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MOORE-PENROSE INVERSE AND DOUBLY COMMUTING ELEMENTS IN C^*-ALGEBRAS

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This work is dedicated to the memory of Professor Angel R. Larotonda, Pucho for all who knew him.

ABSTRACT. In this work it is proved that the Moore-Penrose inverse of the product of n-doubly commuting regular C^*-algebra elements obeys the so-called reverse order law. Conversely, conditions regarding the reverse order law of the Moore-Penrose inverse are given in order to characterize when n-regular elements doubly commute. Furthermore, applications of the main results of this article to normal C^*-algebra elements, to Hilbert space operators and to Calkin algebras will be considered.

1. Introduction

Consider an unitary ring A. An element $a \in A$ will be said *regular* if it has a *generalized inverse* in A, that is if there exists $b \in A$ such that

$$a = aba.$$

A generalized inverse is also termed a *pseudo inverse*.

Note that if a is a regular element of A and b is a generalized inverse of a, then $p = ba$ and $q = ab$ are *idempotents* of A, that is $p = p^2$ and $q = q^2$.

Given $a \in A$ a regular element, a generalized inverse b of a will be called *normalized*, if b is regular and a is a pseudo inverse of b, equivalently if

$$a = aba, \qquad b = bab.$$

Recall that if b is a generalized inverse of a, then $c = bab$ is a normalized pseudo inverse of a.

Next suppose that a is a regular element and b is a normalized generalized inverse of a. In the presence of an involution $*\colon A \to A$, it is possible to enquire if the idempotents p and q are *self-adjoint*, that is whether $(ba)^* = ba$ and $(ab)^* = ab$. In this case b is called the *Moore-Penrose inverse* of a, see [16] where this concept was introduced.

In [10] it was proved that each regular element a in a C^*-algebra A has a uniquely determined Moore-Penrose inverse. The Moore-Penrose inverse of $a \in A$

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will be denote by $a^\dagger$. Therefore, the Moore-Penrose inverse of a regular element $a \in A$ is the unique element that satisfy the following equations:

$$a = aa^\dagger a, \qquad a^\dagger = a^\dagger aa^\dagger, \qquad (a^\dagger a)^* = a^\dagger a, \qquad (aa^\dagger)^* = aa^\dagger.$$

According to the uniqueness of the notion under consideration, if a has a Moore-Penrose inverse, then a^* also has a Moore-Penrose inverse and

$$(a^*)^\dagger = (a^\dagger)^*.$$

Moreover, according to the above equations, if a is a regular element, then $a^\dagger$ also is and

$$(a^\dagger)^\dagger = a.$$

For other properties regarding Moore-Penrose inverses in C^*-algebras, see the works [10], [11], [14] and [16].

On the other hand, an n-tuple $a = (a_1, \ldots, a_n)$ of elements in a C^*-algebra A will be called *doubly commuting*, if $a_i a_j = a_j a_i$ and $a_i a_j^* = a_j^* a_i$, for all i, $j = 1, \ldots, n$, $i \neq j$. For instance, necessary and sufficient for (a, a) to be doubly commuting is that a is a normal element in A.

Doubly commuting operators have been studied in very different contexts, to mention only some of the most relevant works, see for example [1]-[4], [6]-[8] and [12]. In this article, doubly commuting tuples of regular C^*-algebra elements will be consider. The main objective of this work consists in the study of the relationship between such tuples and the Moore-Penrose inverse of the product of the elements in the tuple.

