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SIMULTANEOUS APPROXIMATION WITH LINEAR COMBINATION OF INTEGRAL BASKAKOV TYPE OPERATORS

KAREEM J. THAMER, MAY A. AL-SHIBEEB AND A.I. IBRAHEM

ABSTRACT. The aim of the present paper is to study some direct results in simultaneous approximation for the linear combination of integral Baskakov type operators.

1 INTRODUCTION

Agrawal and Thamer [1] introduced a new sequence of linear positive operators $M_n$called integral Baskakov – type operators to approximate unbounded continuous functions on $[0,\infty)$ and it is defined as follow

Let $\alpha>0$, $f\in C_\alpha[0,\infty)=\{f\in C[0,\infty):|f(t)|\le M(1+t)^\alpha for \quad some$ $M>0\}$.

Then,

(1.1) $M_n(f(t);x)=(n-1)\sum\limits_{\upsilon=1}^{\infty}p_{n,\upsilon}(x)\int\limits_0^\infty p_{n,\upsilon-1}(t)\ f(t)\ dt\ +\ (1+x)^{-n}f(0)$,

where $p_{n,\upsilon}(x)=\begin{pmatrix} n+\upsilon-1\\ \upsilon\end{pmatrix} x^\upsilon\ (1+x)^{-(n+\upsilon)}$, $x\in[0,\infty)$ is the kernel of Lupas operators $L_n(f(t);x)=\sum\limits_{\upsilon=0}^{\infty}\begin{pmatrix} n+\upsilon-1\\ \upsilon\end{pmatrix} x^\upsilon(1+x)^{-(n+\upsilon)}\ f(\upsilon/n)$.

We may also write (1.1) as :

$$M_n(f(t);x)=\int\limits_0^\infty W_n(t,x)f(t)\ dt \quad ,$$

where $W_n(t,x)=(n-1)\sum\limits_{\upsilon=1}^{\infty}p_{n,\upsilon}(x)\ p_{n,\upsilon-1}(t)+(1+x)^{-n}\delta(t)$, $\delta(t)$ being the Dirac delta function.

The space $C_\alpha[0,\infty)$ is normed by $\|f\|_{C_\alpha}=\sup\limits_{0\le t\le\infty}|f(t)|\ (1+t)^{-\alpha}$.

The operator (1.1) was used to study the degree of approximation in simultaneous approximation by Agrawal and Thamer [1]. It turned out that the order of approximation by the operator (1.1) is, at best, $O(n^{-1})$, howsoever smooth the function may be. Thus, if we want to have a better order of approximation, we have to slacken the positivity condition. This is achieved by considering some

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carefully chosen linear combination introduced by May [6] and Rathore [7] of the operator (1.1). The linear combination is defined as follows:

Let $d_0, d_1, \ldots, d_k$ be $(k+1)$ arbitrary but fixed distinct positive integers. Then, following Agrawal and Sinha [3] , the linear combination $M_n(f,k,x)$ of $M_{d_j n}(f;x)$, $j = 0,1,2,\ldots,k$ is given by

$$M_n(f,k,x) = \frac{1}{\Delta} \begin{vmatrix} M_{d_0 n}(f;x) & d_0^{-1} & d_0^{-2} & \ldots & d_0^{-k} \\ M_{d_1 n}(f;x) & d_1^{-1} & d_1^{-2} & \ldots & d_1^{-k} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ M_{d_k n}(f;x) & d_k^{-1} & d_k^{-2} & \ldots & d_k^{-k} \end{vmatrix}, \tag{1.2}$$

where Δ is the Vandermonde determinant obtained by replacing the operator column of the above determinant by the entries 1. We have

$$M_n(f,k,x) = \sum_{j=0}^{k} C(j,k)\, M_{d_j n}(f;x)\ , \tag{1.3}$$

where

$$C(j,k) = \prod_{\substack{i=0 \\ i\neq j}}^{k} \frac{d_j}{d_j - d_i} \quad , \quad k \neq 0 \quad \textit{and} \quad C(0,0) = 1. \tag{1.4}$$

The object of the present paper is to investigate the degree of approximation of the operator $M_n^{(r)}\ (f,k,x)$. First we establish a Voranovskaja type asymptotic formula and then obtain an error estimate in terms of the local modulus of continuity for the operator $M_n^{(r)}\ (f,k,x)$.

