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ON THE RELATIONSHIP BETWEEN DISJUNCTIVE RELAXATIONS AND MINORS IN PACKING AND COVERING PROBLEMS

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Abstract. In 2002, Aguilera et al. analyzed the performance of the disjunctive lift-and-project operator defined by Balas, Ceria and Cornuéjols on covering and packing polyhedra, in the context of blocking and antiblocking duality. Their results generalize Lovász's Perfect Graph Theorem and a theorem of Lehman on ideal clutters. This study motivated many authors to work on the same ideas, providing alternative proofs and analyzing the behaviour of other lift-and-project operators in the same context.

In this paper, we give a survey of the results in the subject and add some new results, showing that the key of the behaviour of the disjunctive operator on these particular classes of polyhedra is the strong relationship between disjunctive relaxations and original relaxations associated to some minors.

1. INTRODUCTION

Many problems in Combinatorial Optimization can be formulated as $0-1$ linear programs, where the set S of feasible solutions may be seen as the set of integral solutions in a polyhedron K, i.e. $S = K \cap \mathbb{Z}^n$.
In spite of optimizing a linear function over S is equivalent to do it over

$$K^* := \text{conv}(S) = \text{conv}(K \cap \mathbb{Z}^n),$$

in the general case, the complete description of K^* by linear inequalities is not known. Moreover, in most of the cases, even though a partial description is found, an exponential number of inequalities is involved.
A polyhedron $K \subset \mathbb{R}^n$ will be called a *relaxation* of $S \subset \mathbb{R}^n$ if $S = K \cap \mathbb{Z}^n$. Given two different relaxations K and K' of S such that $S \subset K' \subset K$, the bound obtained by optimizing over K' is tighter than the one obtained by optimizing over K and we say that the relaxation K is *weaker* than K'.
This fact motivates the definition of operators such that, applied on a relaxation K of S, in each step they obtain a new relaxation $K' \subset K$, arriving to conv(S) in a finite number of iterations.
This is the case of the N and N_+ lift-and-project operators defined by Lovász and Schrijver [12] and the disjunctive operator (BCC) defined by Balas, Ceria and Cornuéjols [4].

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In [1] and [2] the authors analyze the performance of the BCC operator on covering and packing polyhedra in the context of blocking and antiblocking duality, respectively. Their results generalize Lovász's Perfect Graph Theorem and its analogous on ideal clutters, due to Lehman. These results motivated Gerards et al. [8] and more recently, Lipták and Tunçel [11], to give alternative proofs in the case of the generalization of Lovász's Perfect Graph Theorem. In the same way, Leoni and Nasini [10] exposed alternative and simpler proofs for the case of the generalization of Lehman's theorem. Lipták and Tunçel [11] started the analysis of the behaviour of the N and N_+ operators in the same context and, recently, Escalante et al. [6] completed this analysis, proving that similar generalizations do not exist for these operators.

The aim of this paper is to show that the key of the behaviour of the disjunctive operator observed in [1] and [2] is the strong relationship between disjunctive relaxations and original relaxations associated to some particular minors. This relationship becomes clear from the characterization of the extreme points of these relaxations.
This idea was initiated by Nasini [13], working in the particular case of the clique relaxation $QSTAB(G)$ of the stable set polytope $STAB(G)$ in a graph G. Independently, Lipták and Tunçel [11] obtained the same result.
In Section 2, we provide the fundamental definitions that will be treated in the rest of the paper.
In Section 3, we summarize the results on the extreme points of the disjunctive relaxations for set covering and set packing polyhedra. Besides, we show that the results on $QSTAB(G)$ can be extended to more general relaxations of $STAB(G)$.
In Section 4, we summarize the results on the extreme points of the blocker of the disjunctive relaxations for set covering polyhedra. Looking for similar relationships in the context of set packing polyhedra, we show that the antiblocker of the disjunctive relaxations of $STAB(G)$ obtained from $QSTAB(G)$ is also strongly related to the antiblocker of the clique relaxation associated to some particular subgraphs of the graph G.
Finally, in Section 5 we summarize the results obtained in this work and present the conclusions.

2. DEFINITIONS AND PRELIMINARIES

The disjunctive operator defined by Balas, Ceria and Cornuéjols [4] can be applied on polyhedra $K \subset [0,1]^n$. After a lift-and-project iteration of the operator on $j \in \{1,\ldots,n\}$, they obtain

$$P_j(K) = \text{conv}\big(\{x \in K : x_j \in \{0,1\}\}\big).$$

Applying iteratively the operator on a subset of indices $J = \{i_1,..,i_j\} \subset \{1,\ldots,n\}$, they proved that

$$P_{i_1}(P_{i_2}(...(P_{i_j}(K)))) = \text{conv}\big(\{x \in K : x_j \in \{0,1\} \text{ for all } j \in J\}\big) := P_J(K).$$

For any subset $J \subset \{1,\ldots,n\}$, $P_J(K)$ will be called a *disjunctive relaxation* of $K^* := \text{conv}\big(K \cap \mathbb{Z}^n\big)$.