In fact, given an n-tuple $a = (a_1, \ldots, a_n)$ of regular elements in a C^*-algebra A, if the n-tuple a is doubly commuting, then $\prod_{i=1}^n a_i$ is regular and

$$(\prod_{i=1}^{n} a_i)^\dagger \;=\; \prod_{i=0}^{n-1} a_{n-i}^\dagger = \;\prod_{i=1}^{n} a_i^\dagger,$$

in particular, $\prod_{i=1}^n a_i$ complies with the so-called reverse order law for the Moore-Penrose inverse. Moreover, if a is such a tuple, then (a_i, a_j) and (a_i^*, a_j) are doubly commuting pairs, where i, $j \in [\![1, n]\!]$, $i \neq j$. Consequently, $a_i a_j$ and $a_i^* a_j$ are regular and

$$(a_i a_j)^\dagger = (a_j a_i)^\dagger, \qquad (a_i^* a_j)^\dagger = (a_j a_i^*)^\dagger.$$

Conversely, if $a = (a_1, \ldots, a_n)$ is an n-tuple of regular elements in a C^*-algebra A such that the above identities are verified by (a_i, a_j) and (a_i^*, a_j), for all i, $j \in [\![1, n]\!]$, $i \neq j$, then a is a doubly commuting n-tuple of elements in the C^*-algebra A.

It is worth noticing that this characterization consists in an extension to the objects under consideration of the sufficient conditon given by J.J. Koliha in [13; 2.13] for the product of two regular elements to be Moore-Penrose invertible.

In section 2 it will be proved the aforementioned characterization. In section three, on the other hand, some applications will be developed. In fact, three sorts of objects will be considered, namely, tuples of commuting regular normal C^*-algebra elements, tuples of doubly commuting regular Hilbert space operators, and tuples of almost doubly commuting regular Hilbert space operators (that is doubly commuting tuples of regular elements in Calkin algebras).

This article is dedicated to the memory of Professor Angel R. Larotonda, who unfortunately and unexpectedly died on January 2th 2005. Although it is not necessary to comment Professor Larotonda's work as mathematician, for his scientific publications consist in a set of achievements which speak for themselves, a few words about the man deserve to be said. The author knew Professor Larotonda for more than twenty years. During uncountable conversations shared with Professor Larotonda, this researcher always showed his condition of sensible, civilized and cultivated human being, three characteristics that seem to be far from being widespread in this time and in any time.

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2. Main Results

In this section the relationship between the Moore-Penrose inverse and doubly commuting tuples of regular C^*-algebra elements will be studied. In fact, the characterization described in the previous section will be proved. In first place, a property of doubly commuting tuples is discussed.

Remark 2.1. Let $a = (a_1, \ldots, a_n)$ be an n-tuple of elements of a C^*-algebra A. Consider $\pi\colon [\![1, n]\!] \to [\![1, n]\!]$ a permutation, and define an n-tuple $b = (b_1, \ldots, b_n)$ in such a way that b_j is either $a_{\pi(j)}$ or $a^*_{\pi(j)}$. Then, it is easy to prove that a is doubly commuting if and only if b is.

Furthermore, note that the following facts are equivalents:

i) $a = (a_1, \ldots, a_n)$ is a doubly commuting n-tuple of elements of A,

ii) for each $i, j = 1, \ldots, n$, $i \neq j$, $a_{i,j} = (a_i, a_j)$ is a pair of doubly commuting elements of A.

Proposition 2.2. *Let $a = (a_1, \ldots, a_n)$ be an n-tuple of regular elements in a C^*-algebra A. Consider the n-tuple $a^\dagger = (a_1^\dagger, \ldots, a_n^\dagger)$. Then, a is doubly commuting if and only if $a^\dagger$ is.*

Proof. According to Remark 2.1, it is enough to prove that a pair of regular elements (b, c) is doubly commuting if and only if $(b^\dagger, c^\dagger)$ is.

Suppose that (b, c) is a doubly commuting pair of regular elements of A. Then, according to Theorem 5 of [10],

$$b^\dagger c = c b^\dagger.$$

Moreover, since according to Remark 2.1 (b, c^*) is a doubly commuting pair, it is clear that

$$b^\dagger c^* = c^* b^\dagger.$$

Consequently, $(b^\dagger, c)$ is a doubly commuting pair.

However, according to Remark 2.1, $(c, b^\dagger)$ is a doubly commuting pair, and thanks to what has been proved, $(c^\dagger, b^\dagger)$ is a doubly commuting pair. Therefore, according again to Remark 2.1, $(b^\dagger, c^\dagger)$ is a doubly commuting pair.

