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A NOTE ON THE L_1-MEAN ERGODIC THEOREM

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ABSTRACT. Let T be a positive contraction on L_1 of a σ-finite measure space. Necessary and sufficient conditions are given in order that for any f in L_1 the averages $\frac{1}{n}\sum_{i=0}^{n-1} T^i f$ converge in the norm of L_1.

1. Introduction.

Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space and let $L_1 = L_1(\Omega, \mathcal{A}, \mu)$ denote de usual Banach space of all real-valued integrable functions on Ω. A linear operator $T\colon L_1 \to L_1$ is called positive if $f \geq 0$ implies $Tf \geq 0$, and a contraction if $\|T\|_1 \leq 1$, with $\|T\|_1$ denoting the operator norm of T on L_1. We say that the pointwise ergodic theorem (resp. the L_1-mean ergodic theorem) holds for T if for any f in L_1 the ergodic averages

$$A_n(T)f = \frac{1}{n}\sum_{i=0}^{n-1} T^i f$$

converge a.e. on Ω (resp. in L_1-norm).

In 1964, Chacon [1] showed a class of positive contractions on L_1 for which the pointwise ergodic theorem fails to hold and Ito [4] proved that if a positive contraction on L_1 satisfies the L_1-mean ergodic theorem, then it satisfies the pointwise ergodic theorem. (cf. also Kim [5]).

More recently, Hasegawa and Sato [2] proved that if $P_1, \ldots, P_d$ are d commuting positive contractions on L_1 such that each P_i, $1 \leq i \leq d$, satisfies the L_1-mean ergodic theorem, and if $T_1, \ldots, T_d$ are (not necessarily commuting) contractions on L_1 such that $|T_i f| \leq P_i|f|$, for $1 \leq i \leq d$, then the averages

$$A_n(T_1, \ldots, T_d)f = A_n(T_1)\ldots A_n(T_d)f$$

converge a.e. for every f in L_1.

In view of these facts, it seems interesting to find out conditions on T which guarantee the validity of the L_1-mean ergodic theorem.

In order to state our result, we will need some definitions and previous results. Let us fix some notation. We denote the space of all nonnegative functions in L_1 and the space of all nonnegative functions in L_∞ by L_1^+ and L_∞^+, respectively. The adjoint operator of T, which acts on $L_\infty(\Omega, \mathcal{A}, \mu) = L_\infty$, is denoted by T^*. Put

$$S_\infty f = \sum_{i=0}^{\infty} T^i f,\ f \in L_1^+; \qquad S_\infty^* h = \sum_{i=0}^{\infty} {T^*}^i h,\ h \in L_\infty^+.$$

By Hopf decomposition, $\Omega = C \cup D$, where C and $D = \Omega \setminus C$ denote respectively the conservative and the dissipative part of Ω respect to T. We recall that C and D are determined uniquely mod μ by:

C$_1$) For all $f \in L_1^+$, $S_\infty f = \infty$ a.e. on $C \cap \{S_\infty f > 0\}$, and
D$_1$) For all $f \in L_1^+$, $S_\infty f < \infty$ a.e on D.

For any $A \in \mathcal{A}$, χ_A will be the characteristic function of the set A. Let us write

$$A_n(T^*)h = \frac{1}{n}\sum_{i=0}^{n-1} {T^*}^i h, \qquad h \in L_\infty.$$

From the results of Helmberg [3] and Lin and Sine [7] about the relationship between the validity of the L_1-mean ergodic theorem for T and the almost everywhere convergence of the averages $A_n(T^*)h$, $h \in L_\infty$, we have:

Theorem 1.1. [Helmberg; Lin-Sine] *Let T be a positive contraction on L_1. Then the following are equivalent:*

a) *The L_1-mean ergodic theorem holds for T.*
b) *For any h in L_∞, the averages $A_n(T^*)h$ converge a.e.*
c) $\lim {T^*}^n \chi_D = 0$ *a.e. and there exists a nonnegative function f in L_1 satisfying $Tf = f$ and $\{f > 0\} = C$.*

Recall that a subset K of L_1 is called weakly sequentially compact if every sequence $\{\phi_n\}$ of elements in K contains a subsequence $\{\phi_{n_k}\}$ which converges weakly to an element in L_1, that is, there exists ϕ in L_1 such that for any h in L_∞

$$\lim_k \int h\phi_{n_k}\, d\mu = \int h\phi\, d\mu.$$

Kim [5] has proved:

Theorem 1.2. [Kim] *Let T be a positive contraction on L_1. Suppose that the sequence*
$\{A_n(T)w\}_n$ is weakly sequentially compact for some $w > 0$ in L_1. Then for each f in L_1, $\lim A_n(T)f$ exists in the L_1-norm and almost everywhere.

