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ETA SERIES AND ETA INVARIANTS OF $\mathbb{Z}_4$-MANIFOLDS

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ABSTRACT. In this paper, for each $n = 4r + 3$, we construct a family of compact flat spin n-manifolds with holonomy group isomorphic to $\mathbb{Z}_4$ such that the spectrum of the Dirac operator D is asymmetric. For these manifolds we will obtain explicit expressions for the eta series, $\eta(s)$, in terms of Hurwitz zeta functions, and for the eta-invariant, η, associated to D. The explicit expressions will show the meromorphic continuation of $\eta(s)$ to $\mathbb{C}$ is in fact everywhere holomorphic.

INTRODUCTION

If A is a positive self-adjoint elliptic differential operator on a compact n-manifold M, then it has a discrete spectrum, denoted by $Spec_A(M)$, consisting of positive eigenvalues λ with finite multiplicity d_λ. This spectrum can be properly studied by the zeta function $\zeta_A(s) = \sum \lambda^{-s}$, where the sum is taken over the non zero eigenvalues of A and $Re(s) > \frac{n}{d}$, with d the order of A.

If A is no longer positive, then the eigenvalues can now be positive or negative. In this case, the spectrum is said to be *asymmetric* if for some $\lambda \in Spec_A(M)$ we have $d_\lambda \neq d_{-\lambda}$. To study this phenomenon, Atiyah, Patodi and Singer introduced in [**APS**] the "signed" version of the zeta function, namely, the so called *eta series* defined by

$$\eta_A(s) = \sum_{0 \neq \lambda \in Spec_A} \operatorname{sign}(\lambda)\, |\lambda|^{-s}.$$

This series converges for $Re(s) > \frac{n}{d}$ and defines a holomorphic function $\eta_A(s)$ which has a meromorphic continuation to $\mathbb{C}$. It is a non trivial fact that this function is really finite at $s = 0$ (See [**APS2**] for n odd, and [**Gi**], [**Gi2**], for n even). The number $\eta_A(0)$ is a spectral invariant, called the *eta invariant*, which gives a measure of the spectral asymmetry of A.

In this paper, we will take A to be the Dirac operator D. It is a first order elliptic essentially self-adjoint operator defined on spin manifolds, that is, manifolds admitting a spin structure. Consider the eta function associated to D, denoted simply by $\eta(s)$. The determination of $\eta(s)$ and $\eta(0)$ is in general a difficult task and explicit computations of these objects are not easy to find in the literature.

For compact flat spin manifolds (see Preliminaries) we have the following picture. Pfäffle computed the η-invariants in the 3-dimensional case ([**Pf**]). A general

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expression for $\eta(s)$ for an *arbitrary* n-manifold M with a spin structure ε is given in [**MP2**]. There, in the particular case of manifolds with holonomy group $F \simeq \mathbb{Z}_2^k$, $1 \le k \le n-1$, the authors obtained a very explicit expression for $\eta(s)$, in terms of differences of Hurwitz zeta functions $\zeta(s,\alpha) = \sum_{j\ge 0} (j+\alpha)^{-s}$, for $Re(s) > 1$ and $\alpha \in (0,1]$. This allowed to compute the η-invariant simply by evaluation at $s = 0$. Also in [**MP2**], similar results were obtained for a family of compact flat spin p-manifolds with holonomy group $F \simeq \mathbb{Z}_p$, with p prime of the form $4r+3$.

These results led us to expect that the eta series of any compact flat spin manifold with abelian holonomy group should be expressible in terms of differences of Hurwitz zeta functions $\zeta(s,\alpha)$ for $\alpha \in (0,1] \cap \mathbb{Q}$.

The goal of the present paper is to deal with the simplest case not covered in [**MP2**], that is, when $F \simeq \mathbb{Z}_4$. More precisely, we consider a rather large family of compact flat spin manifolds M with holonomy group $\mathbb{Z}_4$ having asymmetric Dirac spectrum and we compute the corresponding eta series and eta invariant for every manifold in the family (in the case of symmetric spectrum one has that $\eta(s) \equiv 0$, see (4.1)). The general expression for $\eta(s)$ is either of the form

$$\eta(s) = \tfrac{C_{M,\varepsilon}}{(8\pi)^s} \left(\zeta(s,\tfrac14) - \zeta(s,\tfrac34)\right)$$

or

$$\eta(s) = \tfrac{C_{M,\varepsilon}}{(8\pi)^s} \Big(a_n\big(\zeta(s,\tfrac18) - \zeta(s,\tfrac78)\big) + b_n\big(\zeta(s,\tfrac38) - \zeta(s,\tfrac58)\big)\Big)$$

where $C_{M,\varepsilon}, a_n, b_n$ are explicit constants depending on M, the spin structure ε of M, and the dimension n of M.

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1. Preliminaries

Flat manifolds. We refer to [**Ch**]. A *Bieberbach group* is a discrete, cocompact, torsion-free subgroup Γ of the isometry group $I(\mathbb{R}^n)$ of $\mathbb{R}^n$. Such Γ acts properly discontinuously on $\mathbb{R}^n$, thus $M_\Gamma = \Gamma\backslash\mathbb{R}^n$ is a compact flat Riemannian manifold with fundamental group Γ. By the Killing-Hopf theorem any such manifold arises in this way. Any element $\gamma \in I(\mathbb{R}^n) = \mathrm{O}(n) \ltimes \mathbb{R}^n$ decomposes uniquely as $\gamma = BL_b$, with $B \in \mathrm{O}(n)$, $b \in \mathbb{R}^n$ and L_b denotes translation by b. The translations in Γ form a normal maximal abelian subgroup L_Λ of finite index, with Λ a lattice in $\mathbb{R}^n$ which is B-stable for each $BL_b \in \Gamma$. As usual, one identifies L_Λ with Λ. The restriction to Γ of the canonical projection $r : \mathrm{I}(\mathbb{R}^n) \to \mathrm{O}(n)$ given by $BL_b \mapsto B$ is a group homomorphism with kernel L_Λ and $F := r(\Gamma)$ is a finite subgroup of $\mathrm{O}(n)$. The group $F \simeq \Lambda\backslash\Gamma$ is called the *holonomy group* of Γ and is isomorphic to the linear holonomy group of the Riemannian manifold M_Γ. The action of F on Λ by conjugation is usually called the *integral holonomy representation* of Γ. By an F-manifold we understand a Riemannian manifold with holonomy group F. In this paper we shall consider $\mathbb{Z}_4$-manifolds which, by the Cartan-Ambrose-Singer theorem, are necessarily flat.

Spin groups. For standard results on spin geometry we refer to [**LM**] or [**Fr**]. Let $Cl(n)$ denote the Clifford algebra of $\mathbb{R}^n$ with respect to the standard inner product $\langle , \rangle$ on $\mathbb{R}^n$ and let $\mathbb{C}l(n) = Cl(n) \otimes \mathbb{C}$ be its complexification. If $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$ then a basis for $Cl(n)$ is given by the set $\{e_{i_1} \cdots e_{i_k} : 1 \le i_1 < \cdots < i_k \le n\}$. One has that $vw + wv + 2\langle v, w\rangle = 0$ holds for all $v, w \in \mathbb{R}^n$, thus $e_ie_j = -e_je_i$ and $e_i{}^2 = -1$ for $i, j = 1, \ldots, n$. Inside the group of units of $Cl(n)$ we have the compact Lie group

$$\mathrm{Spin}(n) = \{v_1 \cdots v_{2k} : \|v_j\| = 1,\ 1 \le j \le 2k\},$$

which is connected if $n \ge 2$ and simply connected if $n \ge 3$. There is a Lie group epimorphism $\mu : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ with kernel $\{\pm 1\}$ given by $v \mapsto (x \mapsto vxv^{-1})$.