Conversely, if $(b^\dagger, c^\dagger)$ is a doubly commuting pair, since $(b^\dagger)^\dagger = b$ and $(c^\dagger)^\dagger = c$, according to the first part of the proof, (b, c) is a doubly commuting pair ∎

Note that in [13] Theorem 5 of [10] was proved using the Drazin inverse, see [13; 2.12]. Therefore, Proposition 2.2 can also be derived using the Drazin inverse, see [13; 2.12] and [13; 2.13].

Remark 2.3. Let $a = (a_1, \ldots, a_n)$ be an n-tuple of regular elements in a C^*-algebra A. Consider $\pi\colon [\![1, n]\!] \to [\![1, n]\!]$ a permutation, and define an n-tuple $b = (b_1, \ldots, b_n)$ in the following way. Given $j = 1, \ldots, n$, b_j is either $a_{\pi(j)}$, $a^*_{\pi(j)}$, $a^\dagger_{\pi(j)}$ or $(a^\dagger_{\pi(j)})^*$. Then, according to Remark 2.1 and to Proposition 2.2, a is doubly commuting if and only if b is.

Furthermore, according to Remark 2.1 and to Proposition 2.2, the following facts are equivalent:

i) a is an n-tuple of doubly commuting regular elements of A,

ii) for each $i, j = 1, \ldots, n$, $i \neq j$, (b_i, b_j) is a pair of doubly commuting regular elements of A.

Next follows the first part of our characterization. In fact, in the following theorem it will be proved that the product of the elements in a doubly commuting tuple of regular elements satisfy the so-called reverse order law for the Moore-Penrose inverse.

Theorem 2.4. *Let $a = (a_1, \ldots, a_n)$ be an n-tuple of doubly commuting regular elements of a C^*-algebra A. Then, $\prod_{i=1}^n a_i$ is regular and*

$$(\prod_{i=1}^n a_i)^\dagger = \prod_{i=0}^{n-1} a^\dagger_{n-i} = \prod_{i=1}^n a^\dagger_i.$$

Proof. Consider b and c two regular elements of A such that the pair (b, c) is doubly commuting. According to [13; 2.13], bc is regular and

$$(bc)^\dagger = c^\dagger b^\dagger.$$

Next consider $a = (a_1, \ldots, a_n, a_{n+1})$ a doubly commuting tuple of regular elements of A. Suppose that $\prod_{i=1}^n a_i$ is regular and

$$(\prod_{i=1}^n a_i)^\dagger = \prod_{i=0}^{n-1} a^\dagger_{n-i}.$$

In order to conclude the proof of the reverse order law for the Moore- Penrose inverse, it is enough to prove that $\prod_{i=1}^{n+1} a_i = (\prod_{i=1}^n a_i) a_{n+1}$ is regular and

$$(\prod_{i=1}^{n+1} a_i)^\dagger = a^\dagger_{n+1} (\prod_{i=0}^{n-1} a_{n-i})^\dagger.$$

Consider $d = \prod_{i=1}^n a_i$. According to Remark 2.1, (a_{n+1}, a_j) is a doubly commuting pair for $j = 1, \ldots, n$, which implies that (d, a_{n+1}) is a doubly commuting pair of regular elements. Therefore, according to the part of the theorem that has been proved, $da_{n+1} = \prod_{i=1}^{n+1} a_i$ is regular, and

$$(\prod_{i=1}^{n+1} a_i)^\dagger = (da_{n+1})^\dagger = a_{n+1}^\dagger d^\dagger = \prod_{i=0}^{n} a_{n+1-i}^\dagger.$$

The last identity is a consequence of [13; 2.13] or Proposition 2.2.