Clearly, $P_{\{1,\ldots,n\}}(K) = K^*$. This last property allows us to talk about the minimum number of iterations needed to find K^*, which is called the *disjunctive index* of K. It is also clear that $K^* = K$ if and only if the disjunctive index of K is equal to zero.

Given a graph G with $N = \{1, \ldots, n\}$ as set of nodes, let us recall that $Q \subset N$ is a *clique* in G if in every pair in Q, its nodes are adjacent. Moreover, $S \subset N$ is a *stable set* in G if in no pair of S, its nodes are adjacent. Clearly, a clique in a graph G is a stable set in its complementary graph $\bar{G}$.
The *stable set polytope* $STAB(G)$ of G is the convex hull of the incidence vector of its stable sets. The *clique relaxation* $QSTAB(G)$ of $STAB(G)$ is defined as

$$QSTAB(G) = \{x \in \mathbb{R}^n_+ : \sum_{i \in Q} x_i \leq 1 \text{ for all } Q \text{ clique in } G\}.$$

In [1], the authors study the behaviour of the disjunctive operator on polyhedra $K = QSTAB(G)$.
Following Lovász's Perfect Graph Theorem in [12], if a graph G is perfect then its *complementary graph* $\overline{G}$ is also perfect. From the polyhedral characterization of perfect graphs given by Chvátal [5], i.e G is perfect if and only if $QSTAB(G) = STAB(G)$, we have that the disjunctive index of $QSTAB(G)$ is zero if and only if the disjunctive index of $QSTAB(\overline{G})$ is zero. Moreover, it is known that if G is minimally non perfect (*mnp*), then $\overline{G}$ is also *mnp* and $QSTAB(G)$ has only one fractional extreme point. It holds that if a graph G is *mnp*, the disjunctive indices of $QSTAB(G)$ and $QSTAB(\overline{G})$ are both equal to one.
These well-known results have been generalized in [1] proving that

Theorem 1.
For any graph G, the disjunctive indices of $QSTAB(G)$ and $QSTAB(\overline{G})$ coincide.

From this result, the disjunctive index of $QSTAB(G)$ may be considered as a measure of the "imperfection" of G. In this sense, Theorem 1 shows that any graph is as "imperfect" as its complementary graph, generalizing Lovász's Perfect Graph Theorem. This result is proved in [1] as a consequence of the behaviour of the disjunctive relaxation on the context of antiblocker duality.
If K^C denotes the *antiblocker* of a polyhedron K defined by

$$K^C = \{\pi : \pi x \leq 1 \text{ for all } x \in K\},$$

it is well known that

$$[QSTAB(G)]^C = STAB(\overline{G}).$$

The proof of Theorem 1 in [1] is a direct corollary of the following stronger result.

Theorem 2.
For any $J \subset \{1, \ldots, n\}$, $P_J\left([P_J(QSTAB(G))]^C\right) = [QSTAB(G)]^C = STAB(\overline{G})$.

Similar results have been obtained in [2], working on set covering polyhedra.
A *clutter* $\mathcal{C}$ is a set of non-comparable subsets —called edges— of a set $N = \{1, \ldots, n\}$, called the vertex set.

Given a clutter $\mathcal{C}$ over N, the *blocker* of $\mathcal{C}$, $b(\mathcal{C})$, is the clutter of minimal *vertex covers* of $\mathcal{C}$, i.e. minimal subsets R of N satisfying

$$|R \cap S| \geq 1, \text{ for all edge } S \text{ of } \mathcal{C}.$$

We will denote by $M(\mathcal{C})$ the $0-1$ matrix (with entries 0 or 1) whose rows are the characteristic vectors of the edges of $\mathcal{C}$. Clearly, $M(\mathcal{C})$ has no dominating rows. Given a clutter $\mathcal{C}$, $x \in \{0,1\}^n$ is the incidence vector of a vertex cover of $\mathcal{C}$ if and only if $M(\mathcal{C})x \geq 1$. The *set covering polyhedron* associated to $\mathcal{C}$ is defined as

$$Q^*(\mathcal{C}) = \text{conv}(Q(\mathcal{C}) \cap \mathbb{Z}^n),$$

where $Q(\mathcal{C}) = \{x \in \mathbb{R}^n_+ : M(\mathcal{C})x \geq 1\}$ is called the original relaxation of $Q^*(\mathcal{C})$. In this case, considering the *blocker* of a polyhedron Q defined as

$$Q^B = \{\pi : \pi x \geq 1 \text{ for all } x \in Q\},$$

and the blocker $b(\mathcal{C})$ of a clutter $\mathcal{C}$, it is well-known [7] that