If B_j is a matrix for $1 \le j \le m$, we will denote by $\mathrm{diag}(B_1, \ldots, B_m)$ the block matrix having the block B_j in the diagonal position j. For $t \in \mathbb{R}$ let $B(t) = \left[\begin{smallmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{smallmatrix}\right]$ and, for $t_1, \ldots, t_m \in \mathbb{R}$, define

$$\begin{aligned} x_0(t_1, \ldots, t_m) &:= \begin{cases} \mathrm{diag}(B(t_1), \ldots, B(t_m)) & \text{if} \quad n = 2m, \\ \mathrm{diag}(B(t_1), \ldots, B(t_m), 1) & \text{if} \quad n = 2m+1, \end{cases} \\ x(t_1, \ldots, t_m) &:= \prod_{j=1}^{m} \big(\cos(t_j) + \sin(t_j)\, e_{2j-1}e_{2j} \big) \in \mathrm{Spin}(n). \end{aligned} \tag{1.1}$$

It is easy to check that, for any $k \in \mathbb{Z}$, the elements $x(t_1, \ldots, t_m)$ satisfy

$$\begin{aligned} x(t_1, \ldots, t_m)^k &= x(kt_1, \ldots, kt_m) \\ -x(t_1, t_2, \ldots, t_m) &= x(t_1 + \pi, t_2, \ldots, t_m). \end{aligned} \tag{1.2}$$

We will also need to fix maximal tori in $\mathrm{Spin}(n)$ and $\mathrm{SO}(n)$. They are respectively given by $T = \{x(t_1, \ldots, t_m) : t_j \in \mathbb{R}, 1 \le j \le m\}$ and by $T_0 = \{x_0(t_1, \ldots, t_m) : t_j \in \mathbb{R}, j = 1, \ldots, m\}$ (see [**LM**] or [**Fr**]). The restriction $\mu : T \to T_0$ is a 2-fold cover and

$$\mu(x(t_1, \ldots, t_m)) = x_0(2t_1, \ldots, 2t_m). \tag{1.3}$$

Spin representations. We consider (L, S) an irreducible complex representation of the Clifford algebra $\mathbb{C}l(n)$, restricted to $\mathrm{Spin}(n)$. The complex vector space S has dimension 2^m with $m := [\frac{n}{2}]$. If n is odd, then (L, S) is irreducible for $\mathrm{Spin}(n)$ and is called the *spin representation*. If n is even, then $\mathrm{S} = \mathrm{S}^+ \oplus \mathrm{S}^-$ where each $\mathrm{S}^\pm$ is irreducible of dimension 2^{m-1}. If $L^\pm$ denote the restricted action of L on $\mathrm{S}^\pm$ then $(L^\pm, \mathrm{S}^\pm)$ are called the *half-spin representations* of $\mathrm{Spin}(n)$. We shall write (L_n, S_n) and $(L_n^\pm, \mathrm{S}_n^\pm)$ for (L, S) and $(L^\pm, \mathrm{S}^\pm)$ when we wish to specify the dimension.

We will make repeatedly use of the following result (see [**MP2**]) which gives the values of the characters χ_{L_n} and $\chi_{L_n^\pm}$ of the spin and half spin representations on T. If $n = 2m$, then

$$\chi_{L_n^\pm}(x(t_1, \ldots, t_m)) = 2^{m-1}\Big(\prod_{j=1}^{m} \cos t_j \pm i^m \prod_{j=1}^{m} \sin t_j \Big). \tag{1.4}$$

Furthermore, $\chi_{L_n}(x(t_1,\ldots,t_m)) = 2^m \prod_{j=1}^m \cos t_j$ for $n = 2m$ or $n = 2m+1$.

Spin structures. If (M,g) is an orientable Riemannian manifold, let $\mathsf{B}(M)$ be the bundle of oriented frames on M and $\pi : \mathsf{B}(M) \to M$ the canonical projection. $\mathsf{B}(M)$ is a principal $\mathrm{SO}(n)$-bundle over M. A *spin structure* on M is an equivariant 2-fold cover $p : \tilde{\mathsf{B}}(M) \to \mathsf{B}(M)$ where $\tilde{\pi} : \tilde{\mathsf{B}}(M) \to M$ is a principal $\mathrm{Spin}(n)$-bundle and $\pi \circ p = \tilde{\pi}$. Such M endowed with a spin structure is called a *spin manifold*.

On compact flat manifolds $M_\Gamma = \Gamma\backslash\mathbb{R}^n$, Γ a Bieberbach group, the spin structures are in a one to one correspondence with group homomorphisms ε commuting the diagram

$$\begin{array}{ccc} & & \mathrm{Spin}(n) \\ & \overset{\varepsilon}{\nearrow} & \big\downarrow \mu \\ \Gamma & \xrightarrow[r]{} & \mathrm{SO}(n) \end{array} \tag{1.5}$$

that is, satisfying $\mu\circ\varepsilon = r$ where $r(\gamma) = B$ if $\gamma = BL_b \in \Gamma$ (see [**Fr**], [**LM**]). We shall denote by (M_Γ,ε) the spin manifold M_Γ endowed with the spin structure ε as in (1.5).

The spectrum of the Dirac operator. If (L,S) is the spin representation, the vector bundle $S(M_\Gamma,\varepsilon) := \Gamma\backslash(\mathbb{R}^n \times \mathrm{S}) \to \Gamma\backslash\mathbb{R}^n = M_\Gamma$ with action $\gamma\cdot(x,w) = (\gamma x, L(\varepsilon(\gamma))(w))$, where $\gamma \in \Gamma$, $w \in \mathrm{S}$, is called the *spinor bundle* of M_Γ. The space $\Gamma^\infty(S(M_\Gamma,\varepsilon))$ of smooth sections of the spinor bundle can be identified with the set $\{f : \mathbb{R}^n \to \mathrm{S} \text{ smooth } : f(\gamma x) = L(\varepsilon(\gamma))f(x)\}$.

One can consider the *Dirac operator* D acting on smooth sections f of $S(M_\Gamma,\varepsilon)$ by $Df(x) = \sum_{i=1}^n e_i \cdot \frac{\partial f}{\partial x_i}(x)$, where $e_i \cdot w = L(e_i)(w)$ for $w \in \mathrm{S}$. One has that D is an elliptic first-order differential operator, which is symmetric and essentially self-adjoint. Furthermore, over compact manifolds, D has a discrete spectrum consisting of real eigenvalues $\pm 2\pi\mu$, $\mu \geq 0$, of finite multiplicity $d_\mu^\pm$. If $f \in \ker D$, f is called a *harmonic spinor*.

In [**MP2**] explicit expressions for the multiplicities $d_\mu^\pm$ for any compact flat spin manifold (M_Γ,ε) with translation lattice Λ and holonomy group F were obtained. We now recall the ingredients for these expressions.

Let $F_1 = \{B \in F = r(\Gamma) : n_B = 1\}$ where $n_B := \dim\ker(B - Id)$. Put $\Lambda_\varepsilon^* = \{u \in \Lambda^* : \varepsilon(\lambda) = e^{2\pi i\lambda\cdot u}, \lambda \in \Lambda\}$, with Λ^* the dual lattice of Λ, and

$$\Lambda_{\varepsilon,\mu}^* = \{u \in \Lambda_\varepsilon^* : \|u\| = \mu\}. \tag{1.6}$$

Now, for each $\gamma = BL_b \in \Gamma$, let $(\Lambda_{\varepsilon,\mu}^*)^B$ denotes the set of elements fixed by B in $\Lambda_{\varepsilon\mu}^*$, that is

$$(\Lambda_{\varepsilon,\mu}^*)^B = \{u \in \Lambda_{\varepsilon,\mu}^* : Bv = v\}. \tag{1.7}$$

Furthermore, for $\gamma \in \Gamma$, let x_γ be a fixed, though arbitrary, element in the maximal torus of $\mathrm{Spin}(n-1)$, conjugate in $\mathrm{Spin}(n)$ to $\varepsilon(\gamma)$. Finally, define a sign $\sigma(u,x_\gamma)$, depending on u and on the conjugacy class of x_γ in $\mathrm{Spin}(n-1)$, in the following way. If $\gamma = BL_b \in \Lambda\backslash\Gamma$ and $u \in (\Lambda_\varepsilon^*)^B \smallsetminus \{0\}$, let $h_u \in \mathrm{Spin}(n)$ such that $h_u\, u\, h_u^{-1} = \|u\|e_n$. Hence, $h_u\varepsilon(\gamma)h_u^{-1} \in \mathrm{Spin}(n-1)$. Take $\sigma_\varepsilon(u,x_\gamma) = 1$ if $h_u\varepsilon(\gamma)h_u^{-1}$ is conjugate to x_γ in $\mathrm{Spin}(n-1)$ and $\sigma_\varepsilon(u,x_\gamma) = -1$ otherwise. As a consequence, $\sigma(-u,x_\gamma)=-\sigma(u,x_\gamma)$ and $\sigma(\alpha u,x_\gamma)=\sigma(u,x_\gamma)$ for every $\alpha>0$ (see Definition 2.3, Remark 2.4 and Lemma 6.2 in [**MP2**] for details).

