REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Volumen 45, Número 2, 2004, Páginas 75–88

ON THE GEOMETRY OF THE CLOSED ORBIT IN A FLAG MANIFOLD

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ABSTRACT. A non compact real form of a complex semisimple Lie group G_c, has only one closed orbit on each complex flag manifold of G_c. This closed orbits form a class of compact homogeneous manifolds which contains properly that of the R-spaces (also called real flag manifolds or orbits of S–representations). In this work we present an extension to closed orbits, of a result known in the case of R-spaces and complex flag manifolds.

1. Introduction

The purpose of this work is to study the geometry of a class of spaces wider than R–spaces, namely closed orbits on complex flag manifolds. These closed orbits has been studied by Wolf in [W]. They are compact homogeneous spaces with abundant geometric properties, the most important one for us here is that every R–space can be viewed as one of these closed orbits, in a convenient complex flag manifold.

Complex flag manifolds are constructed by the choice of a subset Φ of the system of simple roots π of a semisimple complex Lie algebra $\mathfrak{g}_c$. This subset $\Phi \subset \pi$ defines a conjugation class of parabolic subalgebras $\mathfrak{p}_{c\Phi}$, with corresponding parabolic subgroups $P_{c\Phi}$, which will in turn determine the complex flag manifolds $M_c(\Phi) = G_c/P_{c\Phi}$. Here G_c is the simply connected, complex, semisimple Lie group, corresponding to the Lie algebra $\mathfrak{g}_c$ (See 2).

If G is a noncompact real form of G_c, we consider the action of G as a subgroup of G_c, on the manifold $M_c(\Phi)$. This action have a unique closed orbit, this is a compact, homogeneous, and real manifold $M(\Phi)$, uniquely determined by the choice of the subset Φ (see [W] and section 2). It is also important the fact that we can obtain an imbedding of the flag manifold $M_c(\Phi)$ as the orbit of a point E in a compact real form $\mathfrak{g}_u$ of $\mathfrak{g}_c$ and that this point E can be chosen so that the orbit of the noncompact real form G through E is precisely the only closed orbit $M(\Phi)$ in $M_c(\Phi)$, producing an embedding of the closed orbit in $M_c(\Phi)$. We will show in section 3 an example of a closed orbit which is not an R–space.

The result we will prove in this work, is an extension for closed orbits, of a result proved by Sanchez for the case of R–spaces, and previously studied by others (see [T],[S1],[S2]). Sanchez introduced in [S2] a geometrical invariant called the *Index Number* of M, $\#_I(M)$, and he proved the following:

2000 Mathematics Subject Classification. Primary 53C30, Secondary 53C35.

* This research is partially supported by CONICET, SeCYT-UNC, and CIUNT, Argentina.

Theorem A ([S2], pag. 896)**.** *Let M be an R–space, and let $\#_I(M)$ be the Index Number of M. Then*

$$\#_I(M) = \sum_{i \geq 0} \beta_i(M, Z_2)$$

where $\beta_i(M, Z_2)$ is the i–th Betti number of M with coefficients in $\mathbb{Z}_2$

This result and the definition of index number of an R–space, are both an extension of similar definitions and results proved by Chen and Nagano ([Ch-N]) and Takeuchi([T]), namely the definition of 2-number for arbitrary connected riemmanian manifolds, for symmetric spaces, and the k-number for flag manifolds. Chen and Nagano defined the 2–number as the maximal possible cardinal of subsets $A \subset M$ such that for every $x, y \in A$, there exists a closed geodesic γ of M, such that x and y are antipodal to each other along γ. In the case of compact symmetric spaces, it can be seen that it coincides with the maximal possible cardinal of subsets $A \subset M$ such that for every $x \in A$, the geodesic symmetry S_x fixes every point in A, and a similar result holds.

In the case of complex flag manifolds, Sanchez extended this definition using the k–symmetric structures this spaces always have (see [K]), and proved a similar result. Finally for the case of R–spaces, Sanchez defined $k_0(M_c)$ as the smallest integer such that there exists a k–symmetric structure on the complex flag manifold M_c for every $k \geq k_0$, and observed that every R–space M can be isometrically embedded into a complex flag manifold M_c. Using this, he was able to define the Index p of the R–space M as the smallest prime number $p \geq k_0(M_c)$, and the Index number $\#_I(M_c)$ as the maximal possible cardinal of the subsets $A_p \subset M$ such that for every $x \in A_p$, the symmetry θ_x^p of the p–symmetric structure of M_c, fixes every point in A_p.

One of the main problems of our work is to extend the definition of Index number to the case of closed orbits (see definition 4.1). We do this using the fact that every closed orbit can be isometrically embedded into a complex flag manifold, and as in the case of R–spaces, using the k–symmetries of the complex flag manifold to define the Index Number. With this, we prove the following (see theorem 7.2):

Theorem B. *Let $M(\Phi)$ be the closed orbit in the flag manifold $M_c(\Phi)$. Then there exists a flag manifold R, an R–space $M_{\mathfrak{a}}$, and a fibration $\rho : M(\Phi) \longrightarrow M_{\mathfrak{a}}$, such that the flag manifold R is contained in a fiber, and:*

$$\chi(M_c(\Phi)) \quad \geq \quad \#_I(M(\Phi)) \quad \geq \quad \chi(R). \sum_{j \geq 0} \beta_j(M_{\mathfrak{a}}, \mathbb{Z}_2) \tag{1}$$

And if the algebraic condition $L_\Phi \subseteq N_{G_u}(i\mathfrak{a})$ is satisfied (see 2.2, and 4.4 for the definitions), then second inequality turns into the equality:

$$\#_I(M(\Phi)) \quad = \quad \chi(R). \sum_{j \geq 0} \beta_j(M_{\mathfrak{a}}, \mathbb{Z}_2) \tag{2}$$

2. Preliminaries and Basic Facts

In this section we will introduce some definitions and constructions. We will keep this notations and assumptions for the rest of the work.

