REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Volumen 46, Número 1, 2005, Páginas 23–29

THE BERGMAN KERNEL ON TUBE DOMAINS

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ABSTRACT. Let Ω be a bounded strictly convex domain in $\mathbf{R}^n$, and $T_\Omega \subset \mathbf{C}^n$ the tube domain over Ω. In this paper, we show that the Bergman kernel of T_Ω can be expressed easily by an integral formula.

1. Introduction

Let $T \subset \mathbf{C}^n$ be a domain. The Bergman kernel (see for instance [4]) $K : T \times T \longrightarrow \mathbf{C}$ is one of the important holomorphic invariants associated to T, but it is often difficult to compute K. We shall show that if T is a tube domain over a bounded strictly convex domain, then an easy computation leads quickly to its Bergman kernel.

Let $\Omega \subset \mathbf{R}^n$ be a bounded strictly convex domain. We are interested in domains $T = T_\Omega$ of the type

$$T_\Omega = \{x + iy\,;\ \ y \in \Omega\} \subset \mathbf{C}^n, \tag{1.1}$$

known as the tube domain over Ω. Let $K(z,w)$ be the Bergman kernel of T_Ω. The main result of this paper is as follows.

Theorem *The Bergman kernel on the tube domain* T_Ω *is given by*

$$K(z,w) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} \frac{e^{it(z-\bar{w})}}{\gamma_t}dt, \quad \text{where} \quad \gamma_t = \int_\Omega e^{-2tx}dx. \tag{1.2}$$

Given a function h on T_Ω, we shall say that $K(z,w)$ reproduces h if

$$\int_{w\in T_\Omega} K(z,w)h(w)\,dV \;=\; h(z), \tag{1.3}$$

2000 Mathematics Subject Classification : 32A07, 32A25.

Key words and phrases. Bergman kernel, tube domain.

where dV is the Lebesgue measure on T_Ω. Let $A^2(T_\Omega)$ denote the Bergman space, namely the Hilbert space of all holomorphic functions h on T_Ω in which

$$\int_{z\in T_\Omega} |h(z)|^2\, dV. \tag{1.4}$$

The Bergman kernel $K(z,w)$ is uniquely characterized by the following three properties:

(i) $K(z,w) = \overline{K(w,z)}$ for all $z,w \in T_\Omega$;

(ii) $K(z,w)$ reproduces every element in $A^2(T_\Omega)$ in the sense of (1.3);

(iii) $K(-,w) \in A^2(T_\Omega)$ for all $w \in T_\Omega$.

Our formulation of $K(z,w)$ in (1.2) clearly satisfies condition (i). Therefore, to prove the theorem, we need to check conditions (ii) and (iii). This will be done by Propositions 2.4 and 2.5 in the next section.

2. The Bergman kernel on tube domains

Let Ω be a bounded strictly convex domain in $\mathbf{R}^n$, and let T_Ω be the tube domain as defined in (1.1). Since Ω is strictly convex, T_Ω is strictly pseudoconvex.

Let γ_t and $K(z,w)$ be defined as in (1.2). The Bergman space $A^2(T_\Omega)$ consists of holomorphic functions h which satisfy (1.4). In this section, we show that $K(z,w)$ satisfies conditions (ii) and (iii) stated in the Introduction. Let P denote the polynomial functions, and we define

$$Pe^{-z^2} \;=\; \{p(z)e^{-z^2}\ ;\ p(z) \text{ polynomial }\}.$$

Lemma 2.1 *$K(z,w)$ reproduces every element of Pe^{-z^2}.*

Proof: The following identity shall be useful:

$$\int_{\mathbf{R}^n} e^{-p^2x^2+zx}dx = (\frac{\sqrt{\pi}}{p})^n \exp(\frac{z^2}{4p^2}) \;\;,\;\; p>0, z\in \mathbf{C}^n. \tag{2.1}$$

Here x^2, zx, z^2 denote the usual dot product. In deriving this identity, we observe that both sides of (2.1) are holomorphic in $z \in \mathbf{C}^n$. Therefore, it suffices to check that it holds on the totally real subspace $z \in \mathbf{R}^n$. This can be obtained from standard integration tables ([3], 3.323 ♯2).