For n odd, the multiplicity of the eigenvalue $\pm 2\pi\mu$, for $\mu>0$, is given by

$$d^{\pm}_{\mu}(\Gamma,\varepsilon)=\tfrac{1}{|F|}\Bigg(\sum_{\substack{\gamma\,\in\,\Lambda\backslash\Gamma\\ B\,\notin\,F_1}}\ \sum_{u\in(\Lambda^*_{\varepsilon,\mu})^B} e^{-2\pi i u\cdot b}\cdot\chi_{L^{\pm}_{n-1}}(x_\gamma)\ + \sum_{\substack{\gamma\,\in\,\Lambda\backslash\Gamma\\ B\,\in\,F_1}}\ \sum_{u\in(\Lambda^*_{\varepsilon,\mu})^B} e^{-2\pi i u\cdot b}\cdot\chi_{L^{\pm\sigma(u,x_\gamma)}_{n-1}}(x_\gamma)\Bigg), \tag{1.8}$$

while for n even, it is given by the first term in (1.8) (i.e., the sum over $B\notin F_1$) with $L^{\pm}_{n-1}$ replaced by L_{n-1}. For $\mu=0$, with n even or odd, we have that $d_0(\Gamma,\varepsilon)=\frac{1}{|F|}\sum_{\gamma\in\Lambda\backslash\Gamma}\chi_{L_n}(\varepsilon(\gamma))=\dim \mathrm{S}^F$, if $\varepsilon_{|\Lambda}=1$, and $d_0(\Gamma,\varepsilon)=0$, otherwise.

2. A family of spin $\mathbb{Z}_4$-manifolds

In this section, for each $n\geq 3$, we shall construct a family of n-dimensional pairwise non-homeomorphic spin $\mathbb{Z}_4$-manifolds, having asymmetric Dirac spectrum. In this way, we will obtain non trivial eta series and eta invariants, to be computed in the next sections. Put $\tilde J:=\left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right]$. For each $j,l\geq 1$ and $k\geq 0$, we set

$$B_{j,k}:=\operatorname{diag}(\underbrace{\tilde J,\dots,\tilde J}_{j},\underbrace{-1,\dots,-1}_{k},\underbrace{1,\dots,1}_{l}), \tag{2.1}$$

where $2j+k+l=n\geq 3$.

Then $B_{j,k}\in \mathrm{O}(n)$, $B_{j,k}^4=Id$ and $B_{j,k}\in\mathrm{SO}(n)$ if and only if k is even. Let $\Lambda=\mathbb{Z}e_1\oplus\cdots\oplus\mathbb{Z}e_n$ be the canonical lattice of $\mathbb{R}^n$ and for j,k,l as before define the groups

$$\Gamma_{j,k}:=\langle B_{j,k}L_{\frac{e_n}{4}},\Lambda\rangle. \tag{2.2}$$

We have that Λ is $B_{j,k}$-stable. Since $(B_{j,k}^m-Id)\frac{m}{4}e_n=0\in\Lambda$, for $0\leq\ m\leq 3$, and $(\sum_{m=0}^3 B_{j,k}^m)\frac{e_n}{4}=e_n\in\Lambda\smallsetminus(\sum_{m=0}^3 B_{j,k}^m)\Lambda$, by Proposition 2.1 in [**MR**], each $\Gamma_{j,k}$ is a Bieberbach group. In this way, we have a family

$$\mathcal{F}^n=\{M_{j,k}:=\Gamma_{j,k}\backslash\mathbb{R}^n\ :\ 1\leq j\leq[\tfrac{n-1}{2}],\,0\leq k\leq n-2j-1,l\geq 1\} \tag{2.3}$$

of compact flat manifolds with holonomy group $F\simeq\mathbb{Z}_4$. It is easy to see that the cardinality of $\mathcal{F}^n$ is given by $\#\mathcal{F}^n=[\frac{n-1}{2}]\big(n-[\frac{n-1}{2}]-1\big)=o(n^2)$.

Lemma 2.1. *For $M_{j,k}\in\mathcal{F}^n$ we have*

$$H_1(M_{j,k},\mathbb{Z})\simeq\mathbb{Z}^l\oplus\mathbb{Z}_2^{j+k}. \tag{2.4}$$

Hence the manifolds in $\mathcal{F}^n$ are non-homeomorphic to each other.

Proof. We compute $H_1(M_{j,k},\mathbb{Z})\simeq\Gamma_{j,k}/[\Gamma_{j,k},\Gamma_{j,k}]$. For $\gamma=B_{j,k}L_{\frac{e_n}{4}}$, we have

$$\begin{aligned}[\Gamma_{j,k},\Gamma_{j,k}] &= \langle[\gamma,L_{e_i}]=L_{(B_{j,k}-Id)e_i}:1\leq i\leq n\rangle\\ &= \langle L_{-e_1\pm e_2},\dots,L_{-e_{2j-1}\pm e_{2j}},L_{2e_{2j+1}},\dots,L_{2e_{2j+k}}\rangle.\end{aligned}$$

Using this, and the fact that $\gamma^4 = L_{e_n}$, it is easy to see that $H_1(M_{j,k}, \mathbb{Z}) \simeq \mathbb{Z}^l \oplus \mathbb{Z}_2^{j+k}$. Thus, if $M_{j,k}$ and $M_{j',k'}$ are homeomorphic then $l = l'$ and $j+k = j'+k'$. Since $n = 2j + k + l = 2j' + k' + l'$ we have that $j = j'$ and $k = k'$. Therefore, the manifolds in $\mathcal{F}^n$ are non-homeomorphic to each other. □

We now study the existence of spin structures on n-dimensional $\mathbb{Z}_d$-manifolds following [**MP**], where the existence of spin structures on $\mathbb{Z}_2^k$-manifolds was considered. Let Γ be a Bieberbach group with holonomy group $F \simeq \mathbb{Z}_d$ and translation lattice Λ. Then $M_\Gamma = \Gamma\backslash\mathbb{R}^n$ with $\Gamma = \langle \gamma, \Lambda \rangle$ where $\gamma = BL_b$, $B \in \mathrm{O}(n)$, $b \in \mathbb{R}^n$, $B\Lambda = \Lambda$ and $B^d = Id$.

Assume there is a spin structure defined on M_Γ, that is, a group homomorphism $\varepsilon : \Gamma \to \mathrm{Spin}(n)$ such that $\mu \circ \varepsilon = r$. Then, necessarily $\varepsilon(L_\lambda) \in \{\pm 1\}$, for $\lambda \in \Lambda$. Thus, if $\lambda_1, \ldots, \lambda_n$ is a $\mathbb{Z}$-basis of Λ and we set $\delta_i := \varepsilon(L_{\lambda_i})$, for every $\lambda = \sum_i m_i \lambda_i \in \Lambda$ with $m_i \in \mathbb{Z}$, we have $\varepsilon(L_\lambda) = \prod_i \delta_i^{m_i} = \prod_{m_i \text{ odd}} \delta_i$.

For any $\gamma = BL_b \in \Gamma$ we will fix a distinguished (though arbitrary) element in $\mu^{-1}(B)$, denoted by u_B. Thus, $\varepsilon(\gamma) = \sigma\, u_B$, where $\sigma \in \{\pm 1\}$ depends on γ and on the choice of u_B.

The morphism ε is determined by its action on the generators of Γ. Hence, we will identify this morphism with the $(n+1)$-tuple

$$\varepsilon \equiv (\delta_1, \ldots, \delta_n, \sigma\, u_B) \tag{2.5}$$

where $\delta_i = \varepsilon(L_{\lambda_i})$ and σ is defined by the equation $\varepsilon(\gamma) = \sigma u_B$.