2.1. Basic definitions and notations

Let G_c be a complex, simply connected, semisimple Lie group, $\mathfrak{g}_c = Lie(G_c)$. Let $\mathfrak{g} \subseteq \mathfrak{g}_c$ a real noncompact form of $\mathfrak{g}_c$, σ the conjugation in $\mathfrak{g}_c$ with respect to $\mathfrak{g}$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a Cartan decomposition of $\mathfrak{g}$. By definition, this means that there exists a compact real form $\mathfrak{g}_u$ of $\mathfrak{g}_c$ such that:

$$\begin{aligned} \sigma\mathfrak{g}_u &\subseteq \mathfrak{g}_u \\ \mathfrak{k} &= \mathfrak{g} \cap \mathfrak{g}_u, \mathfrak{p} = \mathfrak{g} \cap i\mathfrak{g}_u \\ \mathfrak{g}_u &= \mathfrak{k} \oplus i\mathfrak{p} \end{aligned}$$

Let K be the analytic subgroup of G corresponding to the subalgebra $\mathfrak{k}$, G and G_u the analytic subgroups of G_c corresponding to $\mathfrak{g}$ and $\mathfrak{g}_u$ respectively. It is well known that G_u and K are compact and connected Lie groups. Furthermore, G_u is a maximal compact subgroup of G_c (see [H, pag. 252, teo 1.1]).

Definition 2.2. Let $\mathfrak{a} \subset \mathfrak{p}$ a maximal abelian subspace. We may complete $i\mathfrak{a}$ to a Cartan subalgebra $\mathfrak{h}_u = \mathfrak{t} \oplus i\mathfrak{a} \subset \mathfrak{g}_u$ by taking $\mathfrak{t}$ in the usual fashion, as a Cartan subalgebra of the centralizer of $i\mathfrak{a}$ in $\mathfrak{k}$ (see [M]). By extension of the name usually given to these subalgebras in the noncompact case, we call this Cartan subalgebra $\mathfrak{h}_u$ of $\mathfrak{g}_u$ *minimally compact.*

2.3. Complex flag manifolds, R-spaces, and Closed Orbits

Let π be a system of simple roots of $\mathfrak{g}_c$. Then every subset Φ of π is associated in a canonical way to a parabollic subalgebra $\mathfrak{p}_{c\Phi}$ of $\mathfrak{g}_c$ (see [Hu] for the definitions and proofs). A subgroup P_c of G_c is a parabollic subgroup if its Lie algebra is a parabollic subalgebra. Then a flag manifold M_c may be defined as a homogeneous space of the form $M_c = G_c/P_c$, where G_c is chosen as above (see 2.1), and P_c is a parabollic subgroup.

Remark 2.4. It is important to notice that, by [W, pag. 1132], the compact real form G_u of G_c acts transitively on $M_c(\Phi)$, $M_c(\Phi) = G_u/G_u \cap P_{c\Phi}$, and that $U = G_u \cap P_{c\Phi}$ is the centralizer of a torus. Furthermore, every complex flag manifold $M_c(\Phi)$ of the complex group G_c can be identified with an orbit of its compact real form G_u trough a point of its Lie algebra $\mathfrak{g}_u$: i.e., there exists a point $E \in \mathfrak{k} \oplus i\mathfrak{a} \subseteq \mathfrak{g}_u$ such that $M_c(\Phi)$ can be identified with the orbit $Ad(g).E$ of the adjoint action of G_u on $\mathfrak{g}_u$, through the imbedding $M_c(\Phi) = G_u/U = G_u/G_u \cap P_{c\Phi} \longrightarrow \mathfrak{g}_u$, $gU \mapsto Ad(g).E$. In particular, every complex flag manifold $M_c(\Phi)$ is compact.

Proposition 2.5. *There exists $E \in \mathfrak{k} \oplus i\mathfrak{a} \subseteq \mathfrak{g}_u$ such that:*

1. *The isotropy of the adjoint action of G_u on $\mathfrak{g}_u$ is $U = G_u \cap P_{c\Phi}$. So, we have an imbedding $\overline{f}_E : M_c(\Phi) = G_u/G_u \cap P_{c\Phi} = G_u/U$ in $\mathfrak{g}_u$ given by $gU \mapsto Ad_{G_u}(g).E \subset \mathfrak{g}_u$.*

2. *$G.E$ is the unique closed orbit of the action of G on $M_c(\Phi)$. Furthermore, any maximal compact subgroup K of G acts transitively on $G.E$, and $f_E : K \mapsto Ad_{G_u}(K).E$ gives an imbedding of the closed orbit of G into the algebra $\mathfrak{g}_u$*

See [W] for a proof.

3. Example of a closed orbit which is not an R-space.

In this section we provide an example that illustrates some of the themes considered in the paper. We will present an example of a closed orbit manifold, in the sense defined in the paper, which is not an R-space.

Let $\mathfrak{g}_c = \mathfrak{sl}(4, \mathbb{C})$ which is a Lie algebra of type A_3

$$\mathfrak{g}_c = \{X: \ 4 \times 4 \ \text{ complex matrix}, \ tr\,(X) = 0\}$$

the diagonal matrices in $\mathfrak{g}_c$ form the *standard* Cartan subalgebra. One has the real forms:
$\mathfrak{su}\,(4)$ compact,
$\mathfrak{sl}\,(4, R)$ split
$\mathfrak{su}^*\,(4) \cong \mathfrak{sl}\,(2, H)$
$\mathfrak{su}\,(p, q) = \left\{X \in \mathfrak{sl}(4, \mathbb{C}): \quad I_{p,q}X +^t \overline{X} I_{p,q} = 0\right\}$, $I_{p,q}$ as in [H, p. 444],
$p + q = 4, \ p \geq q > 0$.
There are then two real forms of this last type, namely $\mathfrak{su}\,(2, 2)$, $\mathfrak{su}\,(3, 1)$.
Let us consider the case $\mathfrak{g} = \mathfrak{su}\,(3, 1)$. Then

$$\mathfrak{g} = \left\{ \begin{bmatrix} A & B \\ {}^t\overline{B} & C \end{bmatrix} : \begin{array}{ll} A \in \mathfrak{u}\,(3) & C \in \mathfrak{u}\,(1) = i\mathbb{R} \\ B\ 3 \times 1 & \text{complex } \ (B \in \mathbb{C}^3) \\ trA + C = 0 & \end{array} \right\},$$

and the conjugation of $\mathfrak{g}_c$ with respect to $\mathfrak{g}$ is $\sigma\,(X) = -\left(I_{p,q}{}^t\overline{X} I_{p,q}\right)$. We have

$$\mathfrak{k} = \left\{ \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} : \begin{array}{ll} A \in \mathfrak{u}\,(3) & C \in \mathfrak{u}\,(1) \\ trA + C = 0 & \end{array} \right\} \tag{3}$$

and

$$\mathfrak{p} = \left\{ \begin{bmatrix} 0 & B \\ {}^t\overline{B} & 0 \end{bmatrix} : \ B, \ 3 \times 1 \quad \text{complex} \right\} \cong \mathbb{C}^3.$$

The compact real form of $\mathfrak{g}_c$ is $\mathfrak{g}_u = \mathfrak{su}\,(4) = \mathfrak{k} \oplus i\mathfrak{p}$. Let G and G_u be the analytic subgroups of $Sl\,(4, \mathbb{C})$ corresponding respectively to the subalgebras $\mathfrak{g}$ and $\mathfrak{g}_u$.