$$Q^*(\mathcal{C}) = [Q(b(\mathcal{C}))]^B.$$

In order to analyze the "disjunctive behaviour" over blocking polyhedra, the authors in [2] had to define an extension $\bar{P}_j$ of the disjunctive operator, as

$$\bar{P}_j(Q(\mathcal{C})) := P_j(Q_0(\mathcal{C})) + \mathbb{R}^n_+,$$

where $Q_0(\mathcal{C}) := Q(\mathcal{C}) \cap [0,1]^n$. They showed that $\bar{P}_j$ preserves the main properties of P_j. In particular, for any $J \subset \{1, \ldots, n\}$,

$$\bar{P}_J(Q(\mathcal{C})) = P_J(Q_0(\mathcal{C})) + \mathbb{R}^n_+.$$

The disjunctive index is also well defined, since

$$\bar{P}_{\{1,\ldots,n\}}(Q(\mathcal{C})) = P_{\{1,\ldots,n\}}(Q_0(\mathcal{C})) + \mathbb{R}^n_+ = Q^*_0(\mathcal{C}) + \mathbb{R}^n_+ = Q^*(\mathcal{C}).$$

Ideal clutters, defined by Lehman, are those for which $Q(\mathcal{C})$ is integral, or equivalently, those for which the disjunctive index is zero.
Lehman's theorem on ideal clutters [9] establishes that if a clutter is ideal then so is its blocker. In other words, Lehman's theorem says that the disjunctive index of $Q(\mathcal{C})$ is zero if and only if the disjunctive index of $Q(b(\mathcal{C}))$ is zero. Moreover, it is known that if $\mathcal{C}$ is minimally non ideal (*mni*), then $b(\mathcal{C})$ is also *mni* and $Q(\mathcal{C})$ has only one fractional extreme point. It holds that if a clutter $\mathcal{C}$ is *mni*, the disjunctive indices of $Q(\mathcal{C})$ and $Q(b(\mathcal{C}))$ are both equal to 1.
Lehman's theorem is generalized in [2] in the following sense:

Theorem 3. *For any clutter $\mathcal{C}$, the disjunctive indices of $Q(\mathcal{C})$ and $Q(b(\mathcal{C}))$ coincide.*

Once again, the disjunctive index of $Q(\mathcal{C})$ may be considered as a measure of the "non-idealness" of $\mathcal{C}$, and Theorem 3 shows that any clutter is as "non-ideal" as its blocker, generalizing Lehman's theorem.

The proof of Theorem 3 in [2] is a direct corollary of the following stronger result (the analogous of Theorem 2) in the context of blocker duality.

Theorem 4. *For any* $J \subset \{1,\ldots,n\}$, $\bar{P}_J(\left[\bar{P}_J(Q(\mathcal{C}))\right]^B) = [Q(\mathcal{C})]^B = Q^*(b(\mathcal{C}))$.

Proofs of Theorem 2 and Theorem 4 given respectively in [1] and [2] are based on the characterization of valid inequalities of the disjunctive relaxations. In those proofs and also in the alternative ones given in [8], the relationship between the disjunctives relaxations and some particular minors is not noticed.

3. DISJUNCTIVE RELAXATIONS, EXTREME POINTS AND MINORS

Let $K \subset [0,1]^n$ be a polyhedron and consider for $S \subset N$,

$$\mathcal{H}_S := \left\{x \in \mathbb{R}^n_+ : x_i = 0 \text{ for } i \in S\right\}$$

and

$$\mathcal{G}_S := \left\{x \in \mathbb{R}^n_+ : x_i = 1 \text{ for } i \in S\right\}.$$

If J is a fixed subset of N, $S \subset J$ and $\overline{S} = J \backslash S$, let us define

$$K_S := K \cap \mathcal{G}_S \cap \mathcal{H}_{\overline{S}}.$$

It is not hard to see [1] that

$$P_J(K) = \text{conv}(\bigcup_{S \subset J} K_S)$$

and, if $V(K)$ is the set of extreme points of K,

$$V(P_J(K)) = \bigcup_{S \subset J} V(K_S).$$

Given a graph $G = (N, E)$ with $N = \{1,\ldots,n\}$ and considering $K = QSTAB(G)$, it is clear that K_S is not empty if and only if S is a stable set in G. Moreover, denoting by $\Gamma(S) = \{i \in N : \ i \text{ is adjacent to some } j \in S\}$, if $x \in K_S$, $x_i = 0$ for all $i \in \Gamma(S)$. Therefore, calling $\widehat{S} = \overline{S} \cup \Gamma(S)$, we have

$$K_S = QSTAB(G) \cap \mathcal{G}_S \cap \mathcal{H}_{\widehat{S}}.$$

Given $S \subset J$, we write any $x \in \mathbb{R}^n$ as $x = (\overline{x}, \widetilde{x})$, with $\overline{x} \in \mathbb{R}^{|J \cup \widehat{S}|}$. Therefore, denoting by $G \backslash S$ (respectively $G \ominus S$), the subgraph obtained from G by the deletion (destruction) of nodes in S, the following lemma holds.