Now, since ε is a morphism and $\gamma = BL_b \in \Gamma$, for any $\lambda \in \Lambda$ we have

$$\varepsilon(L_{B\lambda}) = \varepsilon(\gamma L_\lambda \gamma^{-1}) = \varepsilon(\gamma)\varepsilon(L_\lambda)\varepsilon(\gamma^{-1}) = \varepsilon(L_\lambda). \tag{2.6}$$

Therefore, if ε is a spin structure on M_Γ, since $\gamma^d \in L_\Lambda$, then the character $\varepsilon_{|\Lambda} : \Lambda \to \{\pm 1\}$ must satisfy the following conditions for any $\gamma = BL_b \in \Gamma$:

$$\begin{array}{ll} (\varepsilon_1) & \varepsilon(\gamma^d) = (\sigma u_B)^d \\ (\varepsilon_2) & \varepsilon(L_{(B-Id)\lambda}) = 1, \quad \lambda \in \Lambda. \end{array} \tag{2.7}$$

We thus set

$$\hat{\Lambda}(\Gamma) := \{\chi \in \mathrm{Hom}(\Lambda, \{\pm 1\}) \,:\, \chi \text{ satisfies } (\varepsilon_1) \text{ and } (\varepsilon_2)\}. \tag{2.8}$$

The next result says that these necessary conditions for the existence of spin manifolds are also sufficient in the case of manifolds with cyclic holonomy groups. We adapt the proof of Theorem 2.1 in [**MP**] to our case.

Proposition 2.2. *If $\Gamma = \langle BL_b, \Lambda \rangle$ is a Bieberbach group with holonomy group $F = \langle B \rangle \simeq \mathbb{Z}_d$ and σ is as in (2.5), then the map $\varepsilon \mapsto (\varepsilon_{|\Lambda}, \sigma)$ defines a bijective correspondence between the spin structures on M_Γ and the set $\hat{\Lambda}(\Gamma) \times \{\pm 1\}$. Hence, the number of spin structures on M_Γ is either* 0 *or* 2^r *for some* $1 \le r \le n$.

Proof. It suffices to prove that, given $\varepsilon \in \hat{\Lambda}(\Gamma)$, we can extend it into a group homomorphism from Γ to $\mathrm{Spin}(n)$, also called ε, satisfying (1.5).

Let $\gamma = BL_b$. Since Λ is normal in Γ and $B^d = Id$, we see that any $\gamma_0 \in \Gamma$ can be written uniquely as $\gamma_0 = \gamma^k L_\lambda$, with $0 \le k \le d-1$, $\lambda \in \Lambda$. For any choice of

$u_B \in \mu^{-1}(B)$, we set $\varepsilon(\gamma) = \sigma u_B$, with $\sigma \in \{\pm 1\}$, and, for a general element in Γ, we define

$$\varepsilon(\gamma^k L_\lambda) = \varepsilon(\gamma)^k \varepsilon(L_\lambda)$$

for $0 \le k \le d-1$. Thus, we get a well defined map $\varepsilon : \Gamma \to \mathrm{Spin}(n)$ such that $\mu \circ \varepsilon = r$, and we claim it is a group homomorphism.

In fact, note that if $\gamma = BL_b$ then $\gamma^k = B^k L_{b(k)}$ where $b(k) := \sum_{i=0}^{k-1} B^{-i} b$. We have that $\gamma^k \gamma^l = B^{k+l} L_{B^{-l} b(k) + b(l)}$ and also $\gamma^k \gamma^l = \gamma^{k+l} = B^{k+l} L_{b(k+l)}$. Hence, by using these relations and condition (ε_2) we get

$$\begin{aligned} \varepsilon(\gamma^k L_\lambda \gamma^l L_{\lambda'}) &= \varepsilon(B^k L_{b(k)+\lambda} B^l L_{b(l)+\lambda'}) = \varepsilon(B^{k+l} L_{B^{-l}(b(k)+\lambda)+b(l)+\lambda'}) \\ &= \varepsilon(\gamma^{k+l} L_{B^{-l}\lambda+\lambda'}) = \varepsilon(\gamma)^{k+l} \varepsilon(L_\lambda)\varepsilon(L'_\lambda) = \varepsilon(\gamma^k L_\lambda)\varepsilon(\gamma^l L_{\lambda'}) \end{aligned}$$

for any $\lambda, \lambda' \in \Lambda$.

It is clear that the number of spin structures is either 2^r, for some $r \ge 1$, or 0, in case the equations given by conditions (ε_1) and (ε_2) are incompatible ones. Since each of these equations divides by 2 the number of spin structures and the covering torus has exactly 2^n such structures we have that $r \le n$. This completes the proof. □

Since spin manifolds are orientable, we need to restrict ourselves to the manifolds $M_{j,k}$ with $k = 2k_0$ even. We have the following result

Corollary 2.3. *Every orientable $\mathbb{Z}_4$-manifold $M_{j,k}$, $k = 2k_0$, has 2^{n-j} spin structures ε parametrized by the $(n+1)$-tuples $(\delta_1, \ldots, \delta_n, \sigma u_{B_{j,k}})$ as in (2.5) satisfying:*

$$\delta_1 = \delta_2, \ \cdots, \ \delta_{2j-1} = \delta_{2j} \qquad \text{and} \qquad \delta_n = (-1)^j \tag{2.9}$$

where $u_{B_{j,k}} = \left(\frac{\sqrt{2}}{2}\right)^j (1 + e_1 e_2) \cdots (1 + e_{2j-1} e_{2j})\, e_{2j+1} \cdots e_{2j+k}$.

Proof. We first note that $B_{j,k} = x_0(\underbrace{\tfrac{\pi}{2}, \ldots, \tfrac{\pi}{2}}_{j}, \underbrace{\pi, \ldots, \pi}_{k_0}, \underbrace{0, \ldots, 0}_{[\frac{l}{2}]}) \in T_0$ in the notation of (1.1). Hence, $\mu^{-1}(B_{j,k}) = \pm x(\frac{\pi}{4}, \ldots, \frac{\pi}{4}, \frac{\pi}{2}, \ldots, \frac{\pi}{2}, 0 \ldots, 0) \in T$, by (1.3) and we take $u_{B_{j,k}} = x(\frac{\pi}{4}, \ldots, \frac{\pi}{4}, \frac{\pi}{2}, \ldots, \frac{\pi}{2}, 0 \ldots, 0)$.

Let $\gamma_{j,k} = B_{j,k} L_{\frac{e_n}{4}}$. Since $\varepsilon(\gamma_{j,k}) = \sigma u_{B_{j,k}}$ and $\gamma_{j,k}^4 = L_{e_n}$ for every j,k, condition (ε_1) gives

$$\delta_n = \varepsilon(\gamma_{j,k})^4 = u_{B_{j,k}}^4 = x(\underbrace{\pi, \ldots, \pi}_{j}, \underbrace{2\pi, \ldots, 2\pi}_{k_0}, 0, \ldots, 0) = (-1)^j,$$

where we have used (1.2) in the third equality. Moreover, condition (ε_2) gives $1 = \varepsilon(L_{(B_{j,k} - Id) e_{2i-1}}) = \delta_{2i-1}\delta_{2i}$ for $1 \le i \le j$, hence (2.9) holds. Since there are no more relations imposed on the δ_i's the result follows. □

Remark 2.4. It would be natural to consider the larger family $\mathcal{F}^n$ with matrices in (2.1) replaced by

$$B_{j,h,k} = \mathrm{diag}(\underbrace{\tilde{J}, \ldots, \tilde{J}}_{j \ge 1}, \underbrace{J, \ldots, J}_{h \ge 0}, \underbrace{-1, \ldots, -1}_{k \ge 0}, \underbrace{1, \ldots, 1}_{l \ge 0}), \tag{2.10}$$

where $J := \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, $2(j+h)+k+l=n$ and $h+l \neq 0$. That is, consider $\mathcal{G}^n = \{M_{j,h,k} := \Gamma_{j,h,k}\backslash\mathbb{R}^n\}$ where $\Gamma_{j,h,k} = \langle B_{j,h,k} L_{\frac{e_n}{4}}, \Lambda\rangle$ if $l \geq 1$ and $\Gamma_{j,h,k} = \langle B_{j,h,k} L_{\frac{e_{2j+1}}{2}}, \Lambda\rangle$ if $l=0$.