According to [M, p 131] there are two conjugation classes of Cartan subalgebras in $\mathfrak{g}$ namely $b^{(o)}$ and $b^{(1)}$. We are interested in the class $b^{(o)}$ whose elements are minimally compact. An element in this class is

$$\mathfrak{h} = \left\{ \begin{bmatrix} ix & 0 & y \\ 0 & iZ & 0 \\ y & 0 & ix \end{bmatrix} : \begin{array}{ll} x, y & \in \mathbb{R} \\ Z & 2 \times 2 \ diag., \ \text{real} \\ & 2x + trZ = 0 \end{array} \right\}. \tag{4}$$

The complexification of $\mathfrak{h}$ in (4) is

$$\mathfrak{h}_c = \left\{ \begin{bmatrix} x & 0 & y \\ 0 & Z & 0 \\ y & 0 & x \end{bmatrix} : \begin{array}{ll} x, y & \in \mathbb{C} \\ Z & 2 \times 2 \ diag., \text{complex} \\ & 2x + trZ = 0 \end{array} \right\} \tag{5}$$

and $\mathfrak{h}_c$ is conjugated to the standard Cartan subalgebra in $\mathfrak{g}_c$. In the process of conjugation, the matrix $H(x,y,Z) \in \mathfrak{h}_c$ changes into

$$V(x,y,Z) = \begin{bmatrix} x+y & 0 & 0 \\ 0 & Z & 0 \\ 0 & 0 & x-y \end{bmatrix} : \begin{array}{ll} x,y & \in \mathbb{C} \\ Z & 2\times 2 \; diag., \text{complex} \\ & 2x + trZ = 0 \end{array} .$$

Let us denote by z_1, z_2 the two diagonal entries of the complex matrix Z. Let also $e_j(V)$ denote, as usual, the j-th element of the diagonal matrix V and we may write down the simple roots in the usual manner. They are

$$\begin{array}{lll} \beta_1 = & e_1 - e_2 = & x + y - z_1, \\ \beta_2 = & e_2 - e_3 = & z_1 - z_2, \\ \beta_3 = & e_3 - e_4 = & z_2 - x + y. \end{array}$$

We shall call Δ the system of roots and Δ^+ the set of positive ones determined by $\pi = \{\beta_1, \beta_2, \beta_3\}$ and the other three positive roots are

$$\begin{array}{lll} \beta_1 + \beta_2 = & e_1 - e_3 = & x + y - z_2, \\ \beta_2 + \beta_3 = & e_2 - e_4 = & z_1 - x + y, \\ \beta_1 + \beta_2 + \beta_3 = & e_1 - e_4 = & 2y. \end{array}$$

Let σ be the conjugations in $\mathfrak{g}_c$ with respect to $\mathfrak{g}$ mentioned above. Then its effect on the roots which by definition is $\sigma(\beta)(H) = \overline{\beta(\sigma(H))}$ can be seen to be

$$\begin{array}{l} \sigma(\beta_1) = \beta_2 + \beta_3 \\ \sigma(\beta_2) = -\beta_2 \\ \sigma(\beta_3) = \beta_1 + \beta_2 \\ \sigma(\beta_1 + \beta_2 + \beta_3) = \beta_1 + \beta_2 + \beta_3. \end{array} \tag{6}$$

Let us consider in $\mathfrak{h}$ the subspaces

$$\begin{array}{l} \mathfrak{a} = \{H(0,y,0) : y \in \mathbb{R}\} \subset \mathfrak{p} \\ \mathfrak{t} = \{H(ix,0,iZ) : 2x + z_1 + z_2 = 0\} \subset \mathfrak{k} \end{array}$$

it is clear that $\dim_{\mathbb{R}} \mathfrak{a} = 1$, $\dim_{\mathbb{R}} \mathfrak{t} = 2$ and $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}.$. Set as usual $[\mathfrak{h}_c]_R = \sum_{\beta\in\Delta} RH_\beta$, we also have $[\mathfrak{h}_c]_{\mathbb{R}} = i\mathfrak{t} \oplus \mathfrak{a}. \subset \mathfrak{h}_c$ and $\mathfrak{h}_u = i[\mathfrak{h}_c]_{\mathbb{R}} = \mathfrak{t} \oplus i\mathfrak{a}. \subset \mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p} = \mathfrak{su}(4)$. Then

$$\mathfrak{h}_u = \mathfrak{t} \oplus i\mathfrak{a} = \left\{ \begin{bmatrix} -ix & 0 & iy \\ 0 & -iZ & 0 \\ iy & 0 & -ix \end{bmatrix} : \begin{array}{ll} x,y & \in \mathbb{R} \\ Z & 2\times 2 \; diag., \text{real} \\ & 2x + trZ = 0 \end{array} \right\}. \tag{7}$$

To construct the manifold M_c we may take the orbit of any regular element. Let us take now in $\mathfrak{h}_u$ the element $E = H(-ix, iy, -iZ)$ such that $E = E_{\mathfrak{t}} + iE_{\mathfrak{a}.} \in \mathfrak{h}_u$ $= \mathfrak{t} \oplus i\mathfrak{a} \subset \mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p} = \mathfrak{su}(4)$ and no root of Δ^+ vanishes on it. To construct this E we need to solve the following set of equations (the factors i are not relevant)

$$\begin{array}{l} x + y - z_1 = t_1 \\ z_1 - z_2 = t_2 \\ z_2 - x + y = t_3 \end{array} , \qquad \begin{array}{l} x + y - z_2 = t_1 + t_2 \\ z_1 - x + y = t_2 + t_3 \\ 2y = t_1 + t_2 + t_3 \end{array} , \qquad t_j \in \mathbb{R},\ t_j > 0 \ \forall\, j.$$

then a solution for the first system is obviously also a solution for the second one.