2 AUXILIARY RESULTS

Throughout our work, N denotes the set of natural numbers, N^0 integers, and $\langle a,b\rangle$ an open interval containing $[a,b]$.

LEMMA 2.1 [4]. *If for $m \in N^0$ (the set of nonnegative integers), the m^{th} order moment of Lupas operators is defined by*

$$\mu_{n,m}(x) = \sum_{v=0}^{\infty} p_{n,v}(x)\left(\frac{v}{n} - x\right)^m.$$

Hence, $\mu_{n,0}(x) = 1$, $\mu_{n,1}(x) = 0$, and there holds the recurrence relation

$$n\mu_{n,m+1}(x) = x(1+x)\ \left[\mu'_{n,m}(x) + m\mu_{n,m-1}(x)\right]\ , \quad m \in N.$$

Consequently

(i) $\mu_{n,m}(x)$ is a polynomial in x of degree at most m.

(ii) For every $x \in [0,\infty)$, $\mu_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$, where $[\beta]$ denotes the integral part of β.

LEMMA 2.2 [1]. *Let the function* $T_{n,m}(x)$, $m \in N^0$ *be defined as*

$$T_{n,m}(x) = (n-1)\sum_{\upsilon=1}^{\infty} p_{n,\upsilon}(x) \int_0^\infty p_{n,\upsilon-1}(t)\,(t-x)^m\,dt + (-x)^m (1+x)^{-n}.$$

Then,

$$T_{n,0}(x) = 1\,, \quad T_{n,1}(x) = \frac{2x}{n-2}$$

and

$$(n-m-2)\,T_{n,m+1}(x) = x(1+x)\,T'_{n,m}(x) + [(2x+1)\,m + 2x]\,T_{n,m}(x) + \\ + 2mx(1+x)\,T_{n,m-1}(x)\ ,\ \ m \in N.$$

Hence,

(i) $T_{n,m}(x)$ *is a polynomial in* x *of degree* m.

(ii) For every $x \in [0,\infty)$, $T_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$.

(iii) The coefficients of $n^{-(\upsilon+1)}$ *in* $T_{n,2\upsilon+2}(x)$ *and* $T_{n,2\upsilon+1}(x)$ *are given by* $\dfrac{(2\upsilon+2)!\ \{x(1+x)\}^{\upsilon+1}}{(\upsilon+1)!}$ *and* $\dfrac{(2\upsilon+1)!}{\upsilon!}\ \{(\upsilon+1)(1+2x)-1\}\ \{x(1+x)\}^{\upsilon}$.

LEMMA 2.3 [5]. *There exist polynomials* $q_{i,j,r}(t)$ *independent of* n *and* υ *such that*

$$t^r (1+t)^r\ \frac{d^r}{dt^r}\,p_{n,\upsilon}(t) = \sum_{\substack{2i+j \le r \\ i,j \ge 0}} n^i\ (\upsilon - nt)^j\ q_{i,j,r}(t)\ p_{n,\upsilon}(t)\ .$$

LEMMA 2.4 [6].*If* $C(j,k)$, $j = 0,1,2,\ldots,k$ *are defined as in (1.4), then*

$$\sum_{j=0}^{k} C(j,k)\ d_j^{-m} = \begin{cases} 1, & m = 0 \\ 0, & m = 1,\ldots,k \end{cases}.$$