However, this larger family of $\mathbb{Z}_4$-manifolds has trivial eta series unless $h=0$ and $l=1$, i.e. the case previously considered. In fact, we have $n_{B_{j,h,k}} \geq 2$ for every j,h,k with $h+l\geq 2$ and, by Corollary 2.6 in [**MP2**], the spectrum of D is symmetric. The remaining case ($h=1$ and $l=0$) is more involved, but one checks it by proceeding similarly as in the next section.

3. The spectrum of the Dirac operator

Since we are looking for spectral asymmetry of D, by Corollary 2.6 in [**MP2**], we will restrict ourselves to orientable odd dimensional manifolds in $\mathcal{F}^n$ with $F_1 \neq \emptyset$ (recall that $F_1 = \{B \in F : \dim(\mathbb{R}^n)^B = 1\}$). Hence, we fix $n = 2m+1$ and take

$$\mathcal{F}_1^n := \{M_{j,k} \in \mathcal{F}^n \,:\, k = 2k_0,\, l=1\}. \tag{3.1}$$

Also, for each $M_{j,k} \in \mathcal{F}_1^n$ we choose, in the notation of (1.1) and Corollary 2.3, the spin structure

$$\varepsilon_{j,k}^{\sigma} = \big(1,\ldots,1,(-1)^j,\sigma x(\underbrace{\tfrac{\pi}{4},\ldots,\tfrac{\pi}{4}}_{j},\underbrace{\tfrac{\pi}{2},\ldots,\tfrac{\pi}{2}}_{\frac{k}{2}})\big), \qquad \sigma\in\{\pm1\}. \tag{3.2}$$

For simplicity, we will use the following shorter notation

$$x_{k_1,\ldots,k_t}(\theta_1,\ldots,\theta_t) := x(\underbrace{\theta_1,\ldots,\theta_1}_{k_1},\ldots,\underbrace{\theta_t,\ldots,\theta_t}_{k_t}). \tag{3.3}$$

By using expression (1.8), we will explicitly compute the multiplicity of the eigenvalues of the Dirac operator D of the spin manifolds $(M_{j,k},\varepsilon_{j,k}^{\sigma})$.

We shall need the following auxiliary function. Let $\omega(j) := \frac{3-(-1)^j}{2}$, i.e.

$$\omega(j) = \begin{cases} 1 & \text{if } j \text{ is even} \\ 2 & \text{if } j \text{ is odd.} \end{cases} \tag{3.4}$$

Theorem 3.1. *Let $n = 2m+1 = 4r+3$. The $\mathbb{Z}_4$-manifolds $M_{j,k} \in \mathcal{F}_1^n$ with spin structures $\varepsilon_{j,k}^{\sigma}$ as in (3.2) have asymmetric Dirac spectrum and, in the notation of (1.6), the multiplicity of the non zero eigenvalue $\pm 2\pi\mu$ of D is given by*

$$d_{\mu}^{\pm}(\varepsilon_{j,k}^{\sigma}) = \begin{cases} 4^{r-1}|\Lambda_{\varepsilon_{j,k}^{\sigma},\mu}| \pm \sigma\,(-1)^{r+[\frac{t}{\omega(j)}]}\,2^{m-\omega(j)-[\frac{j}{2}]} & \mu = \frac{2t+1}{\omega(j)} \\ 4^{r-1}|\Lambda_{\varepsilon_{j,k}^{\sigma},\mu}| & \textit{otherwise} \end{cases} \tag{3.5}$$

for $k\neq 0$, and by

$$d_{\mu}^{\pm}(\varepsilon_{m,0}^{\sigma}) = \begin{cases} 4^{r-1}|\Lambda_{\varepsilon_{m,0}^{\sigma},\mu}| \pm (-1)^r\,2^{r-1}\big((-1)^t\,2^r + \sigma\,(-1)^{[\frac{t}{2}]}\big) & \mu = \frac{2t+1}{2} \\ 4^{r-1}|\Lambda_{\varepsilon_{m,0}^{\sigma},\mu}| & \textit{otherwise} \end{cases}$$

for $k=0$, where $t\in\mathbb{N}_0$ in both cases and $\omega(j)$ is as in (3.4).

Furthermore, for every $M_{j,k}\in\mathcal{F}_1^n$ there are no non-trivial harmonic spinors, that is, $d_0^{\pm}(\varepsilon_{j,k}^{\sigma}) = 0$.

Proof. For fixed j,k, let $\gamma = B_{j,k}L_{\frac{e_n}{4}}$ be the generator of $\Gamma_{j,k}$. For $1 \le h \le 3$, let $b_h \in \mathbb{R}^n$ be defined by $\gamma^h = B_{j,k}^h L_{b_h}$ and we put

$$S_h^{\pm}(\mu) := \sum_{u \in (\Lambda_{\varepsilon_{j,k}^{\sigma},\mu})^{B^h}} e^{-2\pi i u\cdot b_h}\ \chi_{L_{n-1}^{\pm\sigma(u,x_{\gamma^h})}}(x_{\gamma^h}). \tag{3.6}$$

Since $F_1(\Gamma_{j,k}) \ne \emptyset$, because $n_B = n_{B^3} = 1$, and since $n_{B^2} = 1$ if and only if $k = 0$, the formula in (1.8) now reads

$$d_\mu^{\pm}(\varepsilon_{j,k}^{\sigma}) = \tfrac{1}{4}\Big(2^{m-1}\,|\Lambda_{\varepsilon_{j,k}^{\sigma},\mu}| \ +\ S_1^{\pm}(\mu) + \delta_{k,0}\,S_2^{\pm}(\mu) + S_3^{\pm}(\mu)\Big), \tag{3.7}$$

where $\delta_{k,0}$ is Kronecker's delta function.

Now, since $\varepsilon_{j,k}^{\sigma}(\gamma) \in T$, the maximal torus in $\mathrm{Spin}(n-1)$, we can take $x_\gamma = \varepsilon_{j,k}^{\sigma}(\gamma) = \sigma x_{_{j,k_0}}(\frac{\pi}{4},\frac{\pi}{2})$, $x_{\gamma^2} = x_{_{j,k_0}}(\frac{\pi}{2},\pi)$ and $x_{\gamma^3} = \sigma x_{_{j,k_0}}(\frac{3\pi}{4},\frac{3\pi}{2})$ since

$$x_{\gamma^h} = \varepsilon(\gamma^h) = \varepsilon(\gamma)^h = (\sigma x_\gamma)^h$$

and $x(\theta_1,\dots,\theta_m)^h = x(h\theta_1,\dots,h\theta_m)$ for $h \in \mathbb{N}$ (see (1.2)). Furthermore, since $B_{j,k}\,e_n = e_n$ and for $u = e_n$ we can take $h_u = 1$, then $\sigma(e_n, x_{\gamma^h}) = 1$ for each $1 \le h \le 3$. Moreover, $\sigma(\mu e_n, x_{\gamma^h}) = 1$ and $\sigma(-\mu e_n, x_{\gamma^h}) = -1$, because $\mu > 0$ (see Preliminaries). On the other hand $b_h = \frac{he_n}{4}$.