We may take z_2 arbitrary which yields

$$\begin{array}{ll} z_1 & = t_2 + z_2 \\ y & = \frac{1}{2}t_3 + \frac{1}{2}z_2 + \frac{1}{2}t_1 + t_2 \\ x & = \frac{1}{2}t_1 + \frac{1}{2}z_2 - \frac{1}{2}t_3 \end{array}$$

Then we see that we can get an element

$$E = E_{\mathfrak{t}} + iE_{\mathfrak{a}.} \in \mathfrak{h}_u = \mathfrak{t} \oplus i\mathfrak{a}. \subset \mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p} = \mathfrak{su}(4)$$

where all the positive roots (and hence all the roots) have a non-zero value (positive in fact). We are taking of course

$$E_{\mathfrak{t}} = H(-ix, 0, -iZ), \qquad E_{\mathfrak{a}.} = H(0, y, 0).$$

By construction E is a *regular element* of $\mathfrak{h}_u$ and hence its orbit by the adjoint action of G_u on $\mathfrak{g}_u$ is a principal orbit. The complex flag manifold $M_c = Ad(G_u)E$ is of type $SU(4)/T^3$ and $\dim_{\mathbb{R}}(M_c) = 12$.

Let us consider now the orbit of E by the group $G \simeq SU(3,1)$. We want to determine the Lie algebra of the parabolic subgroup of G that is the isotropy group at E. To that end we need to determine the Borel subalgebra $\mathfrak{P}_E \subset \mathfrak{g}_c$ which is the Lie algebra of the isotropy subgroup of G_c corresponding to $E \in M_c$.

Since, by construction, E is a regular element in $\mathfrak{h}_c = (\mathfrak{h}_u)^{\mathbb{C}}$, it determines a system of simple roots (base) by [Hu, p. 48] and it is clear that the base determined by E is precisely $\pi = \{\beta_1, \beta_2, \beta_3\}$.

Now consider the elements H_β in $\mathfrak{h}_c$ associated to the roots $\beta \in \Delta$. They are contained in $i\mathfrak{h}_u = [\mathfrak{h}_c]_{\mathbb{R}}$ then, if τ denotes the conjugation in $\mathfrak{g}_c$ with respect to $\mathfrak{g}_u$, we have $\tau(H_\beta) = -H_\beta$ for every root $\beta \in \Delta$. Then $\mathfrak{P}_E \cap \tau\mathfrak{P}_E = \mathfrak{P}_E^r = \mathfrak{h}_C$. Now by [W, p. 1131, (2.6), 1.] $\mathfrak{g}_u \cap \mathfrak{P}_E$ is a real form of $\mathfrak{P}_E \cap \tau\mathfrak{P}_E = \mathfrak{P}_E^r = \mathfrak{h}_c$ and this is precisely $\mathfrak{g}_u \cap \mathfrak{P}_E = \mathfrak{h}_u$.

Now we seek to determine the isotropy subalgebra of $\mathfrak{g}$ at this point E in M_c. Again by [W, p. 1131, (2.6), 1.] $\mathfrak{g} \cap \mathfrak{P}_E$ is a real form of $\mathfrak{P}_E \cap \sigma\mathfrak{P}_E$ where now σ is as above the conjugation of $\mathfrak{g}_c$ with respect to $\mathfrak{g}$.

It is clear from (6) that

$$\mathfrak{P}_E \cap \sigma\mathfrak{P}_E = \mathfrak{h}_c \oplus \sum_{\beta \in \Delta^+, \beta \neq \beta_2} \mathfrak{g}_{c\beta}$$

but we can determine directly the *real codimension* of the orbit $G(E)$ by using [W, p. 1133, (2.12), (ii)] which indicates that its value is in this case $|\Phi^u \cap \sigma\Phi^u| = 5$ as follows immediately from (6). This means that the orbit $G(E)$ has real dimension 7.

It remains to be proven that this is in fact the closed orbit and to that end we use [W, p. 1135, (3.4), (iv)] by showing that the maximal compact subgroup $K \subset G$ which is the analytic subgroup corresponding to the subalgebra $\mathfrak{k}$ indicated in (3) acts transitively on this orbit. To reach this conclusion we just need to show that the dimension of the orbit of E by K is also 7 because then the orbit $K(E)$ will be open in $G(E)$ and since K is compact we will have $K(E) = G(E)$.

Since we clearly have $\mathfrak{h}_u \cap \mathfrak{k} = \mathfrak{t}$ we see that the dimension of $K(E)$ is $\dim\mathfrak{k} - \dim\mathfrak{t}$ $= \dim\mathfrak{k} - 2$ and since $\dim\mathfrak{k} = 9$ we get finally that $\dim K(E) = 7$ and so this is the closed orbit.

This closed orbit is not an R-space. In fact, for this group G, one can get essentially one R-space which is the sphere S^5. To get this space one has to take the orbit by K of a non-zero element $E_1 \in \mathfrak{a}$ (or $iE_1 \in i\mathfrak{a}$). For instance we may take, in the above notation, $E_1 = H(0, y, 0)$ with $y \neq 0$ in $\mathbb{R}$. Since $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$ every orbit contains an element of $\mathfrak{a}$ and so the elements of $\mathfrak{a}$ determine the orbits. Furthermore the representation of K on $\mathfrak{p}$ is essentially the canonical one of $U(3)$ on $\mathbb{C}^3$ whose orbits are $\{0\}$ and S^5. Then the closed orbit described above is not an R-space.

4. The index number of the closed orbit in a flag manifold

In this section we will establish the definition of index number of the closed orbit, and prove some results that will be necessary for the proof of the main theorem.

Definition 4.1. *Let $M(\Phi)$ the unique closed G-orbit in the flag manifold $M_c(\Phi)$. We define the Index Number of $M(\Phi)$ as*

$$\begin{aligned}\#_I(M(\Phi)) \quad = \quad & \max\{card(A \cap M(\Phi)), \textit{where } A = \mathfrak{h}_u \cap M_c(\Phi),\ \textit{for } \mathfrak{h}_u \\ & \textit{minimally compact}\}\end{aligned}$$

It is obviously an integer, and we will get that it is finite as a corollary of our main result. In the rest of this section, we will see equivalent ways of obtaining it that allow us to calculate and prove our results.

