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ON THE COHOMOLOGY RING OF FLAT MANIFOLDS WITH A SPECIAL STRUCTURE

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1. *Introduction.*

A Riemannian manifold is said to be *Kähler* if the holonomy group is contained in $U(n)$. It is *quaternion Kähler* if the holonomy group is contained in $Sp(n)Sp(1)$. It is known that quaternion Kähler manifolds of dimension ≥ 8 are Einstein, so the scalar curvature s splits these manifolds according to whether $s > 0$, $s = 0$ or $s < 0$. Ricci flat quaternion Kähler manifolds include hyperkähler manifolds, that is, those with holonomy group contained in $Sp(n)$. Such a manifold can be characterized by the existence of a pair of integrable, anticommuting complex structures, compatible with the Riemannian metric, and parallel with respect to the Levi-Civita connection (see [**Be**], for instance).

The simplest model of hyperkähler manifolds is provided by $\mathbb{R}^{4n}$ with the standard flat metric and a pair J, K of orthogonal anticommuting complex structures. This hyperkähler structure descends to the $4n$-torus $T_\Lambda := \Lambda\backslash\mathbb{R}^{4n}$, for any lattice Λ in $\mathbb{R}^{4n}$. If $M_\Gamma = \Gamma\backslash\mathbb{R}^{4n}$ is a compact flat manifold such that the holonomy action of $F = \Lambda\backslash\Gamma$ centralizes (resp. normalizes) the algebra generated by J, K, then M_Γ inherits a hyperkähler (resp. quaternion Kähler) structure.

In [**DM**] (see also [**JR**] and [**BDM**]) we described a doubling construction for Bieberbach groups which allows to give rather simple examples of quaternion Kähler flat manifolds which admit no Kähler structure.

The purpose of the present paper is to study the real cohomology ring of low dimensional compact flat manifolds endowed with one of these special structures. In particular, we will determine the structure of this ring in the case of all 4-dimensional Kähler flat manifolds and all 8-dimensional compact flat kyperkähler manifolds. We shall make use of the known classification of space groups in dimension 4, given in [**BBNWZ**], and of the classification of flat hyperkähler 8-manifolds due to L. Whitt ([**Wh**]). It turns out that the integral holonomy groups of hyperkähler 8-manifolds are obtained by doubling the holonomy groups of the Kähler flat 4-manifolds and as a consequence we will show that the cohomology ring is an exterior algebra in generators of degree one and two.

In [**Sa**], [**Sa2**] and [**Sa3**], Salamon obtains a family of linear relations among the Betti numbers of general hyperkähler manifolds (see Remark 4.2). In Section 5 we give several examples (5.1 - 5.3) showing that these relations may not hold in the quaternion Kähler case.

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As a second interesting class we study the hyperkähler manifolds obtained by doubling (twice) a Hantzsche-Wendt type manifold (see [**MR**]). This gives, for any $m \geq 3$, a $4n$-dimensional compact flat hyperkähler manifold with holonomy group $\mathbb{Z}_2^{n-1}$. We will show that the cohomology ring is generated by the F-invariant forms of degree 2 and 3, giving a procedure to find the relations. In particular we shall see that this algebra has a complicated structure and, even in the simplest case ($n = 3$) is far from being an exterior algebra, as seen in the 8-dimensional case.

The interest in understanding the structure of the cohomology ring of hyperkähler and quaternion Kähler flat manifolds was stimulated by the study of the Betti numbers of hyperkähler manifolds in the work of Salamon (see [**Sa**], [**Sa2**], [**Sa3**]) and Verbitsky ([**Ve**]).

2. *Hyperkähler and quaternionic Kähler structures on flat manifolds.*

We first recall some basic notions on compact flat manifolds (see [**Ch**] or [**Wo**]). A compact connected flat Riemannian manifold has euclidean space $\mathbb{R}^n$ as its universal covering space and a Bieberbach group Γ as fundamental group (i.e. a discrete cocompact subgroup Γ of $I(\mathbb{R}^n)$ which is torsion-free). If $v \in \mathbb{R}^n$, let L_v denote translation by v. By Bieberbach's first theorem, if Γ is a crystallographic group then $\Lambda = \{v \ : \ L_v \in \Gamma\}$ is a lattice in $\mathbb{R}^n$. We will identify the lattice Λ with the translation lattice $\{L_v : v \in \Lambda\}$, a normal and maximal abelian subgroup of Γ. The quotient $F = \Lambda\backslash\Gamma$ is a finite group, the point group (or holonomy group) of Γ. When Γ is torsion free, the geometric interpretation of $\Lambda\backslash\Gamma$ is that of the holonomy group of the flat Riemannian manifold M.