Since Λ is the canonical lattice $\mathbb{Z}^n$ then $\Lambda^* = \Lambda$ and we can write $\Lambda_\varepsilon^* = \Lambda + u_\varepsilon$ where $u_\varepsilon = \sum_{\{i:\delta_i=-1\}} e_i$. Furthermore, since $\delta_i = 1$ for $1 \le i \le n-1$ and $\delta_n = (-1)^j$ (see (3.2)), we have that $\Lambda^*_{\varepsilon_{j,k}^{\sigma}} = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$ if j is even and $\Lambda^*_{\varepsilon_{j,k}^{\sigma}} = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_{n-1} \oplus (\mathbb{Z}+\frac{1}{2})e_n$ if j is odd. Thus,

$$(\Lambda^*_{\varepsilon_{j,k}^{\sigma}})^{B_{j,k}^h} = \begin{cases} \mathbb{Z}e_n & j \text{ even} \\ (\mathbb{Z}+\frac{1}{2})e_n & j \text{ odd.} \end{cases} \tag{3.8}$$

Clearly, if μ is such that $(\Lambda^*_{\varepsilon_{j,k}^{\sigma},\mu})^{B_{j,k}^h} = \emptyset$ for every $1 \le h \le 3$, only the identity of Γ can give a non-zero contribution to (1.8) and the multiplicity formula now reads $d_\mu^{\pm}(\varepsilon_{j,k}^{\sigma}) = 4^{r-1}\,|\Lambda_{\varepsilon_{j,k}^{\sigma},\mu}|$. Thus, from now on, we will assume that μ satisfies $(\Lambda^*_{\varepsilon_{j,k}^{\sigma},\mu})^{B_{j,k}^h} \ne \emptyset$ for some $1 \le h \le 3$. Then, we have $(\Lambda^*_{\varepsilon_{j,k}^{\sigma},\mu})^{B_{j,k}^h} = \{\pm\mu e_n\}$ with $\mu \in \mathbb{N}$ for j even and $\mu \in \mathbb{N}_0 + \frac{1}{2}$ for j odd and hence we have

$$S_h^{\pm}(\mu) = e^{-\frac{\pi}{2}ih\mu}\ \chi_{L_{n-1}^{\pm}}(x_{\gamma^h})\ +\ e^{\frac{\pi}{2}ih\mu}\ \chi_{L_{n-1}^{\mp}}(x_{\gamma^h}). \tag{3.9}$$

If, furthermore, $\chi_{L_{n-1}^{-}}(x_{\gamma^h}) = -\chi_{L_{n-1}^{+}}(x_{\gamma^h})$ holds, then

$$S_h^{\pm}(\mu) = (e^{-\frac{\pi}{2}ih\mu} - e^{\frac{\pi}{2}ih\mu})\,\chi_{L_{n-1}^{\pm}}(x_{\gamma^h}) = -2\,i\sin(\tfrac{\pi h\mu}{2})\,\chi_{L_{n-1}^{\pm}}(x_{\gamma^h}). \tag{3.10}$$

By (1.4), the characters $\chi_{L_{n-1}^{\pm}}(x_{\gamma^h})$ have the expression

$$\chi_{L_{n-1}^{\pm}}(x_{\gamma^h}) = \sigma^h 2^{m-1}\Big((\cos(\tfrac{h\pi}{4}))^j(\cos(\tfrac{h\pi}{2}))^{k_0} \pm i^m(\sin(\tfrac{h\pi}{4}))^j(\sin(\tfrac{h\pi}{2}))^{k_0}\Big).$$

The explicit values, for $1 \le h \le 3$, are given in the following table:

$\chi_{L^{\pm}_{n-1}}(x_{\gamma^h})$	$k>0$	$k=0$
$h=1$	$\pm\sigma\, 2^{m-1} i^m\, (\frac{\sqrt{2}}{2})^j$	$\sigma\, 2^{m-1}\, (\frac{\sqrt{2}}{2})^m (1 \pm i^m)$
$h=2$	0	$\pm 2^{m-1}\, i^m$
$h=3$	$\pm\sigma\, 2^{m-1}\, i^m\, (\frac{\sqrt{2}}{2})^j (-1)^{k_0}$	$\sigma\, 2^{m-1}\, (\frac{\sqrt{2}}{2})^m\, ((-1) \pm i^m)$

(3.11)

<u>*Case 1: $k > 0$.*</u> Let $h = 1$ or 3. Suppose that j is even, then $\mu \in \mathbb{N}$. If μ is even, $S^{\pm}_h(\mu) = 0$ because $\sin(\frac{\pi h\mu}{2}) = 0$. Let $\mu = 2t + 1$ with $t \in \mathbb{N}_0$. Since $\sin(\frac{(2t+1)h\pi}{2}) = (-1)^{t+[\frac{h}{2}]}$, by (3.10) we have

$$S^{\pm}_h(\mu) = -2\, i\, (-1)^{\frac{\mu-1}{2}+[\frac{h}{2}]}\, \chi_{L^{\pm}_{n-1}}(x_{\gamma^h}).$$

Replacing these values in (3.7) we get

$$\begin{aligned} d^{\pm}_{\mu}(\varepsilon^{\sigma}_{j,k}) &= \tfrac{1}{4}\Big(2^{m-1}|\Lambda_{\varepsilon^{\sigma}_{j,k},\mu}| - 2\, i\, (-1)^t \big(\chi_{L^{\pm}_{n-1}}(x_{\gamma}) - \chi_{L^{\pm}_{n-1}}(x_{\gamma^3})\big)\Big) \\ &= \tfrac{1}{4}\Big(2^{m-1}|\Lambda_{\varepsilon^{\sigma}_{j,k},\mu}| \pm \sigma\, (-1)^{t+1}\, 2^m\, i^{m+1} (\tfrac{\sqrt{2}}{2})^j \big(1 - (-1)^{k_0}\big)\Big) \\ &= 4^{r-1}|\Lambda_{\varepsilon^{\sigma}_{j,k},\mu}| \pm \sigma\, (-1)^{t+r}\, 2^{m-1-[\frac{j}{2}]} \end{aligned} \tag{3.12}$$

where we have used that $m = 2r + 1$ and that k_0 is odd, because j is even and $n = 2(j + k_0) + 1 \equiv 3$ (4).

Now, if j is odd, $\mu = \frac{2t+1}{2}$ with $t \in \mathbb{N}_0$. One has that

$$\sin\big(\tfrac{\pi(2t+1)}{4}\big) = \sin\big(\tfrac{3\pi(2t+1)}{4}\big) = (-1)^{[\frac{t}{2}]}(\tfrac{\sqrt{2}}{2}). \tag{3.13}$$

Then, by using that k_0 is even because j is odd, we get that

$$\begin{aligned} d^{\pm}_{\mu}(\varepsilon^{\sigma}_{j,k}) &= \tfrac{1}{4}\Big(2^{m-1}|\Lambda_{\varepsilon^{\sigma}_{j,k},\mu}| - 2\, i\, (-1)^{[\frac{t}{2}]}(\tfrac{\sqrt{2}}{2})\big(\chi_{L^{\pm}_{n-1}}(x_{\gamma}) + \chi_{L^{\pm}_{n-1}}(x_{\gamma^3})\big)\Big) \\ &= \tfrac{1}{4}\Big(2^{m-1}|\Lambda_{\varepsilon^{\sigma}_{j,k},\mu}| \pm \sigma\, (-1)^{[\frac{t}{2}]+1}\, 2^m\, i^{m+1} (\tfrac{\sqrt{2}}{2})^{j+1} \big(1 + (-1)^{k_0}\big)\Big) \\ &= 4^{r-1}|\Lambda_{\varepsilon^{\sigma}_{j,k},\mu}| \pm \sigma\, (-1)^{r+[\frac{t}{2}]}\, 2^{m-2-[\frac{j}{2}]}. \end{aligned} \tag{3.14}$$

Taking $\mu = \frac{2t+1}{\omega(j)}$, with $\omega(j) = \frac{3-(-1)^j}{2}$, from (3.12) and (3.14) we finally obtain expression (3.5).

<u>*Case 2: $k = 0$.*</u> Then j is odd and $\mu = \frac{2t+1}{2}$ with $t \in \mathbb{N}_0$. We have that

$$S^{\pm}_2(\mu) = \pm(-1)^{t+1}\, 2^m\, i^{m+1} = \pm(-1)^{t+r}\, 2^{2r+1}, \tag{3.15}$$

by (3.10); and, for $h = 1, 3$, by (3.11) we obtain

$$\begin{aligned} S^{\pm}_h(\mu) &= \sigma\, 2^{m-1}(\tfrac{\sqrt{2}}{2})^m \Big(e^{-\frac{\pi}{2} ih\mu}\big((-1)^{[\frac{h}{2}]} \pm i^m\big) + e^{\frac{\pi}{2} ih\mu}\big((-1)^{[\frac{h}{2}]} \mp i^m\big)\Big) \\ &= \sigma\, 2^{m-1}(\tfrac{\sqrt{2}}{2})^m \Big((-1)^{[\frac{h}{2}]} 2\cos(\tfrac{\pi h\mu}{2}) \pm i^m(-2i)\sin(\tfrac{\pi h\mu}{2})\Big). \end{aligned}$$

Then, since $\cos\big(\frac{\pi(2t+1)}{4}\big) - \cos\big(\frac{3\pi(2t+1)}{4}\big) = 0$, by using (3.13) we have

$$\begin{aligned} S^{\pm}_1(\mu) + S^{\pm}_3(\mu) &= \mp\sigma\, 2^{m+1}\, (\tfrac{\sqrt{2}}{2})^{m+1} \sin(\tfrac{\pi(2t+1)}{4}) \\ &= \pm\sigma\, (-1)^{r+[\frac{t}{2}]}\, 2^{r+1}. \end{aligned} \tag{3.16}$$

By introducing the values of (3.15) and (3.16) in (3.7), we get to the expressions in the statement of the proposition.