Lemma 4.2. *Let $N_{G_u}(\mathfrak{h}_u)$ be the normalizer of $\mathfrak{h}_u$ in G_u. Then $\mathfrak{h}_u \cap M_c(\Phi)$ is non empty, and there exists an element $E \in \mathfrak{h}_u \cap M_c(\Phi)$ such that $\mathfrak{h}_u \cap M_c(\Phi) = [N_{G_u}(\mathfrak{h}_u)](E)$.*

Proof:
By [H, pag. 162]), a Cartan subalgebra can be realized as a centralizer, that is, there exists a regular element $X \in \mathfrak{g}_u$ such that $\mathfrak{h}_u = c_{\mathfrak{g}_u}(\{X\}) = \{Y \in \mathfrak{g}_u : [Y, X] = 0\}$. We can then consider the function $f : M_c(\Phi) \longrightarrow \mathbb{R}$, defined by $f(Z) = \langle Z, X\rangle$. This function has a minimum at a point E of the orbit (since $M_c(\Phi)$ is compact). Hence $\frac{d}{dt}\Big|_{t=0} f(Ad(\exp(tY)).E) = 0 \quad \forall\, Y \in \mathfrak{g}_u$, and

$$\frac{d}{dt}\bigg|_{t=0} \langle Ad(\exp(tY)).E, X\rangle \quad = \quad \langle [Y, E], X\rangle = \langle Y, [E, X]\rangle = 0$$

$\forall\ Y \in \mathfrak{g}_u$. Then $[E, X] = 0$, so $E \in \mathfrak{h}_u$ and $\mathfrak{h}_u \cap M_c(\Phi)$ is not empty.

Let us prove now the second part of the statement. In order to do that, we consider $W(\mathfrak{h}_u) = \{Ad(g) : g \in G_u \text{ and } \mathrm{Ad}(g)\mathfrak{h}_u \subseteq \mathfrak{h}_u\}$, the Weyl group of $\mathfrak{h}_u$. We will prove that $A = M_c(\Phi) \cap \mathfrak{h}_u$ is an orbit of this group.

In order to do this, we consider $X, Y \in \mathfrak{h}_u$, such that $X = Ad(g)Y$, $g \in G_u$. Then $c_{\mathfrak{g}_u}(\{X\}) = c_{\mathfrak{g}_u}(\{Ad(g)Y\}) = Ad(g)c_{\mathfrak{g}_u}(\{Y\})$. But using that $\mathfrak{h}_u$ is abelian, we can see that $\mathfrak{h}_u \subset c_{\mathfrak{g}_u}(\{X\}) = \{Z \in \mathfrak{g}_u : [Z, X] = 0\}$, and $Ad(g)\mathfrak{h}_u \subset c_{\mathfrak{g}_u}(\{X\})$. Therefore $\mathfrak{h}_u \subset c_{\mathfrak{g}_u}(\{X\})$ and $Ad(g)\mathfrak{h}_u \subset c_{\mathfrak{g}_u}(\{X\})$, so they both are Cartan subalgebras of $c_{\mathfrak{g}_u}(\{X\})$.

Now if we denote by $(G_u)_X$ the isotropy group of X by the adjoint action of G_u, we know that $c_{\mathfrak{g}_u}(\{X\})$ is the Lie algebra of the connected component of the Identity.

By [H, pag. 248, ii], there exists $Z \in c_{\mathfrak{g}_u}(\{X\})$ such that $Ad(\exp Z)(Ad(g)\mathfrak{h}_u) = \mathfrak{h}_u$, and $Ad(\exp Z)(X) = X$; hence

$$X = Ad(\exp Z)(Ad(g)Y) = Ad(\exp Z.g)(Y)$$

with $Ad(\exp Z.g) \in W(\mathfrak{h}_u)$. So, $X \in [W(\mathfrak{h}_u)].Y, \ \forall\ X \in M_c(\Phi) \cap \mathfrak{h}_u$. That is, $M_c(\Phi) \cap \mathfrak{h}_u$ is an orbit of the Weyl group.

Finally, since the Weyl group is generated by the elements $g \in G_u$ that preserve $\mathfrak{h}_u$, that is, the elements of the normalizer $N_{G_u}(\mathfrak{h}_u)$. So $M_c(\Phi) \cap \mathfrak{h}_u$ is the orbit through any of its points of the group $N_{G_u}(\mathfrak{h}_u)$. This proves the lemma. .

Remark 4.3. *By lemma 4.2,* $A = \mathfrak{h}_u \cap M_c(\Phi) = [N_{G_u}(\mathfrak{h}_u)](E)$, *so* $A \cap M(\Phi) = \mathfrak{h}_u \cap M(\Phi) = [N_{G_u}(\mathfrak{h}_u)](E) \cap M(\Phi)$.

Definition 4.4. *Let* $L_\Phi = \{g \in N_{G_u}(\mathfrak{h}_u) : Ad(g).E \in M(\Phi)\}$

Remark 4.5. *With this definition, it is easy to see that*

$$[N_{G_u}(\mathfrak{h}_u)](E) \cap M(\Phi) = L_\Phi(E) \tag{8}$$

This and remark 4.3 prove that $A \cap M(\Phi) = L_\Phi(E)$ *for all* $\mathfrak{h}_u$ *minimally compact subalgebras. Then,*

$$\#_I(M(\Phi)) = max\{card(L_\Phi(E)), \text{where } \mathfrak{h}_u \text{ is minimally compact}\}$$

and by equation (8), the index number $\#_I(M(\Phi))$ *is*

$$max\{card([N_{G_u}(\mathfrak{h}_u)](E) \cap M(\Phi)), \text{where } \mathfrak{h}_u \text{ is minimally compact}\} \tag{9}$$

Finally, the next result will be very important in the proof of the main theorem. Let

$$[N_{G_u}(\mathfrak{h}_u)]_\sigma = \{g \in N_{G_u}(\mathfrak{h}_u) : (Ad(g) \circ \sigma)X = (\sigma \circ Ad(g))X \ \ \forall\ X \in \mathfrak{h}_u\}$$

where σ is the conjugation of $\mathfrak{g}_c$ with respect to $\mathfrak{g}$ (the noncompact real form of $\mathfrak{g}_c$.

Proposition 4.6. $\#([N_{G_u}(\mathfrak{h}_u)]_\sigma.E) \leq \#_I(M(\Phi))$. *Furthermore, if* $L_\Phi \subseteq N_{G_u}(i\mathfrak{a})$, *then equality holds.*

Proof: Notice that by definition, $[N_{G_u}(\mathfrak{h}_u)]_\sigma$ is a subgroup of $N_{G_u}(\mathfrak{h}_u)$, so the orbit through E of the first group is contained in the orbit through E of the second one. If we were able to prove that $[N_{G_u}(\mathfrak{h}_u)]_\sigma.E \subseteq M(\Phi)$, then

$$[N_{G_u}(\mathfrak{h}_u)]_\sigma.E \subseteq [N_{G_u}(\mathfrak{h}_u)].E \cap M(\Phi)$$

But then, by (9),

$$\#[N_{G_u}(\mathfrak{h}_u)]_\sigma.E \leq \#_I(M(\Phi)),$$

as we wanted to prove.