LEMMA 2.5 [8].*Let* f *be* r *times differentiable on* $[0,\infty)$ *such that* $f^{(r-1)}(t) = O(t^\alpha)$ *for some* α *as* $t \longrightarrow \infty$. *Then for* $r = 1,2,\ldots$ *and* $n > \alpha + r$, *we have*

$$M_n^{(r)}(f(t),x) = \frac{(n+r-1)!\ (n-r-1)!}{(n-1)!\ (n-2)!} \times \\ \times \sum_{\upsilon=1}^{\infty} p_{n+r,\upsilon}(x) \int_0^\infty p_{n-r,\upsilon+r-1}(t)\ f^{(r)}(t)\ dt.$$

LEMMA 2.6 [2]. *For* $r \in N$ *and* n *sufficiently large, there holds*

$$M_n\left((t-x)^r, k, x\right) = n^{-(k+1)}\ \{Q(r,k,x) + o(1)\}\,,$$

where $Q(r,k,x)$ *is a certain polynomial in* x *of degree* r.

3 MAIN RESULTS

In this section we shall state and prove the main results.

Theorem 3.1. *Let $f \in C_\alpha[0,\infty)$ and be bounded on every finite subinterval of $[0,\infty)$ admitting a derivative of order $2k+r+2$ at a fixed point $x \in (0,\infty)$. Let $f(t) = O(t^\alpha)$ as $t \longrightarrow \infty$ for some $\alpha > 0$, then we have*

$$\underset{n\longrightarrow\infty}{Lim}\ n^{k+1}\ \left[M_n^{(r)}(f,k,x) - f^{(r)}(x)\right] = \sum_{i=r}^{2k+r+2} f^{(i)}(x)\ Q(i,k,r,x) \tag{3.1}$$

and

$$\underset{n\longrightarrow\infty}{Lim}\ n^{k+1}\ \left[M_n^{(r)}(f,k+1,x) - f^{(r)}(x)\right] = 0, \tag{3.2}$$

where $Q(i,k,r,x)$ are certain polynomials in x.

Further, the Limits (3.1) and (3.2) hold uniformly in $[a,b]$, if $f^{(2k+r+2)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,\infty)$, $\eta > 0$.

Proof. By the Taylor expansion, we have

$$f(t) = \sum_{i=0}^{2k+r+2} \frac{f^{(i)}(x)}{i\,!}\ (t-x)^i + \varepsilon(t,x)\ (t-x)^{2k+r+2},$$

where $\varepsilon(t,x) \longrightarrow 0 \quad as \quad t \longrightarrow x.$

Thus, using Lemma 2.5, we have for sufficiently large n

$$\begin{aligned} n^{k+1}\left[M_n^{(r)}(f,k,x) - f^{(r)}(x)\right] &= n^{k+1}\left[\sum_{j=0}^{k} C(j,k)\ M_{d_jn}^{(r)}(f;x) - f^{(r)}(x)\right] \\ &= I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 = n^{k+1}\Bigg[\sum_{i=0}^{2k+r+2} \frac{f^{(i)}(x)}{i\,!} \sum_{j=0}^{k} C(j,k)\, \frac{(d_jn-r-2)\,!\ (d_jn+r-1)\,!}{(d_jn-1)\,!\ (d_jn-2)\,!} \\ \times \sum_{\upsilon=1}^{\infty} p_{d_jn+r,\upsilon}(x) \int_0^\infty p_{d_jn-r,\upsilon+r-1}(t)\ \frac{d^r}{dt^r}\ (t-x)^i\, dt - f^{(r)}(x) \\ + (-1)^{2k+r+2}\ \frac{(n+2k+r+1)\,!}{(n-1)\,!}\ (1+x)^{-n-2k-r-2}\, f(0)\Bigg]. \end{aligned}$$

$$\begin{aligned} I_2 = n^{k+1}\Bigg[\sum_{j=0}^{k} C(j,k)\ (d_jn-1)\ \sum_{\upsilon=1}^{\infty} p_{d_jn,\upsilon}^{(r)}(x) \\ \times \int_0^\infty p_{d_jn,\upsilon-1}(t)\, \varepsilon(t,x)\, (t-x)^{2k+r+2}\, dt \end{aligned}$$

$$+(-1)^{2k+r+2}\,\frac{(n+2k+r+1)\,!}{(n-1)\,!}\,(1+x)^{-n-2k-r-2}\,f\,(0)\Bigg].$$