Lemma 5. *Given* $J \subset N$, $S \subset J$ *and* $x = (\overline{x}, \widetilde{x}) \in \mathbb{R}^n$, x *is an extreme point of* K_S *if and only if* $x \in \mathcal{G}_S \cap \mathcal{H}_{\widehat{S}}$ *and* $\widetilde{x}$ *is an extreme point of* $QSTAB((G \backslash \overline{S}) \ominus S)$.

This lemma is the key of the proof in [13] of Theorem 6 below.

Theorem 6. [13] *For any* $J \subset N$, $P_J(QSTAB(G)) = STAB(G)$ *if and only if* $G \backslash J$ *is perfect.*

The generalization of Lovász's Perfect Graph Theorem given by Theorem 1 is a direct consequence of the previous theorem.

The "symmetry" of Theorem 1 and Theorem 3 can be also expressed by similar relationships between the extreme points of the disjunctive relaxations of the set covering polyhedron associated to a clutter $\mathcal{C}$ and those associated to some minors of $\mathcal{C}$.
Given a clutter $\mathcal{C}$ over $N = \{1, \ldots, n\}$ and $j \in N$, $\mathcal{C}/j$ denotes the clutter obtained from $\mathcal{C}$ by the *contraction* of j, i.e. the clutter defined over $N - \{j\}$, whose edges are the minimal elements of $\{S \setminus \{j\} : S \text{ edge of } \mathcal{C}\}$. Also, $\mathcal{C}\backslash j$ denotes the clutter obtained from $\mathcal{C}$ by the *deletion* of j, i.e. the clutter defined over $N - \{j\}$, whose edges are the edges of $\mathcal{C}$ not containing j. Given two disjoints subsets R, S of N, $\mathcal{C}/R\backslash S$ denotes the *minor* of $\mathcal{C}$ obtained by the contraction of nodes in R and deletion of nodes in S. It is known that $b(\mathcal{C}/R\backslash S) = b(\mathcal{C})/S\backslash R$.
In this way, in [10] it is proved that

Theorem 7. [10] *For any $J \subset N$, $\bar{P}_J(Q(\mathcal{C})) = Q^*(\mathcal{C})$ if and only if $\mathcal{C}/\overline{R}\backslash R$ is ideal for every $R \subset J$.*

Let us notice that, meanwhile the integrality of $P_J\,(QSTAB(G))$ involves the perfection of only the subgraph induced by $N\backslash J$ (see Theorem 6), the "symmetric" result for set covering polyhedra involves the idealness of many minors. In other words, in this case there is not a particular minor of $\mathcal{C}$ which guarantees the integrality of $\bar{P}_J(Q(\mathcal{C}))$. This difference should be found in the characterization of the extreme points of the disjunctive relaxations of $Q(\mathcal{C})$. We present here an scheme of the proof of Theorem 7 in [10].
In the following, J will be a fixed subset of N. For $R \subset J$, with $\overline{R} = J\backslash R$, let us consider the polyhedron

$$Q_R := Q(\mathcal{C}) \cap \mathcal{H}_{\overline{R}} \cap \overline{\mathcal{G}}_R,$$

where

$$\overline{\mathcal{G}}_R := \left\{x \in \mathbb{R}^n_+ : x_i \geq 1 \text{ for } i \in R\right\}.$$

It can be easily seen that

$$\bar{P}_J(Q(\mathcal{C})) = \text{conv}(\bigcup_{R \subset J} Q_R).$$

Then, every extreme point of $\bar{P}_J(Q(\mathcal{C}))$ is an extreme point of Q_R, for some $R \subset J$. Moreover, writing any $x \in \mathbb{R}^n$ as $x = (\overline{x}, \widetilde{x})$, where $\overline{x} \in \mathbb{R}^{|J|}$ and denoting by $x^R \in \{0,1\}^{|J|}$ the characteristic vector of $R \subset J$, we have

Lemma 8. *Given $x = (\overline{x}, \widetilde{x}) \in \mathbb{R}^n$ and $R \subset J$, x is an extreme point of Q_R if and only if $\overline{x} = x^R$ and $\widetilde{x}$ is an extreme point of $Q(\mathcal{C}/\overline{R}\backslash R)$.*

The difference with the "stable set case" appears noting that it is not true that for any $R \subset J$, every extreme point of Q_R is an extreme point of $\bar{P}_J(Q(\mathcal{C}))$. However, it can be proved that