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Remark 2.5. Let $a = (a_1, \ldots, a_n)$ be an n-tuple of doubly commuting regular elements in a C^*-algebra A. Consider $\pi\colon [\![1, n]\!] \to [\![1, n]\!]$ a permutation and consider an n-tuple $b = (b_1 \ldots, b_n)$, where b_j is either $a_{\pi(j)}$, $a^*_{\pi(j)}$, $a^\dagger_{\pi(j)}$ or $(a^\dagger_{\pi(j)})^*$, $j = 1, \ldots, n$. Then, b is an n-tuple of doubly commuting regular elements of A. Consequently, according to [13; 2.13] or Proposition 2.2 and to Theorem 2.4, $\prod_{i=1}^{n} b_i$ is regular and

$$(\prod_{i=1}^{n} b_i)^\dagger = \prod_{i=0}^{n-1} b_{n-i}^\dagger = \prod_{i=1}^{n} b_i^\dagger.$$

Next consider two permutations $\pi, \sigma\colon [\![1, n]\!] \to [\![1, n]\!]$. Let $\tau\colon [\![1, n]\!] \to [\![1, n]\!]$ be the permutation $\tau = \pi^{-1}\sigma$, that is $\pi\tau = \sigma$. Next associate to π an n-tuple b as in the previous paragraph. Then, it is possible to associate to σ an n-tuple c such that $b_{\tau(j)} = c_j$, for $j = 1, \ldots, n$. Consequently, according to [13; 2.13] or Proposition 2.2 and to Theorem 2.4,

$$(\prod_{i=1}^{n} b_i)^\dagger = \prod_{i=1}^{n} b_i^\dagger = \prod_{i=1}^{n} c_i^\dagger = (\prod_{i=1}^{n} c_i)^\dagger.$$

For instance, if (a, b) is a pair of doubly commuting regular elements of the C^*-algebra A, the following identities hold.

$$\begin{aligned}
(ab)^\dagger &= b^\dagger a^\dagger = a^\dagger b^\dagger = (ba)^\dagger,\\
(a^*b)^\dagger &= b^\dagger (a^\dagger)^* = (a^\dagger)^* b^\dagger = (ba^*)^\dagger,\\
(ab^*)^\dagger &= (b^\dagger)^* a^\dagger = a^\dagger (b^\dagger)^* = (b^*a)^\dagger,\\
(a^*b^*)^\dagger &= (b^\dagger)^*(a^\dagger)^* = (a^\dagger)^*(b^\dagger)^* = (b^*a^*)^\dagger.
\end{aligned}$$

Consequently,

$$\begin{aligned}
(ab)^\dagger &= (ba)^\dagger = ((a^*b^*)^\dagger)^* = ((b^*a^*)^\dagger)^*,\\
(a^*b)^\dagger &= (ba^*)^\dagger = ((ab^*)^\dagger)^* = ((b^*a)^\dagger)^*.
\end{aligned}$$

Finally, recall that according to Remark 2.1, $a = (a_1, \ldots, a_n)$ is a doubly commuting n-tuple of regular elements of A if and only if (a_i, a_j) is a doubly commuting pair of regular elements of A, for i, $j = 1, \ldots, n$, $i \neq j$. Thererfore, for each such pair the above identities hold.

Next follows the converse of Theorem 2.4.

Theorem 2.6. *Consider $a = (a_1, \ldots, a_n)$ an n-tuple of regular elements in a C^*-algebra A. Suppose that for all $i,\, j = 1, \ldots, n$, $i \neq j$, $a_i a_j$, $a_j a_i$, $a_i^* a_j$ and $a_j a_i^*$ are regular and comply with the following identities:*

$$(a_i a_j)^\dagger = (a_j a_i)^\dagger, \qquad (a_i^* a_j)^\dagger = (a_j a_i^*)^\dagger.$$

Then a is a doubly commuting n-tuple of regular elements of A.