We need to prove that $[N_{G_u}(\mathfrak{h}_u)]_\sigma.E \subseteq M(\Phi)$. Now for [H, pag. 324, 8.8.ii],

$$N_{G_u}(i\mathfrak{a}) = N_K(i\mathfrak{a})U^* \tag{10}$$

where $U^* = \exp(i\mathfrak{a})$ (in particular, $U^* \subset T_u = \exp(\mathfrak{h}_u)$). But in order to be able to apply the theorem of the cite, we need to verify that G_u is simply connected, this follows from hypotesis that G_c is simply connected. Then we have that

$$[N_{G_u}(\mathfrak{h}_u)]_\sigma.E \subset N_K(i\mathfrak{a})U^*(E) = N_K(i\mathfrak{a})(E) \subset M(\Phi).$$

This proves the first part of the statement.

We will now prove the second part. Let $g_0 \in L_\Phi$, that is, $Ad(g_0).E \in M(\Phi)$. If $L_\Phi \subseteq N_{G_u}(i\mathfrak{a})$, then obviously $L_\Phi(E) \subseteq N_{G_u}(i\mathfrak{a})(E)$. But this and equation (10), imply that $N_{G_u}(i\mathfrak{a})(E) = N_K(i\mathfrak{a})(E)$. So, there exists $h \in N_K(i\mathfrak{a})$ such that $Ad(g_0).E = Ad(h).E$. Now using that $h_u = \mathfrak{t} \oplus i\mathfrak{a}$ and $Ad(g_0).E \in h_u$, we can decompose

$$Ad(g_0).E = Ad(h).E = Ad(h)(E_\mathfrak{t}) + Ad(h)(iE_\mathfrak{a}) \tag{11}$$

with $Ad(h)(E_\mathfrak{t}) \in \mathfrak{t}$ and $Ad(h)(iE_\mathfrak{a}) \in i\mathfrak{a}$.

On the other hand, if $\xi = Ad(h).E_\mathfrak{t}$, then

$$\xi = Ad(h).E_\mathfrak{t} \in Ad(h)(\mathfrak{t}) \subseteq Ad(h)(\mathfrak{m}) = \mathfrak{m}$$

so

$$\begin{aligned} \xi &= Ad(h).E_\mathfrak{t} \in \mathfrak{t} \subseteq \mathfrak{m}, \\ \xi &= Ad(h).E_\mathfrak{t} \in Ad(h)(\mathfrak{t}) \subseteq \mathfrak{m} \end{aligned} \tag{12}$$

Let us consider then the orbit through ξ of $Q = [Z_K(i\mathfrak{a})]_e$. This is a complex flag manifold, because Q is a connected, compact Lie group on a point ξ of its Lie Algebra $Lie(Q) = \mathfrak{m}$. And consider $R = Q_\xi$ the isotropy group at ξ, which is compact, connected, and of maximal rank in Q, since it is the centralizer of a torus. Clearly, $\mathfrak{r} = Lie(R) = \{X \in \mathfrak{m} : [X, \xi] = 0\}$. By equations (12), $\mathfrak{t}$ and $Ad(h)(\mathfrak{t})$ are contained in $\mathfrak{r}$, and furthermore, they are subalgebras corresponding to maximal tori in R. In particular they are conjugated, and there exist $q \in R$ such that

$$Ad(qh)(\mathfrak{t}) \;=\; Ad(q)Ad(h)(\mathfrak{t}) = \mathfrak{t} \tag{13}$$

And $Ad(q)(\xi) = \xi$, since $q \in R = Q_\xi$. But using that $q \in Q$, we get that $Ad(q)Ad(h)(iE_\mathfrak{a}) = Ad(h)(iE_\mathfrak{a})$ (and q fixes every element in $i\mathfrak{a}$). Therefore, $Ad(q)Ad(h)(E) = Ad(h)(E)$. This in turn implies that

$$Ad(g_0)(E) = Ad(h)(E) = Ad(qh)(E)$$

Notice besides that equation (13) implies that $Ad(qh)(\mathfrak{t}) = \mathfrak{t}$, $h \in N_K(i\mathfrak{a})$, and $q \in Q$, so $Ad(qh)(i\mathfrak{a}) \subseteq (i\mathfrak{a})$. In particular, qh leaves $i\mathfrak{a}$ invariant, hence $qh \in [N_{G_u}(\mathfrak{h}_u)]_\sigma$. This proves that every element of $L_\Phi(E)$ is an element of the orbit $[N_{G_u}(\mathfrak{h}_u)]_\sigma(E)$, that is,

$$L_\Phi(E) = [N_{G_u}(\mathfrak{h}_u)](E) \cap M(\Phi) = [N_{G_u}(\mathfrak{h}_u)]_\sigma(E) \tag{14}$$

.

5. The fibration ρ and the orbits of $\#([N_{G_u}(\mathfrak{h}_u)]_\sigma.E)$

In view of proposition 4.6, we need to calculate the cardinal of the orbit of $[N_{G_u}(\mathfrak{h}_u)]_\sigma$ through a point $E \in A$. In order to do this, we will define a fibration and prove that the intersection of this orbit with all the fibers have the same number of elements, and that this number can be calculated as the cardinal of a normalizer.

Definition 5.1. *Let $M(\Phi) = Ad(K)(E)$ the closed orbit, and let $M_\mathfrak{a}$ be the R–space $M_\mathfrak{a} = Ad(K)(iE_\mathfrak{a})$. We consider $\rho : M(\Phi) \longrightarrow M_\mathfrak{a}$ the natural fibration given by $\rho(Ad(k)(E)) = \rho(Ad(k)(E_\mathfrak{t} + iE_\mathfrak{a})) = (Ad(k)(iE_\mathfrak{a}))$. (see equation (11))*

Remark 5.2. *Notice that, since $K_\mathfrak{a}$ is the isotropy group the $iE_\mathfrak{a}$ in K, then $K_\mathfrak{a}.E$ is the fiber of the fibration over the point $iE_\mathfrak{a}$ in the R–space $M_\mathfrak{a}$. In fact, if $k_\mathfrak{a} \in K_\mathfrak{a}$, then*

$$k_\mathfrak{a}.E = k_\mathfrak{a}.(E_\mathfrak{t} + iE_\mathfrak{a}) = k_\mathfrak{a}.(E_\mathfrak{t}) + k_\mathfrak{a}(iE_\mathfrak{a}) = k_\mathfrak{a}.(E_\mathfrak{t}) + iE_\mathfrak{a}.$$

Lemma 5.3. *Let for the rest of the paper be $B = N_K(i\mathfrak{a})$. Then*

$$[N_{G_u}(\mathfrak{h}_u)]_\sigma.E \cap K_a.E = N_{K_a \cap B}(\mathfrak{t}).E$$

Proof: We need to calculate $[N_{G_u}(\mathfrak{h}_u)]_\sigma.E \cap$ Fiber through E', where $E' = g.E = Ad(g).E, \quad g \in [N_{G_u}(\mathfrak{h}_u)]_\sigma$. Now

$$[N_{G_u}(\mathfrak{h}_u)]_\sigma.E' = [N_{G_u}(\mathfrak{h}_u)]_\sigma.(g.E) = [N_{G_u}(\mathfrak{h}_u)]_\sigma.E$$

since the orbit of a group through any point of the orbit, is just the same orbit.