Write $w = x + iy$ on T_Ω, where x, y are variables on $\mathbf{R}^n, \Omega$ respectively. Then

$$\begin{aligned}
&\int_{w\in T_\Omega} K(z,w)e^{-w^2}dV \\
&= \tfrac{1}{(2\pi)^n}\int_{(x,y)\in T_\Omega}\int_{t\in\mathbf{R}^n} e^{it(z-(x-iy))}\gamma_t^{-1}e^{-(x+iy)^2}dt\,dx\,dy \\
&= \tfrac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\int_\Omega\Big(\int_{\mathbf{R}^n}\exp(-itx - x^2 - 2ixy)dx\Big)\exp(itz - ty + y^2)\gamma_t^{-1}dy\,dt \\
&= \tfrac{1}{(2\sqrt{\pi})^n}\int_{\mathbf{R}^n}\int_\Omega \exp\big(-\tfrac{(t+2y)^2}{4}\big)\exp(itz - ty + y^2)\gamma_t^{-1}dy\,dt && \text{by } (2.1) \\
&= \tfrac{1}{(2\sqrt{\pi})^n}\int_{\mathbf{R}^n}\int_\Omega \exp\big(-\tfrac{t^2}{4} - 2ty + itz\big)\gamma_t^{-1}dy\,dt \\
&= \tfrac{1}{(2\sqrt{\pi})^n}\int_{\mathbf{R}^n}\exp\big(-\tfrac{t^2}{4} + itz\big)dt \\
&= e^{-z^2}. && \text{by } (2.1)
\end{aligned}$$

Consequently,

$$\int_{w\in T_\Omega} K(z,w)e^{-w^2}\,dV = e^{-z^2} \quad , \quad z \in T_\Omega. \tag{2.2}$$

From (2.2), substitute $z \in T_\Omega$ with $z + c \in T_\Omega$, where $c \in \mathbf{R}^n$. This gives

$$\frac{1}{(2\pi)^n}\int_{w\in T_\Omega}\int_{t\in\mathbf{R}^n}\frac{e^{it(z+c-\bar{w})}}{\gamma_t}e^{-w^2}dt\,dV = e^{-(z+c)^2}.$$

Changing the variable w to $w + c$ in LHS gives

$$\frac{1}{(2\pi)^n}\int_{w\in T_\Omega}\int_{t\in\mathbf{R}^n}\frac{e^{it(z-\bar{w})}}{\gamma_t}e^{-(w+c)^2}dt\,dV = e^{-(z+c)^2}.$$

Apply $\frac{d}{dc}|_{c=0}$ to both sides, we see that $K(z,w)$ reproduces the function $-2ze^{-z^2}$. We carry out the procedure $\frac{d^k}{dc^k}|_{c=0}$ for $k = 1, 2, ...,$ and Lemma 2.1 follows. □

We plan to show that $K(z,w)$ reproduces every element of $A^2(T_\Omega)$. In view of Lemma 2.1, this will follow if we can show that Pe^{-z^2} is a dense subset of the Hilbert space $A^2(T_\Omega)$. This will be established by the next two lemmas. Our strategy is to convert the problem on $A^2(T_\Omega)$ to another Hilbert space $L^2(\mathbf{R}^n, \gamma_t)$ by Lemma 2.2, and obtain a result on $L^2(\mathbf{R}^n, \gamma_t)$ by Lemma 2.3. We then transfer this result back to $A^2(T_\Omega)$, by Proposition 2.4.

Let $L^2(\mathbf{R}^n, \gamma_t)$ denote the Hilbert space of L^2 functions on $\mathbf{R}^n$, with weight γ_t given in (1.2). Namely,

$$L^2(\mathbf{R}^n, \gamma_t) = \{f \; ; \; \int_{\mathbf{R}^n}|f(t)|^2\gamma_t\,dt \; < \; \infty\}.$$

Lemma 2.2 (T. G. Genchev) *The transformation*

$$g(t) \longmapsto \int_{\mathbf{R}^n} e^{izt} g(t)\, dt \tag{2.3}$$

is an isometry from $L^2(\mathbf{R}^n, \gamma_t)$ *to* $A^2(T_\Omega)$, *preserving the Hilbert space norms.*

Proof: Given $h(z) \in A^2(T_\Omega)$, there exists $g(t) \in L^2(\mathbf{R}^n, \gamma_t)$ such that

$$h(z) = \int_{\mathbf{R}^n} e^{izt} g(t)\, dt, \tag{2.4}$$

see [1],[2]. This means that the transformation in (2.3) is surjective. In order to see that (2.3) is well-defined, injective and preserves the norms, we shall prove that in (2.4), $\|h\|_{A^2(T_\Omega)} = \|g\|_{L^2(\mathbf{R}^n,\gamma_t)}$.