Lemma 9. [10] *Let* $x = (x^T, \widetilde{x})$ *be an extreme point of* Q_T*, for some* $T \subset J$ *and* S *a minimal subset of* T *such that* $w = (x^S, \widetilde{x}) \in Q(\mathcal{C})$*. Then,* w *is an extreme point of* Q_S *and also, an extreme point of* $\bar{P}_J(Q(\mathcal{C}))$*.*

Now, we derive Theorem 7, the "symmetric" of Theorem 6:

Proof. (of Theorem 7)
Suppose that for some $R \subset J$, $\mathcal{C}/\overline{R}\backslash R$ is not ideal, i.e. that $Q(\mathcal{C}/\overline{R}\backslash R)$ has some fractional extreme point $\widetilde{x}$. Then, $(x^R, \widetilde{x})$ is a fractional extreme point of Q_R. From Lemma 9, there exists $S \subset R$ such that $(x^S, \widetilde{x})$ is a fractional extreme point of $\bar{P}_J(Q(\mathcal{C}))$, and then $\bar{P}_J(Q(\mathcal{C}))$ is not integral. Therefore, $\bar{P}_J(Q(\mathcal{C})) \neq Q^*(\mathcal{C})$.
To see the converse, if $\mathcal{C}/\overline{R}\backslash R$ is ideal for every $R \subset J$, from Lemma 8 follows that Q_R is an integral polyhedron for every $R \subset J$. Therefore, $\bar{P}_J(Q(\mathcal{C}))$ is also integral. □

Now we come back to Theorem 6. Let us observe that, in other words, this theorem states that for any graph G, the disjunctive index of $QSTAB(G)$ is the minimum number of nodes that it is necessary to delete in G in order to obtain a perfect graph.
A similar result have been found in [3], working on the matching polytope in a graph. Let us recall that a *matching* in a graph $G = (N, E)$ is a subset of edges where any two of them are incident to a same node of G. The matching polytope $MATCH(G)$ is defined as the convex hull of the incidence vector of the matchings in G. The natural original relaxation of $MATCH(G)$ is

$$K_{\leq}(G) = \left\{ x \in \mathbb{R}_+^{|E|} : \sum_{j:(i,j)\in E} x_{ij} \leq 1 \text{ for all } i \in N \right\}.$$

It is known that $K_{\leq}(G) = MATCH(G)$ if and only if G is bipartite. In [3], it is proved that the disjunctive index of $K_{\leq}(G)$ is the minimum number of edges that must be taken off from E in order to obtain a bipartite graph.
This fact does not seem to be surprising if we recall that the matching polytope of a graph G is exactly the stable set polytope of the line graph $L(G)$ of G. However, it cannot be seen as a particular case of Theorem 6 because the relaxation $K_{\leq}(G)$ of $STAB(L(G))$ is, in general, weaker than $QSTAB(L(G))$.
In a more general context, in a *set packing problem* we are given a clutter $\mathcal{C}$ over a set N and optimize over $K^*(\mathcal{C})$, the convex hull of $0-1$ vectors in $K(\mathcal{C}) = \left\{ x \in \mathbb{R}_+^{|N|} : M(\mathcal{C})x \leq 1 \right\}$.
If we consider the *associated graph* $G(\mathcal{C})$ of $\mathcal{C}$ defined by a set of nodes equal to N, the vertex set of $\mathcal{C}$, and where two nodes in $G(\mathcal{C})$ are adjacent if there exists an edge of $\mathcal{C}$ that contains both of them, it is not hard to see that $K^*(\mathcal{C}) = STAB(G(\mathcal{C}))$ and $K(\mathcal{C})$ is a relaxation, weaker than $QSTAB(G(\mathcal{C}))$.
When $K(\mathcal{C}) = QSTAB(G(\mathcal{C}))$ we say that the clutter is *conformal*. For this family of clutters we have the results on the disjunctive relaxations presented in Section 2.

The rest of this section is devoted to generalize Theorem 6 for other families of clutters. We are looking for families of clutters $\mathbf{C}$ where, for any $\mathcal{C}$ in $\mathbf{C}$, the disjunctive index of the relaxation $K(\mathcal{C})$ coincides with the minimum number of nodes we must delete from $G(\mathcal{C})$ in order to obtain a graph G', such that $G' = G(\mathcal{C}')$ for some $\mathcal{C}' \in \mathbf{C}$ and $K(\mathcal{C}') = STAB(G')$.
Given a family $\mathbf{C}$ of clutters and a graph G, let us say that G is a ***C**-graph* if $G = G(\mathcal{C})$ for some $\mathcal{C} \in \mathbf{C}$.
We need a consistent definition for the family we are looking for. For this purpose, we have to impose some conditions.