Proof. According to Remark 2.1, it is enough to prove that (a_i, a_j) is doubly commuting for all $i,\, j = 1, \ldots, n$, $i \neq j$. However, since $a_i a_j$ and $a_j a_i$ are regular and $(a_i a_j)^\dagger = (a_j a_i)^\dagger$, it is clear that $a_i a_j = a_j a_i$. Similarly, $a_i^* a_j = a_j a_i^*$. Therefore, (a_i, a_j) is a doubly commuting pair, $i,\, j = 1, \ldots, n$, $i \neq j$. ∎

Remark 2.7. In [13] sufficient conditions for a product to be Moore-Penrose inversible were given. Actually, thanks to the Theorem 2.6 it is now possible to state the following characterization. Let a and b be two regular elements in a C^*-algebra A. If (a, b) is a doubly commuting pair, then ab, ba, a^*b and ab^* are regular and

$$(ab)^\dagger = (ba)^\dagger, \qquad (a^*b)^\dagger = (ba^*)^\dagger.$$

On the other hand, if a and b are two regular elements such that ab, ba, a^*b and ab^* are regular and they comply with the above identities, then (a, b) is a doubly commuting pair. Furthermore, in this case ab and a^*b comply the reverse order law for the Moore-Penrose inverse.

3. Some applications

In this section several applications of the main results of this work will be considered. In first place, commuting tuples of normal elements in a C^*-algebra will be studied.

Remark 3.1. Let $a = (a_1, \ldots, a_n)$ be a doubly commuting n-tuple of elements in a C^*-algebra A. Let $\alpha = (m_1, \ldots, m_n)$ be an n-tuple of nonnegative entire numbers and define the n-tuple

$$a_\alpha = (a_1^{m_1}, \ldots, a_n^{m_n}).$$

Then, it is not difficult to prove that a_α is a doubly commuting n-tuple of elements of A.

On the other hand, if $b \in A$ is a regular normal element of A, then b^n is regular and $(b^n)^\dagger = (b^\dagger)^n$, for all $n \in \mathbb{N}$.

In fact, since b is regular, $b_n = (b, b, \ldots, b)$ (n-times) is a doubly commuting n-tuple of regular elements of A. Therefore, according to Theorem 2.4, b^n is regular and $(b^n)^\dagger = (b^\dagger)^n$.

In the following theorem the characterization of the previous section will be applied to commuting tuples of regular normal elements. This result is an extension of [13; 2.14] to the case under consideration.

Theorem 3.2. *Let $a = (a_1, \ldots, a_n)$ be a commuting n-tuple of regular normal elements in a C^*-algebra A. Let $\alpha = (m_1, \ldots, m_n)$ be an n-tuple of nonnegative entire numbers. Then $\prod_{i=1}^n a_i^{m_i}$ is regular and*

$$(\prod_{i=1}^{n} a_i^{m_i})^\dagger = \prod_{i=1}^{n} (a_i^\dagger)^{m_i}.$$

Proof. First of all recall that according to the Flugede-Putman Theorem, [5; IX, 6.7] or [9; 9.6.7], $a = (a_1, \ldots, a_n)$ is a doubly commuting tuple.

Next consider, as in Remark 3.1, the n-tuple

$$a_\alpha = (a_1^{m_1}, \ldots, a_n^{m_n}).$$

Then, according to Remark 3.1, a_α is a doubly commuting tuple of regular elements of A. Consequently, according to Theorem 2.4, Remark 2.5 and Remark 3.1 again, the proof of the Theorem is concluded.

∎

Remark 3.3. Let $a = (a_1, \ldots, a_n)$ be a commuting n-tuple of regular normal elements in a C^*-algebra A. Consider, as in Remark 2.3, $\pi\colon [\![1, n]\!] \to [\![1, n]\!]$ a permutation, and define an n-tuple $b = (b_1, \ldots, b_n)$, where given $j = 1, \ldots, n$, b_j is either $a_{\pi(j)}$, $a^*_{\pi(j)}$, $a^\dagger_{\pi(j)}$ or $(a^\dagger_{\pi(j)})^*$. Next consider the n-tuple $b_\beta = (b_1^{m_1}, \ldots, b_n^{m_n})$, where $\beta = (m_1, \ldots, m_n)$ is an n-tuple of nonnegative integers. Then, according to Proposition 2.2, Remark 3.1, Theorem 3.2 and Theorem 10 of [10], b_β is an n-tuple of doubly commuting regular normal elements of A. Therefore, according to Theorem 3.2, $\prod_{i=1}^n b_i^{n_i}$ is regular and