Let Γ be a Bieberbach group with holonomy group F and translation lattice $\Lambda \subset \mathbb{R}^n$. Let $\phi : F \to \mathbb{R}^n$ be a 1-cocycle modulo Λ, that is, ϕ satisfies $\phi(B_1B_2) = B_2^{-1}\phi(B_1) + \phi(B_2)$, modulo Λ, for each $B_1, B_2 \in F$. Then ϕ defines a cohomology class in $H^1(F; \Lambda\backslash\mathbb{R}^n) \simeq H^2(F; \Lambda)$ and one may associate to ϕ a crystallographic group with holonomy group F and translation lattice Λ. Furthermore, this group is torsion-free if and only if the class of ϕ is a special class (see [**Ch**]).

Definition 2.1. Let Γ be a Bieberbach group with holonomy group F and translation lattice $\Lambda \subset \mathbb{R}^n$. Let $\phi : F \to \mathbb{R}^n$ be any 1-cocycle modulo Λ. We let $d_\phi\Gamma$ be the subgroup of $I(\mathbb{R}^{2n})$ generated by elements of the form $\left[\begin{smallmatrix} B & 0 \\ 0 & B \end{smallmatrix}\right] L_{(\phi(B),b)}$ and $L_{(\lambda,\mu)}$, for $\gamma = BL_b \in \Gamma$ and $(\lambda, \mu) \in \Lambda \oplus \Lambda$.

We point out that if the holonomy group F of Γ centralizes a complex structure on $\mathbb{R}^n$, then M_Γ is Kähler. We now review a procedure to construct compact flat manifolds endowed with a Kähler, hyperkähler or quaternionic Kähler structure. We refer to [**DM**] for the details. This method will be used in later sections.

Proposition 2.2. *Let Γ, ϕ and $d_\phi\Gamma$ be as in Definition 2.1. Then*

(i) $d_\phi\Gamma$ is a Bieberbach group with holonomy group F, translation lattice $\Lambda \oplus \Lambda$ and $d_\phi\Gamma\backslash\mathbb{R}^{2n}$ is a Kähler compact flat manifold.

(ii) If $\Gamma\backslash\mathbb{R}^n$ has a locally invariant Kähler structure, then $d_\phi\Gamma\backslash\mathbb{R}^{2n}$ is hyperkähler. In particular, if $\phi' : F \to \mathbb{R}^{2n}$ is any 1-cocycle modulo $\Lambda \oplus \Lambda$, then $d_{\phi'}d_\phi\Gamma\backslash\mathbb{R}^{4n}$ is hyperkähler.

Remark 2.3. Benson-Gordon have proved ([**BG**]) that if N is a simply connected nilpotent Lie group, Γ is a discrete cocompact subgroup of N, and $M = \Gamma\backslash N$ has a Kähler structure (J, g) (with g positive definite) then M is a torus. The above proposition says that there are plenty of compact flat riemannian Kähler manifolds other than tori.

Remark 2.4. In general, there are many choices of ϕ as in Proposition 2.2. In this paper we shall work with ϕ the 1-cocycle associated to Γ, as in [**BDM**]. We will denote $d_\phi\Gamma$ by $d\Gamma$ in this case.

For many Bieberbach groups Γ one can enlarge $d\Gamma$ into a Bieberbach group $d_q\Gamma$ in such a way that some element in the holonomy group of $d_q\Gamma$ anticommutes with the complex structure J_{2n} in $\mathbb{R}^{2n}$. By repeating the procedure twice, one gets a Bieberbach group such that any element in the holonomy group will either commute or anticommute with each one of a pair of anticommuting complex structures, hence the quotient manifold will be a quaternion Kähler flat manifold which in general, will not be Kähler.