Definition 10. *A family of clutters* $\mathbf{C}$ *is hereditary if it satisfies the following conditions:*

1. *if* $\mathcal{C}$ *and* $\mathcal{C}'$ *are in* $\mathbf{C}$ *and* $G(\mathcal{C}) = G(\mathcal{C}')$*, then* $\mathcal{C} = \mathcal{C}'$*;*
2. *if* $\mathcal{C} \in \mathbf{C}$ *and* i *is a vertex of* $\mathcal{C}$*, then* $\mathcal{C} \setminus i$ *is in* $\mathbf{C}$*.*

Then, we have

Definition 11. *Given an hereditary family of clutters* $\mathbf{C}$ *and* G *a* ***C**-graph such that* $G = G(\mathcal{C})$ *with* $\mathcal{C} \in \mathbf{C}$*,* G *is* ***C**-perfect if and only* $K(\mathcal{C}) = STAB(G(\mathcal{C}))$*.*

Condition 1 in Definition 10 implies that **C**-perfection in Definition 11 is well defined. Besides, it is not hard to see that $G(\mathcal{C} \backslash i) = G(\mathcal{C}) \backslash \{i\}$. Then, Condition 2 implies that any node induced subgraph of a **C**-graph is a **C**-graph, and moreover, any node induce subgraph of a **C**-perfect graph is **C**-perfect.
According to Definition 11, when $\mathbf{C}$ is the family of conformal clutters, a **C**-perfect graph is a perfect graph. Also, when $\mathbf{C}$ is the family of clutters with at most two vertices in each edge, for any clutter $\mathcal{C}$ in $\mathbf{C}$ we have,

$$K(\mathcal{C}) = FRAC(G(\mathcal{C})) := \{x \in \{0,1\}^{|N|} : x_i + x_j \leq 1, \text{ for all } (i,j) \text{ edge in } G(\mathcal{C})\},$$

and it is known that **C**-perfect graphs are bipartite graphs.
This definition also includes $K_{\leq}(G)$ as a relaxation of $MATCH(G) = STAB(L(G))$. In this case, the elements of $\mathbf{C}$ have vertex set equal to the set of edges of a graph $G = (N, E)$, and the edges are the subsets of E incident in G to a node in N. Besides, $K(\mathcal{C}) = K_{\leq}(G)$ and $G(\mathcal{C}) = L(G)$. It is known that $L(G)$ is **C**-perfect if and only if G is bipartite.
Now, the generalization of Theorem 6 will be formulated as

Theorem 12. *If* $\mathcal{C}$ *belongs to an hereditary family of clutters* $\mathbf{C}$ *and* $G = G(\mathcal{C}) = (N, E)$*, for any* $J \subset N$*,* $P_J(K(\mathcal{C})) = STAB(G)$ *if and only if* $G \backslash J$ *is* ***C**-perfect.*

To prove Theorem 12, it only remains to recall that

$$V(P_J(K(\mathcal{C}))) = \bigcup_{S \subset J} V(K_S),$$

where $K_S = K(\mathcal{C}) \cap \mathcal{G}_S \cap \mathcal{H}_{\overline{S}}$ and notice that again, K_S is not empty if and only if S is a stable set in $G(\mathcal{C})$. Moreover, if $x \in K_S$, $x_i = 0$ for all $i \in \Gamma(S)$ in $G(\mathcal{C})$. Therefore, calling $\widehat{S} = \overline{S} \cup \Gamma(S)$, we have

$$K_S = K(\mathcal{C}) \cap \mathcal{G}_S \cap \mathcal{H}_{\widehat{S}}.$$

It is not difficult to see that $x = (\overline{x}, \widetilde{x}) \in \mathbb{R}^n$ with $\overline{x} \in \mathbb{R}^{|J \cup \widehat{S}|}$ is an extreme point of K_S if $x \in \mathcal{G}_S \cap \mathcal{H}_{\widehat{S}}$ and $\widetilde{x}$ is an extreme point of $K(\mathcal{C}\backslash(S \cup \widehat{S}))$. Moreover, $G(\mathcal{C}\backslash(S \cup \widehat{S})) = G(\mathcal{C})\backslash(S \cup \widehat{S}) = (G(\mathcal{C})\backslash\overline{S}) \ominus S$.

Proof. (of Theorem 12)
Clearly, $P_J(K(\mathcal{C})) = STAB(G(\mathcal{C}))$ if and only $P_J(K(\mathcal{C}))$ is an integral polyhedron. Equivalently, from the previous observations, $K(\mathcal{C}\backslash(S \cup \widehat{S})$ has to be integral, for any $S \subset J$. Equivalently, $K(\mathcal{C}\backslash(S \cup \widehat{S})) = STAB(G(\mathcal{C})\backslash(S \cup \widehat{S}))$. Finally, $P_J(QSTAB(G(\mathcal{C}))) = STAB(G(\mathcal{C}))$ if and only if $(G(\mathcal{C})\backslash\overline{S}) \ominus S$ is **C**-perfect for all $S \subset J$. Since $(G(\mathcal{C})\backslash\overline{S}) \ominus S$ is a node induced subgraph of $G(\mathcal{C})\backslash J$, the result follows. □

Now, for any graph G, the results for $QSTAB(G)$ and $K_{\leq}(G)$ may be seen as particular cases of the previous theorem. We also have that the disjunctive index of $FRAC(G)$ is the minimum number of nodes we must delete from G in order to obtain a bipartite graph.