$$(\prod_{i=1}^{n} b_i^{n_i})^\dagger = \prod_{i=1}^{n} (b_i^\dagger)^{n_i}.$$

For example, if (a, b) is a pair of commuting regular normal elements in a C^*-algebra A, then

$$\begin{array}{llll}
a^k b^l, & (a^k)^* b^l, & a^k (b^l)^*, & (a^k)^* (b^l)^*, \\
(a^k)^\dagger b^l, & a^k (b^l)^\dagger, & (a^k)^\dagger (b^l)^\dagger, & ((a^k)^\dagger)^* ((b^k)^\dagger)^*, \\
((a^k)^\dagger)^* b^l, & (a^k)^\dagger (b^l)^*, & ((a^k)^\dagger)^* (b^l)^*, & a^k ((b^l)^\dagger)^*, \\
(a^k)^* (b^l)^\dagger, & (a^k)^* ((b^l)^\dagger)^*, & ((a^k)^\dagger)^* (b^l)^\dagger, & (a^k)^\dagger ((b^l)^\dagger)^*
\end{array}$$

are regular elements of A, for k and $l \in \mathbb{N}$. Furthermore, their Moore-Pensrose inverses can be calculated according to the first part of the present Remark.

Next the main results of the present work will be applied to n-tuples of regular Hilbert space operators.

Let H be a Hilbert space and consider $A = L(H)$, the C^*-algebra of all bounded and linear maps defined in H. Recall that $T \in L(H)$ is regular as an operator if and only if T is a regular element of A. Moreover, necessary and sufficient for $T \in L(H)$ to be a regular operator is the fact that the range of T, $R(T)$, is a closed subspace of H, see for example [9; 3.8]. Note that in this case the Moore-Penrose inverse can be described in a direct way.

In fact, consider $T \in L(H)$ a bounded Hilbert space operator with closed range. Define $S \in L(H)$ as follows:

$$S\colon R(T) \to N(T)^{\perp}, \qquad S\mid_{R(T)} \equiv \tilde{T}^{-1},$$
$$S\colon R(T)^{\perp} \to H, \qquad S\mid_{R(T)^{\perp}} \equiv 0,$$

where $\tilde{T} = T \mid_{N(T)^{\perp}}^{R(T)}\colon N(T)^{\perp} \to R(T)$, that is the restriction to $N(T)^{\perp}$ of T. Then, it is not diffuclt to prove that

$$T = TST, \quad S = STS, \quad (TS)^* = TS, \quad (ST)^* = ST,$$

that is, S is the Moore-Penrose inverse of T.

On the other hand, an n-tuple of continuous linear maps defined in H, $T = (T_1, \ldots, T_n)$, is doubly commuting as operators if and only if it is doubly commuting as elements of $A = L(H)$. In the following theorems the relationship between doubly commuting tuples of regular operators and the Moore-Penrose inverse will be studied.

Theorem 3.4. *Let $T = (T_1, \ldots, T_n)$ be an n-tuple of regular Hilbert space operators. Then, if the n-tuple T is doubly commuting, $\prod_{i=1}^{n} T_i$ is a regular operator and*

$$(\prod_{i=1}^{n} T_i)^{\dagger} = \prod_{i=0}^{n-1} T_{n-i}^{\dagger} = \prod_{i=1}^{n} T_i^{\dagger}.$$

Conversely, if for all i, $j = 1, \ldots, n$, $i \neq j$, T_iT_j, T_jT_i, $T_i^*T_j$ and $T_jT_i^*$ are regular operators which comply with the following identities:

$$(T_iT_j)^{\dagger} = (T_jT_i)^{\dagger}, \qquad (T_i^*T_j)^{\dagger} = (T_jT_i^*)^{\dagger},$$

then, T is a doubly commuting n-tuple of regular Hilbert space operators.