It's clear that

$$(-1)^{2k+r+2}\,\frac{(n+2k+r+1)\,!}{(n-1)\,!}\,(1+x)^{-n-2k-r-2}\,f\,(0)\longrightarrow 0 \quad as \quad n\longrightarrow\infty.$$

Let $I_1 = I_3 + I_4$, where

$$I_3=\Bigg[n^{k+1}\sum_{i=r+1}^{2k+r+2}\frac{f^{(i)}\,(x)}{i!}\,\sum_{j=0}^{k}C\,(j,k)\,\frac{(d_jn-r-2)\;!\;(d_jn+r-1)\;!}{(d_jn-1)\;!\;(d_jn-2)\;!}$$

$$\times\sum_{\upsilon=1}^{\infty}p_{d_jn+r,\upsilon}\,(x)\int_0^{\infty}p_{d_jn-r,\upsilon+r-1}\,(t)\,\frac{d^r}{dt^r}\,(t-x)^i\,dt\Bigg].$$

$$I_4=n^{k+1}\left[f^{(r)}\,(x)\sum_{j=0}^{k}C\,(j,k)\,\frac{(d_jn-r-2)\;!\;(d_jn+r-1)\,!}{(d_jn-1)\,!\,(d_jn-2)\,!}-f^{(r)}\,(x)\right].$$

Thus, by (1.4),

$$I_4=n^{k+1}f^{(r)}\,(x)\left[\sum_{j=0}^{k}C\,(j,k)\,\frac{(d_jn-r-2)\;!\;(d_jn+r-1)\,!}{(d_jn-1)\,!\,(d_jn-2)\,!}-1\right].$$

Now, in view of Lemma 2.4, we have

$$I_4=f^{(r)}\,(x)\;K\,(r,k)+o\,(1)\;,\quad n\longrightarrow\infty\;,$$

where $K\,(r,k)$ is a constant depending only on r and k.

Next, by Lemma 2.4 and Lemma 2.6,we get

$$I_3=\sum_{i=r+1}^{2k+r+2}f^{(i)}\,(x)\;Q\,(i,k,r,x)+o\,(1),\quad n\longrightarrow\infty.$$

Thus

$$I_1\longrightarrow f^{(r)}\,(x)\;K\,(r,k)+\sum_{i=r+1}^{2k+r+2}f^{(i)}\,(x)\;Q\,(i,k,r,x)$$

$$=\sum_{i=r}^{2k+r+2}f^{(i)}\,(x)\;Q\,(i,k,r,x)\quad as\quad n\longrightarrow\infty.$$

Now we must prove that $I_2\longrightarrow 0 \quad as \quad n\longrightarrow\infty$. For this, it is sufficient to prove that

$$I\equiv x^r n^{k+1}M_n^{(r)}\left(\varepsilon\,(t,x)\;(t-x)^{2k+r+2};x\right)\longrightarrow 0\quad as\quad n\longrightarrow\infty.$$

Using Lemma 2.3, we get

$$|I|\leq n^{k+1}\,(n-1)\;M\,(x)\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}n^i\sum_{\upsilon=1}^{\infty}p_{n,\upsilon}\,(x)\;|\upsilon-nx|^j$$

$$\times \int_0^\infty p_{n,\upsilon-1}(t)\ |\varepsilon(t,x)|\ \left|(t-x)^{2k+r+2}\right| dt,$$

where $M(x) = \sup|q_{i,j,r}(x)|$, and then applying the Schwarz inequality we get:

$$|I| \le n^{k+1}(n-1)M(x) \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \left\{\sum_{\upsilon=1}^\infty p_{n,\upsilon}(x)\ (\upsilon - nx)^{2j}\right\}^{1/2}$$

$$\times\left\{\sum_{\upsilon=1}^\infty p_{n,\upsilon}(x)\left(\int_0^\infty p_{n,\upsilon-1}(t)\ |\varepsilon(t,x)|\ \left|(t-x)^{2k+r+2}\right| dt\right)^2\right\}^{1/2}.$$