Finally, the claim concerning to the multiplicity of the 0-eigenvalue follows directly from the expressions just after (1.4) and (1.8), respectively. □

Remark 3.2. From the multiplicity formulae obtained in Proposition 3.1 we see that, generically, there are no Dirac isospectrality between manifolds in $\mathcal{F}_1^n$.

On the other hand, if $m = 2r$, that is, if $n = 4r + 1$, then the spectrum of D is symmetric with multiplicities given by $d_\mu^{\pm}(\varepsilon_{j,k}^\sigma) = 4^{r-1}|\Lambda_{\varepsilon_{j,k}^\sigma,\mu}|$ for every pair j, k. This follows by (3.12) and (3.14) in the proof of Theorem 3.1, in the case $k > 0$, and by (3.7) for $k = 0$, since both (3.15) and (3.16) vanish in this case. Hence, for a fixed n, $\{(M_{2j,k}, \varepsilon_{2j,k}^\sigma) : 1 \leq j \leq r\}$, is a set of m mutually D-isospectral $\mathbb{Z}_4$-manifolds, and similarly for $\{(M_{2j+1,k}, \varepsilon_{2j,k}^\sigma) : 1 \leq j \leq r\}$.

4. Eta series and eta invariants of $\mathbb{Z}_4$-manifolds

In general, for a differential operator A having positive and negative eigenvalues we can decompose the spectrum $Spec_A(M) = \mathcal{S} \dot\cup \mathcal{A}$ where $\mathcal{S}$ and $\mathcal{A}$ are the symmetric and the asymmetric components of the spectrum, respectively. That is, if $\lambda = 2\pi\mu$, $\lambda \in \mathcal{S}$ if and only if $d_\mu^+(M) = d_\mu^-(M)$. We say that $Spec_A(M)$ is symmetric if $\mathcal{A} = \emptyset$.

As a measure of the spectral asymmetry of a differential operator A on a manifold M, Atiyah, Patodi and Singer introduced the *eta series*

$$\eta_A(s) = \sum_{\substack{\lambda \in Spec_D(M) \\ \lambda \neq 0}} \frac{\operatorname{sign}(\lambda)}{|\lambda|^s} = \frac{1}{(2\pi)^s} \sum_{\mu \in \frac{1}{2\pi}\mathcal{A}} \frac{d_\mu^+(M) - d_\mu^-(M)}{|\mu|^s} \tag{4.1}$$

generalizing the zeta functions for the Laplacian. It is known that this series converges absolutely for $Re(s) > \frac{n}{d}$, where d is the order of A, and defines a holomorphic function $\eta_A(s)$ in this region, having a meromorphic continuation to $\mathbb{C}$ that is holomorphic at $s = 0$ ([**APS2**], [**Gi2**]). The *eta invariant* is defined by $\eta_A := \eta_A(0)$. It is known that if $n \not\equiv 3 \mod (4)$ then $\eta(s) \equiv 0$ for every Riemannian manifold M (see [**Fr**]).

Now, we let $A = D$, the Dirac operator. By using the results obtained in the previous section we shall compute the expression for the eta series and the values of the η-invariants for the spin $\mathbb{Z}_4$-manifolds considered. We have the following result

Theorem 4.1. *Let $n = 2m + 1 = 4r + 3$. The eta series for the $\mathbb{Z}_4$-manifolds $M_{j,k} \in \mathcal{F}_1^n$ with spin structures $\varepsilon_{j,k}^\sigma$ as in (3.2) are given for $k > 0$ by*

$$\eta_{\varepsilon_{j,k}^\sigma}(s) = \frac{C_{j,\sigma}}{(8\pi)^s} \sum_{h=0}^{2\omega(j)-1} (-1)^{[\frac{h+\omega(j)}{2}]} \, \zeta\big(s, \tfrac{2h+1}{4\omega(j)}\big) \tag{4.2}$$

with $C_{j,\sigma} = \sigma\,(-1)^r\,2^{m+1-\omega(j)-[\frac{j}{2}]}$, *and by*

$$\eta_{\varepsilon^\sigma_{m,0}}(s) = \frac{(-1)^r\,2^r}{(8\pi)^s}\sum_{h=0}^{3}\big(\sigma + (-1)^{[\frac{h+1}{2}]}\,2^r\big)\,\zeta\big(s, \tfrac{2h+1}{8}\big) \tag{4.3}$$

where $\zeta(s,\alpha) = \sum_{j=0}^{\infty}(j+\alpha)^{-s}$ *is the Hurwitz zeta function for* $Re(s) > 1$, $\alpha \in (0,1]$, *and* $\omega(j)$ *is as defined in (3.4).*

Furthermore, the meromorphic continuation of $\eta_{\varepsilon^\sigma_{j,k}}(s)$ *to* $\mathbb{C}$ *is everywhere holomorphic for all manifolds* $M_{j,k} \in \mathfrak{F}^n_1$.

Note. Observe that, for $k > 0$, all the eta functions $\{\eta_{\varepsilon^\sigma_{2j,k}}(s)\}$ are mutually proportional and the same happens with $\{\eta_{\varepsilon^\sigma_{2j+1,k}}(s)\}$.

Proof. By Proposition 3.1 we have that

$$d^+_\mu(\varepsilon^\sigma_{j,k}) - d^-_\mu(\varepsilon^\sigma_{j,k}) = \begin{cases} \sigma\,(-1)^{r+[\frac{t}{\omega(j)}]}\,2^{m+1-\omega(j)-[\frac{j}{2}]} & k > 0 \\ (-1)^r\,2^r\big(\sigma\,(-1)^{[\frac{t}{2}]} + (-1)^t 2^r\big) & k = 0. \end{cases} \tag{4.4}$$

Now, by (4.1) and (4.4), in the case $k > 0$, we have

$$\eta_{\varepsilon^\sigma_{j,k}}(s) = \frac{C_{j,\sigma}}{(2\pi)^s}\sum_{t=0}^{\infty}\frac{(-1)^{[\frac{t}{\omega(j)}]}}{(\frac{2t+1}{\omega(j)})^s} \tag{4.5}$$

where $C_{j,\sigma} = \sigma\,(-1)^r\,2^{m+1-\omega(j)-[\frac{j}{2}]}$.

If j is even, the series in (4.5) equals

$$\sum_{t=0}^{\infty}\frac{(-1)^t}{(2t+1)^s} = \frac{1}{4^s}\left(\sum_{t=0}^{\infty}\frac{1}{(t+\frac14)^s} - \sum_{t=0}^{\infty}\frac{1}{(t+\frac34)^s}\right) = \frac{1}{4^s}\big(\zeta(s,\tfrac14) - \zeta(s,\tfrac34)\big).$$

where we have separated the contributions of $2t$ and $2t+1$, and hence in this case

$$\eta_{\varepsilon^\sigma_{j,k}}(s) = \frac{\sigma\,(-1)^r\,2^{m-[\frac{j}{2}]}}{(8\pi)^s}\big(\zeta(s,\tfrac14) - \zeta(s,\tfrac34)\big). \tag{4.6}$$

For j odd, the series in (4.5) now equals

$$\begin{aligned}\sum_{t=0}^{\infty}\frac{(-1)^{[\frac{t}{2}]}}{(t+\frac12)^s} &= \frac{1}{4^s}\sum_{t=0}^{\infty}\frac{1}{(t+\frac18)^s} + \frac{1}{(t+\frac38)^s} - \frac{1}{(t+\frac58)^s} - \frac{1}{(t+\frac78)^s} \\ &= \frac{1}{4^s}\left(\zeta(s,\tfrac18) + \zeta(s,\tfrac38) - \zeta(s,\tfrac58) - \zeta(s,\tfrac78)\right).\end{aligned}$$

where we have separated the contributions of $4t+h$, with $0 \le h \le 3$, and hence

$$\eta_{\varepsilon^\sigma_{j,k}}(s) = \frac{C_{j,\sigma}}{(8\pi)^s}\sum_{h=0}^{3}(-1)^{[\frac{h}{2}]}\,\zeta(s, \tfrac{2h+1}{8}). \tag{4.7}$$

By putting together expressions (4.6) and (4.7) we get formula (4.2).

