii). By considering the iterated $\mathbf{P^1}$-bundle over X corresponding to $(s_1, \ldots, s_n)$, the theorem reduces to the fact that X itself is irreducible and smooth over k, which follows from the connectedness of G and the transitivity of the action of G on X. The assertion about the dimension is clear. The fact that $O(s_1, \ldots, s_n)$ is open in $\overline{O}(s_1, \ldots, s_n)$ follows easily from the definitions. Since $\overline{O}(s_1, \ldots, s_n)$ is irreducible and $O(s_1, \ldots, s_n)$ is nonempty and open, it is also dense. □

Now let $w = s_1 \ldots s_n$, $s_i \in S$, $i = 1, \ldots, n$, be a *reduced decomposition* of $w \in W$. We have a commutative diagram

$$\begin{tikzcd} \overline{O}(s_1, \ldots, s_n) \arrow[rr, "\phi"] \arrow[dr, "\varpi"'] & & \overline{O(w)} \arrow[dl, "\pi"] \\ & X & \end{tikzcd} \qquad (2)$$

where ϕ is the morphism defined by $\phi(B_0, \ldots, B_n) = (B_0, B_n)$ (See [D-L], 9.1, p.148 and [D-L], 1.2(a), p.106), and ϖ and π are the first projections. In the following corollary, by *desingularization*, we mean with Grothendieck ([EGA], IV, 7.9.1) a proper birational morphism (consequently surjective) of a nonsingular variety into another, possibly singular, variety.

Corollary 5. *With the notations above, always assuming that* $w = s_1 \ldots s_n$ *is a reduced decomposition, we have:*

(i) The morphism $\phi : \overline{O}(s_1, \ldots, s_n) \longrightarrow \overline{O(w)}$ *is a desingularization.*

(ii) For any $x \in X$*, the restriction* $\phi_x : \varpi^{-1}(x) \longrightarrow \pi^{-1}(x)$ *of* ϕ *to the fibers of* ϖ *and* π *over* x *is a desingularization.*

(i) This result is known and appears in a more precise form in [D-L], 9.1, p.148. We sketch an argument in the present context. It is clear that the morphism ϕ is proper, being a morphism of projective varieties. Moreover ϕ induces an isomorphism $\phi^0 : O(s_1, \ldots, s_n) \xrightarrow{\sim} O(w)$ since $w = s_1 \ldots s_n$ is a reduced decomposition ([D-L], p.106). The rest follows from Corollary 4.

(ii) As in (i), the morphism ϕ_x is proper, being a morphism of projective varieties. The commutative diagram (2) restricts to

$$\begin{tikzcd} O(s_1,\dots,s_n) \arrow[rr, "\overset{\phi^0}{\sim}"] \arrow[dr, "\varpi^0"'] & & O(w) \arrow[dl, "\pi^0"] \\ & X & \end{tikzcd}$$

In this situation, the restriction $\phi_x^0 : (\varpi^0)^{-1} \longrightarrow (\pi^0)^{-1}$ of ϕ^0 to the fibers over x is also an isomorphism. On the other hand

$$(\varpi^0)^{-1}(x) = \varpi^{-1}(x) \cap O(s_0,\dots,s_n)$$

is nonempty and open, and consequently dense in $\varpi^{-1}(x)$. Similarly

$$(\pi^0)^{-1}(x) = \pi^{-1}(x) \cap O(w)$$

is nonempty and open, and consequently dense in $\pi^{-1}(x)$. $\square$

Remark 6. For $w \in W$, let $S(w)$ be the image of the Bruhat cell $C(w) = BwB$ under the morphism $G \longrightarrow G/B \longrightarrow X$, $g \mapsto gB \mapsto gBg^{-1}$. The closure $\overline{S(w)}$ is called the *Schubert variety* corresponding to w. Then $S(w) = \{{}^{bw}B \mid b \in B\}$ and, if $x_B = B \in X$,

$$(\pi^0)^{-1}(x_B) = \{({}^gB, {}^{gw}B) \mid {}^gB = B\} = \{B\} \times S(w).$$

Thus we can identify $\overline{S(w)} = \pi^{-1}(x_B)$ and we can regard the morphism ϕ_{x_B} as a desingularization of the Schubert variety $\overline{S(w)}$.

Now, let G be defined over the finite field $\mathbf{F_q}$, and let $s_i \in S$.

Corollary 7. *The Deligne-Lusztig variety $\overline{X}(s_1,\dots,s_n)$ is smooth over $\mathbf{F_q}$.*

Taking into account Corollary 4, the proof in [D-L], Lemma 9.11, p.151, extends to this case. $\square$

Finally, as promised at the beginning, we show how to construct a new class of error-correcting codes. Let D be a divisor on $\overline{Y} = \overline{X}(s_1,...,s_n)$, which is smooth over $\mathbf{F_q}$ by Corollary 7. We denote by

$$L(D) = \{f \in \mathbf{F}_q(\overline{Y})^* \mid \operatorname{div}(f) + D \geq 0\} \cup \{0\}$$

the Riemann-Roch space associated to $\overline{Y}$. Let $E = \{P_1, ..., P_n\}$ be a subset of $\overline{Y}(\mathbf{F}_q)$ disjoint from $\operatorname{supp}(D)$. The set

$$C = \{(f(P_1),...,f(P_n)) \mid f \in L(D)\}.$$

is the *algebraic-geometric code* associated to $\overline{Y}/\mathbf{F}_q$, D, and E. If $(s_1,\dots,s_n) \in W$ is not a reduced decomposition of $s_1 \dots s_n$, this code is not, as far as we are able to determine, included in the family of error-correcting codes constructed in [Han3] arising from Deligne-Lustig varieties of higher rank.

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