Since $\varepsilon(t,x)\longrightarrow 0 \quad as \quad t\longrightarrow x$, for a given $\varepsilon>0$ there exists a $\delta>0$ such that $|\varepsilon(t,x)|<\varepsilon$, whenever $0<|t-x|<\delta$, and for $|t-x|\ge\delta$ there exists a constant C such that $|\varepsilon(t,x)|\le C|t-x|^\beta$, where β is an integer $\ge \max(\alpha, 2k+r+2)$.

Hence, as $\int_0^\infty p_{n,\upsilon-1}(t)\,dt = \frac{1}{n-1}$, we have

$$\left(\int_0^\infty p_{n,\upsilon-1}(t)\ |\varepsilon(t,x)|\ \left|(t-x)^{2k+r+2}\right| dt\right)^2 \le$$

$$\le \left(\int_0^\infty p_{n,\upsilon-1}(t)\ dt\right)\left(\int_0^\infty p_{n,\upsilon-1}(t)\ (\varepsilon(t,x))^2\ (t-x)^{4k+2r+4}\, dt\right).$$

$$\le \frac{1}{n-1}\left[\int_{0<|t-x|<\delta} p_{n,\upsilon-1}(t)\ \varepsilon^2\ (t-x)^{4k+2r+4}\, dt\right.$$

$$\left.+\int_{|t-x|\ge\delta} p_{n,\upsilon-1}(t)\ C^2\ (t-x)^{4k+2r+2\beta+4}\ dt\right].$$

Now, by Lemma 2.2, we get

$$\sum_{\upsilon=1}^\infty p_{n,\upsilon}(x)\left(\int_0^\infty p_{n,\upsilon-1}(t)\ |\varepsilon(t,x)|\ \left|(t-x)^{2k+r+2}\right| dt\right)^2 \le$$

$$\le \frac{1}{n-1}\sum_{\upsilon=1}^\infty p_{n,\upsilon}(x)\int_0^\infty p_{n,\upsilon-1}(t)\ \varepsilon^2\ (t-x)^{4k+2r+4}\, dt$$

$$+\frac{C^2}{n-1}\sum_{\upsilon=1}^\infty p_{n,\upsilon}(x)\int_0^\infty p_{n,\upsilon-1}(t)\ (t-x)^{4k+2r+2\beta+4}\, dt.$$

$$\leq \varepsilon^2 \frac{1}{n-1}\left[T_{n,4k+2r+4}(x)-(-x)^{4k+2r+4}(1+x)^{-n}\right]$$
$$+\frac{C^2}{n-1}\left[T_{n,4k+2r+2\beta+4}(x)-(-x)^{4k+2r+2\beta+4}(1+x)^{-n}\right].$$
$$=\varepsilon^2 O\left(n^{-(2k+r+2)}\right)+O\left(n^{-(2k+r+\beta+2)}\right).$$

By Lemma 2.1, we have

$$|I| \leq n^{k+1}M(x)\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i+j}O\left(n^{-j/2}\right)O\left(n^{-(2k+r+2)/2}\right)$$
$$\times\left\{\varepsilon^2+O\left(n^{-\beta}\right)\right\}^{1/2}.$$
$$=O(1)\left\{\varepsilon^2+O\left(n^{-\beta}\right)\right\}^{1/2}$$
$$\leq \varepsilon O(1).$$

Since $\varepsilon > 0$ is arbitrary, it follows that $I \longrightarrow 0$ as $n \longrightarrow \infty$. The assertion (3.2) follows along similar lines by using Lemma 2.4 for $k+1$ in place of k.