If $g = hv, \quad v \in U^*, h \in B$, so $g.E = Ad(g)E = Ad(h)E = h.E$ and the fiber of ρ through $Ad(h)E = h(\text{Fiber through E})h^{-1} = hK_ah^{-1}$ (Isotropy groups of points in the same orbit are conjugated). But then,

$$[N_{G_u}(\mathfrak{h}_u)]_\sigma.E' \cap (hK_ah^{-1}).E' = [N_{hK_ah^{-1}\cap B}(\mathfrak{t})].E'$$

since $E' = Ad(h)E$, $[N_{G_u}(\mathfrak{h}_u)]_\sigma.E' = [N_{G_u}(\mathfrak{h}_u)]_\sigma.E$.

And

$$N_{hK_ah^{-1}\cap B}(\mathfrak{t}) = hN_{K_a\cap B}(\mathfrak{t})h^{-1}$$

Now using that $E' = Ad(h)E$,

$$\begin{aligned} N_{h(K_\mathfrak{a}\cap B)h^{-1}}(E') &= \left(h\left(N_{K_\mathfrak{a}\cap B}(\mathfrak{t})\right)h^{-1}\right)(E') \\ &= Ad(h)N_{K_\mathfrak{a}\cap B}(\mathfrak{t})(E) \end{aligned}$$

The number of points of the orbit $[N_{G_u}(\mathfrak{h}_u)]_\sigma(E)$ over the fibers of ρ through $iE_\mathfrak{a}$ and $iE'_\mathfrak{a}$ are the same. So, if γ is the number of fibers involved,

$$\#\left[[N_{G_u}(\mathfrak{h}_u)]_\sigma(E)\right] \quad = \quad \#\left[N_{K_\mathfrak{a}\cap B}(\mathfrak{t})(E)\right].\gamma \tag{15}$$

since all the orbits have the same number of points. But
$\gamma = \#\left[[N_{G_u}(\mathfrak{h}_u)]_\sigma(iE_\mathfrak{a})\right]$. So by Theorem A and equality 14, we have that

$$\gamma = \#\left[N_{G_u}(\mathfrak{h}_u)\right]_\sigma(iE_\mathfrak{a}) = \sum_{j\geq 0}\beta_j(M_\mathfrak{a}, \mathbb{Z}_2) \tag{16}$$

.

6. The cardinal of $N_{K_{\mathfrak{a}}\cap B}(\mathfrak{t})$

In view of the results of last section, se will need to calculate the cardinal of $N_{K_{\mathfrak{a}}\cap B}(\mathfrak{t})$.

Lemma 6.1. *Let $\mathfrak{m}$ be the Lie algebra of $C = Z_K(i\mathfrak{a})$, $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{m}$. Then there exist a finite number of elements $k_1, \dots, k_r \in B = N_K(i\mathfrak{a})$, such that*

$$N_{K_{\mathfrak{a}}\cap B}(\mathfrak{t}) = N_{C_e}(\mathfrak{t}) \cup \left[\cup_{j=1}^{r} k_j N_{C_e}(\mathfrak{t})\right]$$

Proof: By [H, pag 284], the connected components of the identity of B and C are the same, so $C_e = (K_{\mathfrak{a}} \cap B)_e = B_e$. But since $K_{\mathfrak{a}}$ and B are compact groups such that $C \subseteq K_{\mathfrak{a}} \cap B \subseteq B$, they all have a finite number of connected components, and they are all left translations of C_e by adequate elements $h \in B$. Now using that disjoint groups produce disjoint normalizers, we have that

$$\begin{aligned} N_{K_{\mathfrak{a}}\cap B}(\mathfrak{t}) &= N_{(K_{\mathfrak{a}}\cap B)_e \cup \left[\cup_{j=1}^{r} k_j (K_{\mathfrak{a}}\cap B)_e\right]}(\mathfrak{t}) \\ &= N_{C_e \cup \left[\cup_{j=1}^{r} k_j C_e\right]}(\mathfrak{t}) \\ &= N_{C_e}(\mathfrak{t}) \cup \left[\cup_{j=1}^{r} N_{k_j C_e}\right](\mathfrak{t}) \end{aligned} \tag{17}$$

Obviously, if we could replace all $k_j's$ for another element in the same connected component of $K_{\mathfrak{a}} \cap B$ as k_j, and such that they belong to $N_{K_{\mathfrak{a}}\cap B}(\mathfrak{t})$, then $N_{k_j C_e} = k_j N_{C_e}$ forall j, and the lemma is proved.

Let then be $\mathfrak{t}_j = Ad(k_j)(\mathfrak{t})$. Now in general $\mathfrak{t}_j \neq \mathfrak{t}$, but since $Ad(k_j)$ preserves $\mathfrak{m} = \mathfrak{t} + i\mathfrak{a}$, $\mathfrak{t}_j$ is another Cartan subalgebra of $\mathfrak{m}$. Then we can choose $u_j \in C_e = (K_{\mathfrak{a}} \cap B)_e$ such that $Ad(u_j)(\mathfrak{t}_j) = \mathfrak{t}$ (recall that all Cartan subalgebras are conjugated by elements in the connected component of the identity of the group). If $h_j = u_j k_j$ for all j, then h_j belongs to the same connected component of $K_{\mathfrak{a}} \cap B$ as k_j, and

$$Ad(h_j)(\mathfrak{t}) = Ad(u_j k_j)(\mathfrak{t}) = Ad(u_j)Ad(k_j)(\mathfrak{t}) = Ad(u_j)(\mathfrak{t}_j) = \mathfrak{t}$$

as we wanted to prove. .

7. Main Theorem

In this section we complete the proof of the main theorem of this work.