Proof. It is a consequence of Theorems 2.4 and 2.6. ∎

Theorem 3.5. *Let $T = (T_1, \ldots, T_n)$ be an n-tuple of commuting regular normal operators defined in a Hilbert space H. Next consider the n-tuple $b_\beta = (b_1^{m_1}, \ldots, b_n^{m_n})$, where $\beta = (m_1, \ldots, m_n)$ is an n-tuple of nonnegative integers. Then, $\prod_{i=1}^{n} T_i^{n_i}$ is regular and*

$$(\prod_{i=1}^{n} T_i^{n_i})^{\dagger} = \prod_{i=0}^{n-1} (T_{n-i}^{\dagger})^{n_i} = \prod_{i=1}^{n} (T_i^{\dagger})^{n_i}.$$

Proof. It is a consequence of Theorem 3.2. ∎

The last application of the results of the previous section concerns Calkin algebras.

Recall that if H is a Hilbert space and $K(H)$ denotes the closed ideal of all compact operators defined in H, then the Calkin algebra of H, $C(H) = L(H)/K(H)$, is a C^*-algebra. Moreover, tha natural quotient map $\pi\colon L(H) \to C(H)$ is a C^*-algebra morphism, see [5; 4] or [15; 4.1.16].

Furthermore, note that if $T \in L(H)$ is a regular operator, then $\pi(T)$ is a regular element of $C(H)$. In addition, it is not difficult to prove that if $T^\dagger$ is the Moore-Penrose inverse of a regular operator T, then $\pi(T)^\dagger = \pi(T^\dagger)$.

On the other hand, recall that an n-tuple $T = (T_1, \ldots, T_n)$ is said to be *almost commuting (resp. almost doubly commuting)*, if $\pi(T) = (\pi(T_1), \ldots, \pi(T_n))$ is a commuting tuple (resp. a doubly commuting tuple) in the C^*-algebra $C(H)$, that is if $T_iT_j - T_jT_i$ (resp. $T_iT_j - T_jT_i$ and $T_iT_j^* - T_j^*T_i$) belong to $K(H)$, for i, $j = 1, \ldots, n$, $i \neq j$, see [6], [7] and [8]. In the following theorems the relationship between n-tuples of almost doubly commuting regular operators and the Moore-Penrose inverse will be studied.

Theorem 3.6. *Let $T = (T_1, \ldots, T_n)$ be an n-tuple of regular operators defined in the Hilbert space H. Suppose that $\prod_{i=1}^n T_i$ is regular. Then, if T is an almost doubly commuting tuple of operators,*

$$(\prod_{i=1}^n T_i)^\dagger - \prod_{i=1}^n T_i^\dagger \in K(H).$$

Conversely, if for all i, $j = 1, \ldots, n$, $i \neq j$, T_iT_j, T_jT_i, $T_i^*T_j$ and $T_jT_i^*$ are regular operators such that

$$(T_iT_j)^\dagger - (T_jT_i)^\dagger \in K(H), \qquad (T_i^*T_j)^\dagger - (T_jT_i^*)^\dagger \in K(H),$$

then T is an almost doubly commuting n-tuple of regular Hilbert space operators.

Proof. It is a consequence of Theorem 3.4 and the above recalled facts regarding Calkin algebras.

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Theorem 3.7. *Let $T = (T_1, \ldots, T_n)$ be an n-tuple of almost commuting regular normal operators defined in a Hilbert space H. Next consider the n-tuple $T_\beta = (T_1^{m_1}, \ldots, T_n^{m_n})$, where $\beta = (m_1, \ldots, m_n)$ is an n-tuple of nonnegative integers, and suppose that $\prod_{i=1}^n T_i^{n_i}$ is regular . Then, T_β is an n-tuple of almost doubly commuting regular operators defined in H, and*

$$(\prod_{i=1}^n T_i^{n_i})^\dagger - \prod_{i=1}^n (T_i^\dagger)^{n_i} \in K(H).$$

Proof. It is a consequence of Theorems 3.2, 3.5 and the recalled facts regarding Calkin algebras.

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Remark 3.8. It iw worth noticing that Theorems 3.2 and 3.4-3.7 can be extended to other tuples of C^*-algebra elements and of Hilbert space operators respectively in the same way that it was done in Remarks 2.3 and 2.5, that is considering permutations and new tuples defined by Moore-Penrose inverses and adjoints.

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