The last assertion follows, due to the uniform continuity of $f^{(2k+r+2)}$ on $[a,b] \subset R_+$ (enabling δ to became independent of $x \in [a,b]$) and the uniform of $o(1)$ term in the estimate of I_3 and I_4 (because, in fact, it is a polynomial in x).

The next result provides an estimate of degree approximation in $M_n^{(r)}(f;x) \longrightarrow f^{(r)}(x)$, $r \in N^0$.

Theorem 3.2. *Let $1 \leq p \leq 2k+2$ and $f \in C_\alpha[0,\infty)$ be bounded on every finite subinterval of $[0,\infty)$. Let $f(t) = O(t^\alpha)$ as $\longrightarrow \infty$ for some $\alpha > 0$.If $f^{(p+r)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,\infty)$, $\eta > 0$, then for n sufficiently large*

$$\left\|M_n^{(r)}(f,k,x)-f^{(r)}\right\| \leq \max\left(C_1 n^{-p/2}\omega_{f^{(p+r)}}\left(n^{-1/2}\right), C_2 n^{-(k+1)}\right),$$

where $\omega_{f^{(p+r)}}(\delta)$ denotes the modulus continuity of $f^{(p+r)}$ on $(a-\eta, b+\eta)$, $C_1 = C_1(k,p,r)$, $C_2 = C_2(k,p,r,f)$ and $\|.\|$ denotes the sup-norm on $[a,b]$.

Proof: For $x \in [a,b]$ and $t \in [0,\infty)$, by the hypothesis we have

$$(3.3) f(t)=\sum_{i=0}^{p+r}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{\left(f^{(p+r)}(\xi)-f^{(p+r)}(x)\right)}{(p+r)!}(t-x)^{(p+r)}(1-\chi(t))$$
$$+h(t,x)\,\chi(t),$$

where ξ lies between t and x , and $\chi(t)$ is the characteristic function of the set $[0,\infty)\setminus(a-\eta, b+\eta)$, $\eta > 0$.Operating on this equality by $M_n^{(r)}(.,k,x)$ and breaking the right hand side into three parts I_1 , I_2 and I_3 say, corresponding to the three terms on the right hand side of (3.3) as in the proof of Theorem 3.1, we have

$$I_1 = \sum_{i=0}^{p+r} \frac{f^{(i)}(x)}{i\,!} M_n^{(r)}\left((t-x)^i, k, x\right)$$
$$= f^{(r)}(x) + O\left(n^{-(k+1)}\right), \text{uniformly for all } x \in [a,b].$$

To estimate I_2, we have for every $\delta > 0$

$$\left|f^{(p+r)}(\xi) - f^{(p+r)}(x)\right| \le \omega_{f^{(p+r)}}(|\xi - x|)$$
$$\le \omega_{f^{(p+r)}}(|t-x|)$$
$$\le \left(1 + \frac{|t-x|}{\delta}\right)\omega_{f^{(p+r)}}(\delta). \tag{3.4}$$

Since

$$I_2 = \sum_{j=0}^{k} C(j,k)(d_j n - 1)\sum_{\upsilon=0}^{\infty} p_{d_j n,\upsilon}^{(r)}(x)$$
$$\times \int_0^\infty p_{d_j n,\upsilon-1}(t)\,\frac{\left(f^{(p+r)}(\xi) - f^{(p+r)}(x)\right)}{(p+r)\,!}(t-x)^{(p+r)}(1-\chi(t))\,dt.$$