Definition 2.5. Let Γ be a Bieberbach group with holonomy group $F \simeq \mathbb{Z}_2^k$, with translation lattice Λ and such that $b \in \frac{1}{2}\Lambda$ for any $\gamma = BL_b \in \Gamma$. Set $E_n = \left[\begin{smallmatrix} I & \\ & -I \end{smallmatrix}\right] \in \mathrm{I}(\mathbb{R}^{2n})$. Set $d_q(\Gamma, v) = \langle d\Gamma, E_n L_{(v,0)}\rangle$, where $v \in \mathbb{R}^n$.

Under rather general conditions, $d_q(\Gamma, v)$ contains $d\Gamma$ as a normal subgroup of index 2, and $v \in \mathbb{R}^n$ can be chosen so that $d_q(\Gamma, v)$ is torsion free, so $M_{d_q(\Gamma,v)}$ is a compact flat manifold with holonomy group $F \times \mathbb{Z}_2$ having as a double cover the Kähler manifold $M_{d\Gamma}$. Furthermore F commutes with J, but E_n only anticommutes with J. If we use this construction twice we get a Bieberbach group $d_q^2(\Gamma, v, u) := d_q(d_q(\Gamma, v), u) \subset \mathrm{I}(\mathbb{R}^{4n})$ such that the holonomy group normalizes two anticommuting complex structures, J_1, J_2, on $\mathbb{R}^{4n}$, hence $d_q^2(\Gamma, v, u)\backslash\mathbb{R}^{4n}$ will be a quaternion Kähler manifold.

In the next results we give conditions on $v \in \mathbb{R}^n$ that ensure that $d_q(\Gamma, v)$ is torsion free. We also note that if n is even, $M_{d_q(\Gamma,v)}$ will always be orientable. This construction will be used in Section 5.

Theorem 2.6. *Let Γ as above. Then*

(i) *If $v \in \mathbb{R}^n$ is such that $2v \in \Lambda$ and satisfies*

$$(B - I)v \in \Lambda \text{ for each } \gamma = BL_b \in \Gamma,$$

then $d_q\Gamma$ is a crystallographic group with translation lattice $\Lambda \oplus \Lambda$ and holonomy group $\mathbb{Z}_2^{k+1}$. Furthermore, $d_q\Gamma$ is torsion-free if and only if $v \notin \Lambda$ and for each $\gamma = BL_b \in \Gamma$ we have:

$$(B + I)(\phi(B) + v) \in \Lambda \setminus (B + I)\Lambda, \text{ or } (B - I)b \notin (B - I)\Lambda.$$

(ii) *If every element in the holonomy group F commutes or anticommutes with a translation invariant complex structure and v satisfies the conditions in (i), then $d_q(\Gamma, v)\backslash\mathbb{R}^{2n}$ is quaternion Kähler.*

3. *Cohomology of Kähler compact flat manifolds of dimension* 4.

In the computation of cohomology, in this and in later sections, we will make much use of the following result of H.Hiller ([**Hi**]):

Theorem 3.1. *Let* Γ *be a Bieberbach group and* $M_\Gamma = \Gamma\backslash\mathbb{R}^n$. *If* K *is a field such that the characteristic of* K *does not divide* $|F|$, *then the cohomology ring of* M_Γ *with coefficients in* K *is given by*

$$H^*(M_\Gamma, \mathbb{K}) \simeq (\bigwedge^*(\Lambda \otimes K))^F.$$

Let Γ be a 4-dimensional Bieberbach group with holonomy group F. It is not hard to see that in order for $M = \Gamma\backslash\mathbb{R}^4$ to be Kähler, it is necessary and sufficient that F commutes with a complex structure J on $\mathbb{R}^4$. Using the classification of compact flat manifolds of dimension 4 in [**BBNWZ**] we see that those groups Γ_i with non trivial holonomy group which have such property have cyclic holonomy groups F of order $2, 3, 4$ or 6, and have the form $\Gamma_i = \langle\gamma_i, \Lambda_i\rangle$, with $\gamma_i = \sigma_i L_{b_i}$, $1 \le i \le 7$ as follows:

$$F \simeq \mathbb{Z}_2 : \quad \sigma_1 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 1 & & & 1 \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}, \quad b_1 = b_2 = \tfrac{e_1}{2}$$

$$F \simeq \mathbb{Z}_4 : \quad \sigma_3 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{bmatrix}, \qquad \sigma_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad b_3 = b_4 = \tfrac{e_1}{4}$$

$$F \simeq \mathbb{Z}_3 : \sigma_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \sigma_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad b_5 = b_6 = \tfrac{e_1}{3}$$

$$F \simeq \mathbb{Z}_6 : \sigma_7 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & -1 \\ & & 1 & 1 \end{bmatrix}, \qquad b_7 = \tfrac{e_1}{6}.$$

We note that in the case of the torus T^4, the cohomology ring is an exterior algebra generated by elements of order 1, and the Poincaré polynomial is $p(t) = (t+1)^4$. For general flat Kähler 4-manifolds we have:

Theorem 3.2. *If* M_Γ *is a* 4*-dimensional Kähler flat manifold which is not a torus, the cohomology ring is an exterior algebra in* $\{e_1, e_2, \eta_1, \eta_2\}$ *where the* e_i $(i = 1, 2)$ *have degree 1 and the* η_j $(j = 1, 2)$ *have degree* 2. *Furthermore, in all cases one has* $\beta_0 = \beta_4 = 1$, $\beta_1 = \beta_2 = \beta_3 = 2$ *and the Poincaré polynomial is given by* $p(t) = (t^2 + 1)(t + 1)^2$, $1 \le i \le 7$.

Proof. To determine the real cohomology rings of the Kähler flat manifolds of dimension 4, we need to compute the F-invariants in each degree, for each Bieberbach group Γ_i in the family considered above.

We shall carry out this computation only in the case of the group Γ_6. The other cases are similar and their verification will be left to the reader.

It is easy to see that in degree 1, the fixed space is spanned by the elements e_1, e_2.

Assume now that $\eta = \sum_{1\le i<j\le 4} a_{ij} e_i \wedge e_j$ satisfies $\sigma_6\eta = \eta$. Now

$$\sigma_6\eta = a_{12}e_1 \wedge e_2 + a_{13}e_1 \wedge (-e_2 - e_3 + e_4) + a_{14}e_1 \wedge (-e_3) + a_{23}e_2 \wedge (-e_3 + e_4) + a_{24}e_2 \wedge (-e_3) + a_{34}(-e_2 + e_4) \wedge (-e_3).$$

Now, $\sigma_6\eta = \eta$ implies $a_{12} = a_{12} - a_{13}$ and $a_{13} = -a_{13} - a_{14}$, thus $a_{13} = a_{14} = 0$. Also, it follows that $a_{23} = -a_{23} - a_{24} + a_{34}$ and $a_{24} = a_{23}$, thus $3a_{23} = a_{34}$, hence the σ_6-fixed space in degree 2 is spanned by the invariant 2-forms $e_1 \wedge e_2$ and $e_2 \wedge e_3 + e_2 \wedge e_4 + 3e_3 \wedge e_4 = (e_2 + 3e_3) \wedge (e_3 + e_4)$, as asserted.

We now turn into degree 3. Let

$$\eta = ae_1 \wedge e_2 \wedge e_3 + be_1 \wedge e_2 \wedge e_4 + ce_1 \wedge e_3 \wedge e_4 + de_2 \wedge e_3 \wedge e_4$$

with $a, b, c, d \in \mathbb{R}$. Now

$$\sigma_6\eta = (-a - b + c)e_1 \wedge e_2 \wedge e_3 + ae_1 \wedge e_2 \wedge e_4 + ce_1 \wedge e_3 \wedge e_4 + de_2 \wedge e_3 \wedge e_4.$$

Thus $\sigma_6\eta = \eta$ implies $a = b$, $c = 3a$. Thus we get that the space of F-invariants in degree 3 is generated by the 3-forms $e_2 \wedge e_3 \wedge e_4$ $(a = 1, d = 0)$ and $e_1 \wedge e_2 \wedge (e_3 + e_4) + 3e_1 \wedge e_3 \wedge (e_3 + e_4) = e_1 \wedge \eta_6$. This completes the verification for Γ_6.