Write $z = x + iy$, and (2.4) gives

$$h(-x+iy) = \int_{\mathbf{R}^n} e^{-ixt} (e^{-yt} g(t))\, dt.$$

Thus for every fixed $y \in \Omega$, $h(-x+iy)$ is the Fourier transform of $e^{-yt}g(t)$. By Plancherel's theorem,

$$\int_{\mathbf{R}^n} |h(-x+iy)|^2\, dx = \int_{\mathbf{R}^n} |e^{-yt} g(t)|^2 dt.$$

Apply $\int_\Omega dy$ to both sides, we get

$$\begin{aligned} \|h\|^2_{A^2(T_\Omega)} &= \textstyle\int_\Omega \int_{\mathbf{R}^n} |h(-x+iy)|^2\, dx\, dy \\ &= \textstyle\int_\Omega \int_{\mathbf{R}^n} |e^{-yt} g(t)|^2\, dt\, dy \\ &= \textstyle\int_{\mathbf{R}^n} |g(t)|^2 \gamma_t\, dt \\ &= \|g\|^2_{L^2(\mathbf{R}^n,\gamma_t)}. \end{aligned}$$

This proves Lemma 2.2. □

Next we define

$$Pe^{-\frac{t^2}{4}} = \{p(t)e^{-\frac{t^2}{4}}\ ;\ p(t) \text{ polynomial } \}.$$

Lemma 2.3 $Pe^{-\frac{t^2}{4}}$ *is dense in the Hilbert space* $L^2(\mathbf{R}^n, \gamma_t)$.

Proof: We shall show that

$$\{t^k e^{-\frac{t^2}{4}}\},\ |k| = 0, 1, 2, ... \tag{2.5}$$

is a complete basis of $L^2(\mathbf{R}^n, \gamma_t)$, and the lemma follows.

Let $f(t) \in L^2(\mathbf{R}^n, \gamma_t)$, and suppose that

$$\int_{\mathbf{R}^n} t^k e^{-\frac{t^2}{4}} f(t)\gamma_t \, dt \;=\; 0 \tag{2.6}$$

for all $|k| = 0, 1, 2, ...$ It is easy to see that $e^{-\frac{t^2}{2}}\gamma_t^2$ is bounded, so

$$\begin{aligned} \textstyle\int_{\mathbf{R}^n} |e^{-\frac{t^2}{4}} f(t)\gamma_t|^2 \gamma_t \, dt \quad &= \textstyle\int_{\mathbf{R}^n} |f(t)|^2 \gamma_t (e^{-\frac{t^2}{2}}\gamma_t^2) \, dt \\ &\le c \textstyle\int_{\mathbf{R}^n} |f(t)|^2 \gamma_t \, dt \\ &< \infty, \quad \text{as } f(t) \in L^2(\mathbf{R}^n, \gamma_t). \end{aligned}$$

Therefore, $e^{-\frac{t^2}{4}} f(t)\gamma_t \in L^2(\mathbf{R}^n, \gamma_t)$. Let

$$h(z) = \int_{\mathbf{R}^n} e^{izt} e^{-\frac{t^2}{4}} f(t)\gamma_t \, dt. \tag{2.7}$$

By Lemma 2.2, $h(z) \in A^2(T_\Omega)$. We may assume that $0 \in T_\Omega$, and consider the power series expansion

$$h(z) = \sum a_k z^k$$

near 0. Then

$$\begin{aligned} a_k \quad &= \tfrac{1}{k!} h^{(k)}(0) \\ &= \tfrac{1}{k!} i^{|k|} \textstyle\int_{\mathbf{R}^n} t^k e^{-\frac{t^2}{4}} f(t)\gamma_t \, dt \quad && \text{by (2.7)} \\ &= 0. && \text{by (2.6)} \end{aligned}$$

So $h(z)$ vanishes near 0. However, since $h(z)$ is holomorphic, it follows that $h \equiv 0$. By Lemma 2.2,

$$e^{-\frac{t^2}{4}} f(t)\gamma_t \;\equiv\; 0.$$

Since $e^{-\frac{t^2}{4}}$ and γ_t are always positive,

$$f \;\equiv\; 0.$$

This shows that the functions in (2.5) form a complete basis of $L^2(\mathbf{R}^n, \gamma_t)$, and the lemma is proved. □