4. DISJUNCTIVE RELAXATIONS IN THE CONTEXT OF POLYHEDRAL DUALITY

The relationship between extreme points and minors has been also found in the context of "polyhedral duality" for set covering polyhedra [10]. The extreme points of the blocker of a disjunctive relaxation are characterized in the following theorem:

Theorem 13. [10] *Let $S \subset J$ and $\alpha = (\alpha^S, \widetilde{\alpha})$. Then, $\alpha \in \left[\bar{P}_J(Q(\mathcal{C}))\right]^B$ if and only if $\widetilde{\alpha} \in \left[Q(\mathcal{C}/S\backslash\overline{S})\right]^B$. Moreover, α is an extreme point of $\left[\bar{P}_J(Q(\mathcal{C}))\right]^B$ if and only if $\widetilde{\alpha}$ is an extreme point of $\left[Q(\mathcal{C}/S\backslash\overline{S})\right]^B$.*

Theorem 4 easily follows from the previous result.

In this section, we will find similar relationships between the antiblocker of a disjunctive relaxation of $STAB(G)$ and some particular subgraphs of a graph G.

In the following, J will be denote a fixed subset of N. Let $\alpha = (\overline{\alpha}, \widetilde{\alpha}) \in \mathbb{R}^n$ with $\overline{\alpha} \in \mathbb{R}^{|J|}$ and $\alpha \in [STAB(G)]^C \cap \mathcal{G}_S \cap \mathcal{H}_{\overline{S}}$ for some $S \subset J$.
From the observations made by Gerards, Maróti and Schrijver in [8], we know that if $\alpha_i > 0$ and $j \in S$ $(j \neq i)$, then (i, j) is an edge of G. In particular, since $\alpha_i = 1$ for all $i \in S$; S is a clique in G. Moreover, if $i \notin J$ and $\alpha_i > 0$, then i is a node of $G \bigtriangledown S := \overline{\overline{G} \ominus S}$.
Now, let α be an extreme point of $[P_J(QSTAB(G))]^C \cap \mathcal{G}_S \cap \mathcal{H}_{\overline{S}}$, for some $S \subset J$ and $J \subset N$. Since $[P_J(QSTAB(G))]^C \subset [STAB(G)]^C$, $\alpha \in [STAB(G)]^C \cap \mathcal{G}_S \cap \mathcal{H}_{\overline{S}}$. Then, we have $\widetilde{\alpha} \in [QSTAB(G\backslash J)]^C$ and $\widetilde{\alpha}_i = 0$, when i is not a node of $G \bigtriangledown S$. In the following theorem, we prove that the converse also holds, i.e.

Theorem 14. *Let $\alpha = (\overline{\alpha}, \widetilde{\alpha}) \in [STAB(G)]^C \cap \mathcal{G}_S \cap \mathcal{H}_{\overline{S}}$. If $\widetilde{\alpha} \in [QSTAB(G\backslash J)]^C$ and $\widetilde{\alpha}_i = 0$ when i is not a node of $G \bigtriangledown S$, then $\alpha \in [P_J(QSTAB(G))]^C$. Moreover,* *if α is an extreme point of $[P_J(QSTAB(G))]^C$ then $\widetilde{\alpha}$ is an extreme point of $[QSTAB(G\backslash J)]^C$.*

Proof. Let $x = (\overline{x}, \widetilde{x})$ be an extreme point of $P_J(QSTAB(G))$ with $\overline{x} \in \{0,1\}^{|J|}$ and $\widetilde{x} \in QSTAB(G\backslash J)$. Let us define $R = \{i \in J : \overline{x}_i = 1\}$. We need to show that $\alpha x \leq 1$. Since $\alpha x = |R \cap S| + \widetilde{\alpha}\widetilde{x}$, clearly $\alpha x \leq 1$ if $|R \cap S| = 0$. If $|R \cap S| \neq 0$, then $|R \cap S| = 1$ and it only remains to prove that $\widetilde{\alpha}\widetilde{x} = 0$.
Let $R \cap S = \{k\}$ and $i \in N\backslash J$ with $\widetilde{\alpha}_i > 0$. Since (i,k) is an edge of G, $\widetilde{x}_i + \overline{x}_k \leq 1$, obtaining that $\widetilde{x}_i = 0$.
For the second part, let us assume that $\widetilde{\alpha}$ is a convex combination of two points $\widetilde{\alpha}^1$ and $\widetilde{\alpha}^2$ in $[QSTAB(G\backslash J)]^C$. Defining $\alpha^1 = (\overline{\alpha}, \widetilde{\alpha}^1)$ and $\alpha^2 = (\overline{\alpha}, \widetilde{\alpha}^2)$, from the previous results we have $\alpha^i \in [P_J(QSTAB(G))]^C$ for $i = 1, 2$ and α is a convex combination of α^1 and α^2. □

¿From this characterization follows an easy new alternative proof for Theorem 2.