On the other hand, for $k = 0$, by (4.1) and (4.4), we have

$$\eta_{\varepsilon^\sigma_{m,0}}(s) = \frac{(-1)^r\,2^r}{(2\pi)^s}\sum_{t=0}^{\infty}\frac{\sigma\,(-1)^{[\frac{t}{2}]} + (-1)^t 2^r}{(t+\frac12)^s}.$$

The series above equals

$$\sum_{t=0}^{\infty} \frac{(\sigma+2^r)}{(4t+\frac{1}{2})^s} + \frac{(\sigma-2^r)}{(4t+\frac{3}{2})^s} - \frac{(\sigma-2^r)}{(4t+\frac{5}{2})^s} - \frac{(\sigma+2^r)}{(4t+\frac{7}{2})^s}$$

where again we have separated the cases $4t+h$, $0 \le h \le 3$. Now, proceeding as before, we obtain

$$\eta_{\varepsilon^\sigma_{m,0}}(s) = \tfrac{(-1)^r 2^r}{(8\pi)^s} \Big((\sigma+2^r)\big(\zeta(s,\tfrac{1}{8}) - \zeta(s,\tfrac{7}{8})\big) + (\sigma-2^r)\big(\zeta(s,\tfrac{3}{8}) - \zeta(s,\tfrac{5}{8})\big)\Big). \quad (4.8)$$

From here it is clear that (4.3) holds.

The last assertion clearly follows from the explicit expressions for eta series obtained in (4.6), (4.7) and (4.8) since the Hurwitz zeta function $\zeta(s,\alpha)$ has a simple pole at $s=1$ with residue 1 (see [**Ap**]). □

Corollary 4.2. *The eta invariants of the spin $\mathbb{Z}_4$-manifolds $M_{j,k} \in \mathcal{F}^n_1$ with spin structures $\varepsilon^\sigma_{j,k}$ are given by*

$$\eta_{\varepsilon^\sigma_{j,k}}(0) = \begin{cases} \sigma\,(-1)^{r+[\frac{\omega(j)}{2}]}\, 2^{m-1-[\frac{j}{2}]} & k>0,\ 1 \le j < 2r+1 \\ (-1)^r\, 2^r\, (\sigma + 2^{r-1}) & k=0,\ j=2r+1. \end{cases} \quad (4.9)$$

where $\omega(j)$ is as in (3.4). *In particular $\eta_{\varepsilon^\sigma_{j,k}}(0) \in \mathbb{Q} \smallsetminus \{0\}$.*

Proof. This is a consequence of the expressions given in Theorem 4.1 and the fact that $\zeta(0,\alpha) = \frac{1}{2} - \alpha$ for every $\alpha \in (0,1]$. □

We now illustrate the results in the lowest dimensions considered, that is $n=3$ and $n=7$.

Example 4.3. For $n=3$ there is only one manifold in $\mathcal{F}^3_1$, namely $M_{1,0}$ where $B_{1,0} = \left[\begin{smallmatrix} \tilde{J} & \\ & 1 \end{smallmatrix}\right]$. Since $r=0$ and $k=0$, by (4.3) we have

$$\eta_{\varepsilon^+_{1,0}}(s) = \tfrac{2}{(8\pi)^s}\big(\zeta(s,\tfrac{1}{8}) - \zeta(s,\tfrac{7}{8})\big), \qquad \eta_{\varepsilon^-_{1,0}}(s) = \tfrac{-2}{(8\pi)^s}\big(\zeta(s,\tfrac{3}{8}) - \zeta(s,\tfrac{5}{8})\big)$$

and by (4.9)

$$\eta_{\varepsilon^+_{1,0}}(0) = \tfrac{3}{2}, \qquad \eta_{\varepsilon^-_{1,0}}(0) = -\tfrac{1}{2}.$$

This is in agreement with the values obtained in [**Pf**].

Example 4.4. For $n=7$ there are 3 manifolds in $\mathcal{F}^7_1$. They are $M_{1,4}$, $M_{2,2}$ and $M_{3,0}$ where $B_{1,4} = \begin{bmatrix} \tilde{J} & & & \\ & -I & & \\ & & -I & \\ & & & 1 \end{bmatrix}$, $B_{2,2} = \begin{bmatrix} \tilde{J} & & & \\ & \tilde{J} & & \\ & & -I & \\ & & & 1 \end{bmatrix}$, $B_{3,0} = \begin{bmatrix} \tilde{J} & & & \\ & \tilde{J} & & \\ & & \tilde{J} & \\ & & & 1 \end{bmatrix}$ and $-I = \left[\begin{smallmatrix} -1 & \\ & -1 \end{smallmatrix}\right]$. Now, by (4.2) and (4.3) we get

$$\begin{aligned}
\eta_{\varepsilon^\sigma_{1,4}}(s) &= \tfrac{-4\sigma}{(8\pi)^s}\Big(\big(\zeta(s,\tfrac{1}{8}) - \zeta(s,\tfrac{7}{8})\big) + \big(\zeta(s,\tfrac{3}{8}) - \zeta(s,\tfrac{5}{8})\big)\Big) \\
\eta_{\varepsilon^\sigma_{2,2}}(s) &= \tfrac{-4\sigma}{(8\pi)^s}\big(\zeta(s,\tfrac{1}{4}) - \zeta(s,\tfrac{3}{4})\big) \\
\eta_{\varepsilon^+_{3,0}}(s) &= \tfrac{-2}{(8\pi)^s}\Big(3\big(\zeta(s,\tfrac{1}{8}) - \zeta(s,\tfrac{7}{8})\big) - \big(\zeta(s,\tfrac{3}{8}) - \zeta(s,\tfrac{5}{8})\big)\Big) \\
\eta_{\varepsilon^-_{3,0}}(s) &= \tfrac{-2}{(8\pi)^s}\Big(-\big(\zeta(s,\tfrac{1}{8}) - \zeta(s,\tfrac{7}{8})\big) - 3\big(\zeta(s,\tfrac{3}{8}) - \zeta(s,\tfrac{5}{8})\big)\Big)
\end{aligned}$$

and, again by (4.9), also

$$\eta_{\varepsilon^\sigma_{1,4}}(0) = -4\sigma, \qquad \eta_{\varepsilon^\sigma_{2,2}}(0) = -2\sigma, \qquad \eta_{\varepsilon^+_{3,0}}(0) = -4, \qquad \eta_{\varepsilon^-_{3,0}}(0) = 3.$$

Remark 4.5. To conclude, we conjecture that for a compact flat manifold of dimension $4r+3$, with a "nice" integral holonomy representation, the eta series $\eta(s)$ can be put in terms of differences of Riemann-Hurwitz zeta functions $\zeta(s,\alpha)$, where $\alpha \in (0,1] \cap \mathbb{Q}$, and that the meromorphic continuation to $\mathbb{C}$ is holomorphic everywhere. Hence, from this expression, the η-invariant is easily computed simply by evaluation at $s=0$. More precisely, we claim that the eta series has the expression

$$\eta_{\Gamma,\varepsilon}(s) = \frac{C_{\Gamma,\varepsilon}}{(2\pi)^s} \sum_{j=1}^{N} f_{j,\Gamma,\varepsilon}(s)\, \big(\zeta(s,\alpha_j) - \zeta(s, 1-\alpha_j)\big)$$

where $N < |F|$, $C_{\Gamma,\varepsilon}$ is a constant depending on M and on the spin structure ε and each $f_{j,\Gamma,\varepsilon}(s)$ is an entire function (trigonometric or constant). The results in this paper, together with those in [**MP2**] bring support to this conjecture. We plan to get deeper into this question in the future.

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