We combine Lemmas 2.1, 2.2 and 2.3 to show that:

Proposition 2.4 *The function $K(z, w)$ of (1.2) reproduces every element of $A^2(T_\Omega)$.*

Proof: We claim that Pe^{-z^2} is dense in $A^2(T_\Omega)$. Identity (2.1) implies that

$$\int_{\mathbf{R}^n} e^{izt-\frac{t^2}{4}}dt = (2\sqrt{\pi})^n e^{-z^2}.$$

Applying $\frac{d}{dz}$ repeatedly to both sides, we see that if $p(t)$ is a polynomial, then

$$\int_{\mathbf{R}^n} e^{izt}p(t)e^{-\frac{t^2}{4}}dt = p_1(z)e^{-z^2} \in A^2(T_\Omega),$$

for another polynomial $p_1(z)$. Namely, the isometry in Lemma 2.2 sends every

$$p(t)e^{-\frac{t^2}{4}} \in Pe^{-\frac{t^2}{4}} \subset L^2(\mathbf{R}^n, \gamma_t)$$

to some

$$p_1(z)e^{-z^2} \in Pe^{-z^2} \subset A^2(T_\Omega).$$

By Lemma 2.3, $Pe^{-\frac{t^2}{4}}$ is dense in $L^2(\mathbf{R}^n, \gamma_t)$. Hence Pe^{-z^2} is dense in $A^2(T_\Omega)$ as claimed.

By Lemma 2.1, $K(z,w)$ reproduces every element of Pe^{-z^2}. Therefore, since Pe^{-z^2} is dense in $A^2(T_\Omega)$, it follows that $K(z,w)$ reproduces every element of $A^2(T_\Omega)$. □

With this result, condition (ii) of $K(z,w)$ stated in the Introduction is verified. Therefore, to prove the theorem, it remains only to check condition (iii). This is done by the following proposition.

Proposition 2.5 *For each fixed* $w \in T_\Omega$,

$$K(-,w) \in A^2(T_\Omega).$$

Proof: Fix $w \in T_\Omega$, and consider $K(-,w)$. For any fixed $\xi \in \mathbf{R}^n$, formula (1.2) satisfies

$$K_\Omega(z,w) = K_{\Omega+\xi}(z+i\xi, w+i\xi)$$

for all $z, w \in T_\Omega$. Further, $z+\xi, w+\xi \in T_\Omega$ and $K(z,w) = K(z+\xi, w+\xi)$. Therefore, without loss of generality, we may assume that $w = 0 \in T_\Omega \subset \mathbf{C}^n$ in the statement of this proposition. We want to show that $K(z,0) \in A^2(T_\Omega)$. But

$$K(z,0) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} \frac{e^{izt}}{\gamma_t}dt,$$

and with Lemma 2.2, we get

$$\begin{aligned}\|K(z,0)\|^2_{L^2(T_\Omega)} &= \tfrac{1}{(2\pi)^n}\|\tfrac{1}{\gamma_t}\|^2_{L^2(\mathbf{R}^n,\gamma_t)}\\ &= \tfrac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\tfrac{1}{\gamma_t}dt\\ &= K(0,0)\\ &< \infty, \quad \text{as } 0\in T_\Omega.\end{aligned}$$

We have shown that, for each $w \in T_\Omega$, $K(-,w) \in A^2(T_\Omega)$. □

This result verifies condition (iii) of the Introduction, and the theorem follows.

References

[1] T. G. Genchev, Integral representations for functions holomorphic in tube domains, *C. R. Acad. Bulgare Sci.* **37** (1984), 717-720.

[2] T. G. Genchev, Paley-Wiener type theorems for functions in Bergman spaces over tube domains, *J. Math. Anal. and Appl.* **118** (1986), 496-501.

[3] I. S. Gradshteyn, I. M. Ryzhik, *Table of Integrals, Series and Products,* Academic Press, (1980).

[4] S. Krantz, *Function Theory of Several Complex Variables, 2ed.* Wadsworth & Brooks/Cole, Pacific Grove 1992.

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Recibido: 6 de mayo de 2004
Aceptado: 28 de marzo de 2005