Proof. (of Theorem 2)
Since $STAB(\overline{G}) \subset P_J([P_J(QSTAB(G))]^C)$, it suffices to prove that the polyhedron $P_J([P_J(QSTAB(G))]^C)$ is integral. If $\alpha = (\overline{\alpha}, \widetilde{\alpha})$ is any extreme point of it, then α is an extreme point of $[P_J(QSTAB(G)]^C$. Moreover, $\alpha \in [STAB(G)]^C \cap \mathcal{G}_S \cap \mathcal{H}_{\overline{S}}$, for some $S \subset J$. From Theorem 14, we have that $\widetilde{\alpha}$ is an extreme point of $[QSTAB(G\backslash J)]^C = STAB(\overline{G\backslash J})$ and then, $\widetilde{\alpha}$ is integral. □

5. SUMMARY AND CONCLUSIONS

In order to make the symmetry of the results between the "packing" and "covering" cases clearer and to completely understand the relationship between the disjunctive relaxations and the original problem associated to some minors, let us summarize the results on the extreme points of the disjunctive relaxations and those of their "dual polyhedra".
Given a clutter $\mathcal{C}$ over $N = \{1, \ldots, n\}$ and $J \subset N$, let

$$K(\mathcal{C}) = \left\{x \in \mathbb{R}_+^{|N|} : M(\mathcal{C})x \leq 1\right\} \text{ and}$$

$$Q(\mathcal{C}) = \left\{x \in \mathbb{R}_+^{|N|} : M(\mathcal{C})x \geq 1\right\}.$$

First, for the disjunctive relaxations, we have:

1. $x = (\overline{x}, \widetilde{x}) \in \mathbb{R}^n$ with $\overline{x} \in \mathbb{R}^{|J \cup \widehat{S}|}$ is an extreme point of $P_J(K(\mathcal{C}))$ if and only if there exists $S \subset J$ such that $x \in \mathcal{G}_S \cap \mathcal{H}_{\widehat{S}}$ and $\widetilde{x}$ is an extreme point of $K(\mathcal{C}')$ with $G(\mathcal{C}') = (G(\mathcal{C})\backslash\overline{S}) \ominus S$;
2. $x = (\overline{x}, \widetilde{x}) \in \mathbb{R}^n$ with $\overline{x} \in \mathbb{R}^{|J|}$ is an extreme point of $P_J(Q(\mathcal{C}))$ if and only if there exists $S \subset J$ such that $\overline{x} = x^S$ and $\widetilde{x}$ is an extreme point of $Q(C/\overline{S}\backslash S)$.

For the "dual polyhedra" of the disjunctive relaxations, let $\alpha = (\overline{\alpha}, \widetilde{\alpha}) \in \mathbb{R}^n$ with $\overline{\alpha} \in \mathbb{R}^{|J|}$. We have:

1. When $\mathcal{C}$ is the clutter of maximal cliques in a graph G:
   α is an extreme point of $[P_J(QSTAB(G))]^C$ if and only if there exists $S \subset J$ such that $\overline{\alpha} \in \mathcal{G}_S \cap \mathcal{H}_{\overline{S}}$ and $\widetilde{\alpha}$ is an extreme point of $[QSTAB(G\backslash J)]^C$;

2. α is an extreme point of $\left[\bar{P}_J(Q(\mathcal{C}))\right]^B$ if and only if there exists $S \subset J$ such that $\overline{\alpha} = \alpha^S$ and $\widetilde{\alpha}$ is an extreme point of $\left[Q(\mathcal{C}/S\backslash\overline{S})\right]^B$.

We believe that the aim of the paper is achieved: the results obtained reveal that the way the disjunctive operator behaves is due to the fact that the extreme points of a disjunctive relaxation and those of its "dual polyhedron" are strongly related to the same combinatorial problem on a minor of the original clutter. From these results we can say that the BCC operator "preserves" the original combinatorial structure of the problem. The "negative" results obtained analyzing the behaviour of other lift-and-project operators in the same context ([11] and [6]) are due to the fact that the N and N_+ operators do not share this "combinatorial" property of the BCC operator.

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