Theorem 7.1. *Let $M(\Phi)$ be the closed orbit in the flag manifold $M_c(\Phi)$. Then there exists a flag manifold $R = [K_{\mathfrak{a}} \cap N_K(i\mathfrak{a})].E$, an R–space M_a, and a fibration $\rho : M(\Phi) \longrightarrow M_{\mathfrak{a}}$, such that the flag manifold R is contained in the fiber through the point $p = iE_{\mathfrak{a}}$ of the fibration ρ, and:*

$$\chi(M_c(\Phi)) \quad \geq \quad \#_I(M(\Phi)) \quad \geq \quad \#_I(R).\#_I(M_{\mathfrak{a}}) \tag{18}$$

In particular, $\#_I$ is always finite (bounded above by the Euler number of $M_c(\Phi)$). Furthermore, if $L_\Phi \subseteq N_{G_u}(i\mathfrak{a})$ then equality holds in (18), that is,

$$\#_I(M(\Phi)) \quad = \quad \#_I(R).\#_I(M_{\mathfrak{a}}) \tag{19}$$

Reformulating this theorem in terms of the results from [S1] and [S2], we get that

Theorem 7.2. *Let $M(\Phi)$ be the closed orbit in the flag manifold $M_c(\Phi)$. Then there exists a flag manifold $R = [K_{\mathfrak{a}} \cap N_K(i\mathfrak{a})].E$, an R–space M_a, and a fibration $\rho : M(\Phi) \longrightarrow M_{\mathfrak{a}}$, such that the flag manifold R is contained in a fiber, and:*

$$\chi(M_c(\Phi)) \quad \geq \quad \#_I(M(\Phi)) \quad \geq \quad \chi(R).\sum_{j\geq 0}\beta_j(M_{\mathfrak{a}},\mathbb{Z}_2) \tag{20}$$

And if $L_\Phi \subseteq N_{G_u}(i\mathfrak{a})$ then equality holds, that is,

$$\#_I(M(\Phi)) \quad = \quad \chi(R).\sum_{j\geq 0}\beta_j(M_{\mathfrak{a}},\mathbb{Z}_2) \tag{21}$$

Proof: Theorem 7.2 is just a restatement of 7.1 using the results of Sanchez cited in the introduction. So we only need to prove theorem 7.1. In order to prove it, we consider $M_{\mathfrak{a}}$, and $\rho : M(\Phi) \longrightarrow M_{\mathfrak{a}}$ defined as above, and $R = (K_{\mathfrak{a}} \cap B).E = (K_{\mathfrak{a}} \cap N_K(i\mathfrak{a})).E$. Notice that $R = (K_{\mathfrak{a}} \cap N_K(i\mathfrak{a})).E \subseteq K_{\mathfrak{a}}.E$, and by remark 5.2, this is the fiber through the point $iE_{\mathfrak{a}}$ of the fibration. So, the flag manifold R is contained in the fiber over $iE_{\mathfrak{a}}$ of the fibration ρ as stated.

Now $A = \mathfrak{h}_u \cap M_c(\Phi)$ is the set of fixed points by the action of the torus $T_u = \exp(\mathfrak{h}_u) \subset (G_u)_E$, that is, $A = \mathfrak{h}_u \cap M_c(\Phi) = F(T_u, M_c(\Phi))$ (cf. [S2]). In particular, it is a finite set. Then $\chi(F(T_u, M_c(\Phi))) = \beta_0(F(T_u, M_c(\Phi)) = n$, since the rest of the Betti numbers vanish. Using a general theorem (cf. [S2]), we have that

$$\chi(M_c(\Phi)) = \chi(F(T_u, M_c(\Phi))) = \#F(T_u, M_c(\Phi)) = n = \#\mathfrak{h}_u \cap M_c(\Phi)$$

By remark 4.3,

$$\begin{aligned}
\#_I(M(\Phi)) \quad &= \quad \max\{\mathrm{card}(\mathfrak{h}_u \cap M(\Phi)), \text{where } \mathfrak{h}_u \text{ is minimally compact}\} \\
&\leq \quad \max\{\mathrm{card}(\mathfrak{h}_u \cap M_c(\Phi)), \text{where } \mathfrak{h}_u \text{ is minimally compact}\} \\
&= \quad \chi(M_c(\Phi))
\end{aligned}$$

This proves our first inequality.

As for the second inequality, notice that by lemma 6.1, we can calculate $\#\left[N_{K_{\mathfrak{a}}\cap B}(\mathfrak{t})\right](E)$, in terms of $\#\left[N_{C_e}(\mathfrak{t})\right](E)$:

$$\begin{aligned}
\#\left[\left[N_{K_{\mathfrak{a}}\cap B}(\mathfrak{t})\right](E)\right] \quad &= \quad \#\left[\left[N_{K_{\mathfrak{a}}\cap B}(\mathfrak{t})\right](E_{\mathfrak{t}})\right] \\
&= \quad \#\left[\left[N_{C_e}(\mathfrak{t}) \cup \left\{\cup_{j=1}^r k_j N_{C_e}(\mathfrak{t})\right\}\right](E_{\mathfrak{t}})\right] \\
&= \quad (r+1)\#\left[N_{C_e}(\mathfrak{t})(E)\right] \\
&= \quad (r+1)\#_I(C_e(E_{\mathfrak{t}})) \\
&= \quad \#_I\left[(K_{\mathfrak{a}} \cap B)(E_{\mathfrak{t}})\right] \\
&= \quad \#_I\left[(K_{\mathfrak{a}} \cap B)(E)\right] \\
&= \quad \#_I(R)
\end{aligned}$$

But then (15), using this formula and(16), becomes

$$\begin{aligned}\#\left[[N_{G_u}(\mathfrak{h}_u)]_\sigma(E)\right] &= \#_I\left[(K_\mathfrak{a}\cap B)(E)\right].\#\left[[N_{G_u}(\mathfrak{h}_u)]_\sigma(iE_\mathfrak{a})\right] \\ &= \#_I(R)\#_I(M_\mathfrak{a})\end{aligned}$$

Finally, by proposition 4.6, $\#([N_{G_u}(\mathfrak{h}_u)]_\sigma.E) \leq \#_I(M(\Phi))$, and if $L_\Phi \subseteq N_{G_u}(i\mathfrak{a})$, then equality holds. This concludes the proof of the theorem. .

Acknowledgements: Part of this work comes from the first author's Ph. D. thesis, written under the direction of the second author. We also want to thank Adrian Will for many helpful comments and discussions.

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Recibido: 12 de junio de 2003
Revisado: 27 de julio de 2004