Using (3.4) and Lemma 2.3, we have

$$|I_2| \le \frac{1}{(p+r)\,!}\sum_{j=0}^{k}|C(j,k)|\sum_{\upsilon=0}^{\infty}\left|p_{d_j n,\upsilon}^{(r)}(x)\right|$$
$$\times \int_0^\infty p_{d_j n,\upsilon-1}(t)\left(1 + \frac{|t-x|}{\delta}\right)|t-x|^{(p+r)}\,\omega_{f^{(p+r)}}(\delta)\,dt$$
$$\le \frac{\omega_{f^{(p+r)}}(\delta)}{(p+r)\,!}\sum_{j=0}^{k}|C(j,k)|\sum_{\substack{2i+s\le r\\ i,s\ge 0}}(d_j n)^i\,\frac{|q_{i,s,r}(x)|}{x^r(1+x)^r}$$
$$\times \sum_{\upsilon=1}^{\infty} p_{d_j n,\upsilon}(x)\,|(\upsilon - d_j n x)|^s \int_0^\infty p_{d_j n,\upsilon-1}(t)\left(|t-x|^{p+r} + \frac{1}{\delta}\,|t-x|^{p+r+1}\right)dt.$$

Putting $K = \sup_{x\in[a,b]}\ \sup_{\substack{2i+s\le r\\ i,s\ge 0}} \frac{|q_{i,s,r}(x)|}{x^r(1+x)^r}$, then applying Schwarz inequality for summation and for integral and Lemmas 2.1 and 2.2 as in the proof of theorem 3.1, we get

$$|I_2| \le K\left[O\left(n^{-p/2}\right) + \frac{1}{\delta}\,O\left(n^{-(p+1)/2}\right)\right]\omega_{f^{(p+r)}}(\delta).$$

Choosing $\delta = n^{-1/2}$, it follows that

$$I_2 = \omega_{f^{(p+r)}}\left(n^{-1/2}\right) O\left(n^{-p/2}\right),$$

where $O-$term holds uniformly in $x \in [a,b]$.

For $x \in [a,b]$ and $t \in [0,\infty) \setminus (a-\eta, b+\eta)$, we can choose a $\delta > 0$ in such a way that $|t-x| \geq \delta$. Hence

$$|I_3| \leq \sum_{j=0}^{k} |C(j,k)| (d_j n - 1) \sum_{\substack{2i+s \leq r \\ i,s \geq 0}} (d_j n)^i \frac{|q_{i,s,r}(x)|}{x^r (1+x)^r}$$

$$\times \sum_{v=1}^{\infty} p_{d_j n, v}(x) |(v - d_j n x)|^s \int_{|t-x| \geq s} p_{d_j n, v-1}(t) |h(t,x)| \, dt.$$

Now, for $|t-x| \geq \delta$ we can find a positive constant M such that $|h(t,x)| \leq M |t-x|^\gamma$, where γ is any integer$\geq \max(\alpha, 2k+r+2)$.

Hence, by Schwarz inequality, Lemmas 2.1 and 2.2 we have

$$|I_3| \leq M \sum_{j=0}^{k} |C(j,k)| (d_j n - 1) \sum_{\substack{2i+s \leq r \\ i,s \geq 0}} (d_j n)^i \frac{|q_{i,s,r}(x)|}{x^r (1+x)^r}$$

$$\times \sum_{v=1}^{\infty} p_{d_j n, v}(x) |(v - d_j n x)|^s \int_{|t-x| \geq s} p_{d_j n, v-1}(t) |t-x|^\gamma \, dt.$$

$$= O\left(n^{(r-\gamma)/2}\right) \quad = O\left(n^{-(k+1)}\right) \text{ uniformly in } x \in [a,b].$$

The required result follows on combining the estimates of I_1, I_2 and I_3.

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Kareem J. Thamer
Department of Mathematics,
College of Education-Amran,
Sana'a University,
Maeen Post Office, Box (13475), Sana'a – Republic of Yemen.
k_alabdullah2005@yahoo.com

May A. Al-Shibeeb
Rayed, P.O.Box (46379),
Post Code 11532, Saudia Arabia King Dom.
maey9999@hotmail.com

A.I. Ibrahem
Department of Mathematics,
College of Science,
Basrah University,
Basrah – Iraq.

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