The purpose of this paper is to prove the following result:

Theorem A. *Let T be a positive contraction on L_1. Then the following assertions are equivalent:*

i) *The L_1-mean ergodic theorem holds for T .*
ii) $\lim {T^*}^n \chi_D = 0$ *a.e. and there exists w in L_1^+ such that $\{w > 0\} = C$ and the sequence $\{A_n(T)w\}$ is weakly sequentially compact.*
iii) *There exists w in L_1^+ such that the averages $A_n(T)w$ converge a.e. to a function w_0 and* $\lim {T^*}^n \chi_{\{w_0=0\}} = 0$ *a.e.*

2. The proofs.

We refer the reader to Krengel's book [6] for a proof of the following properties related with Hopf decomposition of Ω:

P_1) For all $h \in L_\infty^+$, $S_\infty^* h = \infty$ a.e. on $C \cap \{S_\infty^* h > 0\}$.
P_2) If $h \in L_\infty^+$ and $T^* h \leq h$, then $T^* h = h$ a.e. on C.
P_3) $T^* \chi_D \leq \chi_D$ a.e.

By P_2) and P_3) we see that $T^* \chi_C = 1$ a.e. on C.
We start with the following lemmas.

Lemma 2.1. *Let f be a function in L_1^+ such that the averages $A_n(T)f$ converge a.e. Let us denote the limit function by f_0. Then $Tf_0 = f_0$ a.e.*

Proof. Because of the identity

$$TA_n(T)f = \frac{n+1}{n} A_{n+1}(T)f - \frac{f}{n},$$

we have $\lim TA_n(T)f = f_0$ a.e. Since $f_0 \in L_1$, the sequence

$$A_n(T)f \wedge f_0 := \min\{A_n(T)f, f_0\}$$

converges to f_0 in L_1-norm by Lebesgue's theorem. Then $T(A_n(T)f \wedge f_0)$ converges to Tf_0 in L_1-norm. It follows that $Tf_0 \leq f_0$ a.e. Then $S_\infty(f_0 - Tf_0) \leq f_0$ a.e. and by statement C_1) we see that $f_0 = Tf_0$ a.e. on C.

By D_1) we have $f_0 = 0$ a.e. on D and therefore $Tf_0 = 0$ on D. The lemma is proved. □

Lemma 2.2. *Let f in L_1^+ such that $Tf = f$ and let $C_0 = \{f > 0\}$, $D_0 = \{f = 0\}$. Then we have:*

i) *$T^* \chi_{D_0} \leq \chi_{D_0}$ a.e.*
ii) *$C_0 \subset C$ and $T^* \chi_{C_0} = \chi_{C_0} + h$, with $h \in L_\infty^+$ such that $\{h > 0\} \subset D$ and $S_\infty^* h \leq 1$ a.e.*

Proof. Being T^* a positive contraction on L_∞, i) follows from

$$0 = \int \chi_{D_0} f \, d\mu = \int f T^* \chi_{D_0} \, d\mu.$$

Statement C_1) implies $C_0 \subset C$ and from

$$\int f \, d\mu = \int f \chi_{C_0} \, d\mu = \int f T^* \chi_{C_0} \, d\mu,$$

we conclude that $T^* \chi_{C_0} = 1$ a.e. on C_0. Then $T^* \chi_{C_0} = \chi_{C_0} + h$, with h in L_∞^+ and $\{h > 0\} \subset D_0$. From this, it is easy to see that for all n

$$T^{*n} \chi_{C_0} = \chi_{C_0} + \sum_{k=0}^{n-1} T^{*k} h.$$

Consequently, $S_\infty^* h = \lim T^{*n} \chi_{C_0} - \chi_{C_0} \leq 1$ a.e. and by property P_1), $\{h > 0\} \subset D$. □

We are now ready to prove our result.

Proof of Theorem A. By virtue of theorem 1.1 it is sufficient to prove that the following assertions are equivalent:

i) $\lim T^{*n}\chi_D = 0$ a.e. and there exists w in L_1^+ such that $\{w > 0\} = C$ and the sequence $\{A_n(T)w\}$ is weakly sequentially compact.

ii) There exists w in L_1^+ such that the averages $A_n(T)w$ converge a.e. to a function w_0 and $\lim T^{*n}\chi_{\{w_0=0\}} = 0$ a.e.

iii) $\lim T^{*n}\chi_D = 0$ a.e. and there exists w in L_1^+ satisfying $Tw = w$ and $\{w > 0\} = C$.

The implications iii)$\Rightarrow$ ii) and iii)$\Rightarrow$ i) are immediate.

i)$\Rightarrow$ iii) By the mean ergodic theorem of Yosida and Kakutani [8], the sequence $\{A_n(T)w\}$ converges in L_1-norm to a function $w_0 \in L_1^+$ such that $Tw_0 = w_0$. Put $C_0 = \{w_0 > 0\}$ and $D_0 = \{w_0 = 0\}$. By i) of lemma 2.2 there exists the a.e. $\lim T^{*n}\chi_{D_0} = u$ and $\{u > 0\} \subset D_0$. Since $T^*u = u$ (see e.g. [3]) we have:

$$0 = \int uw_0\, d\mu = \int u \lim A_n(T)w\, d\mu = \lim \int uA_n(T)w\, d\mu = \int uw\, d\mu$$

where the third equality follows from the fact that $\{A_n(T)w\}$ being L_1-norm convergent is weakly convergent in L_1.

Then $\{u > 0\} \subset D$. Moreover, from $\lim T^{*n}\chi_D = 0$ a.e., we can see that $u = 0$ a.e.

On the other hand, from ii) of lemma 2.2 we obtain $T^{*n}\chi_{C_0} = \chi_{C_0}$ on C, for all n. As $T^{*n}\chi_C = 1$ on C, we deduce that for all n $T^{*n}\chi_{C\setminus C_0} = \chi_{C\setminus C_0}$ on C. Therefore, for all n, $\chi_{C\setminus C_0} = T^{*n}\chi_{C\setminus C_0} \le T^{*n}\chi_{D_0}$ a.e. on C. Thus $\mu(C\setminus C_0) = 0$ and iii) follows.

ii)$\Rightarrow$ iii) By lemma 2.1, $Tw_0 = w_0$. Put $C_0 = \{w_0 > 0\}$ and $D_0 = \{w_0 = 0\}$. By ii) of lemma 2.2, $D \subset D_0$. Then $\lim T^{*n}\chi_D = 0$ a.e. and $\lim T^{*n}\chi_{C\setminus C_0} = 0$ a.e. Now, iii) follows as in i)$\Rightarrow$ iii). □

Remarks.

1. In iii) of Theorem A, the condition $\lim T^{*n}\chi_{\{w_0=0\}} = 0$, can not be replaced by $\lim T^{*n}\chi_D = 0$. To see this, take $\tau\colon \Omega \to \Omega$ an ergodic, conservative measure preserving transformation with respect to μ, where μ is σ-finite and infinite. Then, the operator $Tf = f_0\tau$ satisfies the pointwise ergodic theorem, but the L_1-mean ergodic theorem does not hold for T.
2. Suppose the L_1-mean ergodic theorem holds for T. Then $\lim T^{*n}\chi_D = 0$ a.e. For each g in L_∞, we denote by g^* the a.e. limit of $A_n(T^*)g$. Now, let h in L_∞. Since $|A_n(T^*)(h\chi_D)| \le \|h\|_\infty A_n(T^*)\chi_D$, we have
$$h^* = \lim A_n(T^*)(h\chi_C) = (h\chi_C)^*.$$
Put $h_C^* = (h\chi_C)^*$. Then, for all n
$$h_C^* = T^{*n}h_C^* = T^{*n}(\chi_C h_C^*) + T^{*n}(\chi_D h_C^*),$$
and we conclude that $h^* = \lim T^{*n}(\chi_C h_C^*)$ a.e.

In fact, we have:

Proposition 2.3. *The following assertions are equivalent:*

i) $\lim T^{*n}\chi_D = 0$ *a.e.*

ii) *Let* u *in* L_∞. *Then* $T^*u = u$ *if and only if* $u = \lim T^{*n}(\chi_C u)$ *and* $T^*(\chi_C u) = \chi_C u$ *on* C.

Sketch of proof.

i)⇒ ii) follows from $T^*\chi_D \leq \chi_D$ a.e. and the fact that $u = \lim T^{*n}(\chi_C u)$ implies $T^*u = u$.

ii)⇒ i) let $h = \lim T^{*n}\chi_D$. Then $T^*h = h$ and $\{h > 0\} \subset D$.

□

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