In the remaining cases the invariants are computed similarly. We now give a table that lists the F-invariants in each degree, for each group.

Degree	1	2	3	4
$\Gamma_{1,3,5,7}$	e_1	$e_1 \wedge e_2$	$e_1 \wedge e_3 \wedge e_4$	e
	e_2	$e_3 \wedge e_4$	$e_2 \wedge e_3 \wedge e_4$	
Γ_2	e_1	$e_1 \wedge e_2$	$e_1 \wedge e_3 \wedge e_4$	e
	e_2	$e_3 \wedge (-e_1 + 2e_4)$	$e_2 \wedge e_3 \wedge (-e_1 + 2e_4)$	
Γ_4	e_1	$e_1 \wedge e_2$	$e_1 \wedge \eta_4$	e
	e_2	$(e_1 + e_2 - 2e_3) \wedge (-e_3 + e_4) := \eta_4$	$e_2 \wedge \eta_4$	
Γ_6	e_1	$e_1 \wedge e_2$	$e_2 \wedge e_3 \wedge e_4$	e
	e_2	$(e_2 + 3e_3) \wedge (e_3 + e_4) := \eta_6$	$e_1 \wedge \eta_6$	

Here $e = e_1 \wedge e_2 \wedge e_3 \wedge e_4$.

The assertions on the Betti numbers and on the structure of the ring follow immediately from the information in the table, thus the theorem follows. □

4. *The cohomology ring of hyperkähler flat* 8*-manifolds.*

By doubling the 4-dimensional Bieberbach groups listed in the previous section we obtain a family of 8-dimensional hyperkähler flat manifolds. In [**Wh**] L. Whitt gives a full classification of such manifolds, showing there are 12 diffeomorphism classes. This classificaton shows in particular, that the holonomy representations of all such manifolds are obtained by doubling the holonomies of Kähler 4-manifolds. The goal of this section will be to determine the cohomology ring of this family. We first need to recall Whitt's classification. For simplicity of notation we shall set

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \ J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ L = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},$$
$$D = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \ N = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \ E = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Furthermore, let E_{ij} be the 2×2 matrix with 1 in the (i,j) place and 0 otherwise.

According to [**Wh**], Theorem 4.3, the holonomy group of Γ is cyclic with generator given by $\sigma_0 = I$, or by one of the following:

$$F \simeq \mathbb{Z}_2 : \sigma_1 = \begin{bmatrix} I_4 & 0 \\ 0 & -I_4 \end{bmatrix}, \quad \sigma_2 = \left[\begin{array}{c|c} I_4 & 0\ L \\ & 0\ L \\ \hline 0 & -I_4 \end{array}\right], \quad \sigma_3 = \left[\begin{array}{c|c} I_4 & 0\ E \\ & 0\ X \\ \hline 0 & -I_4 \end{array}\right],$$
$$\sigma_4 = \left[\begin{array}{c|c} I_4 & L\ E_{21} \\ & L\ E_{12} \\ \hline 0 & -I_4 \end{array}\right],$$

$$F \simeq \mathbb{Z}_4 : \sigma_5 = \left[\begin{array}{c|cc} I_4 & & \\ \hline & J & \\ & & J \end{array}\right], \quad \sigma_6 = \left[\begin{array}{c|cc} I_4 & 0 & L \\ & 0 & L \\ \hline & J & \\ & & J \end{array}\right], \quad \sigma_7 = \left[\begin{array}{c|cc} I_4 & L & E_{12} \\ & L & 0 \\ \hline & J & \\ & & J \end{array}\right],$$

$$F \simeq \mathbb{Z}_3 : \ \sigma_8 = \left[\begin{array}{c|cc} I_4 & 0 & L \\ & 0 & L \\ \hline & N & \\ & & N \end{array}\right], \quad \sigma_9 = \left[\begin{array}{c|cc} I_4 & L & E_{22} \\ & L & 0 \\ \hline & N & \\ & & N \end{array}\right], \quad \sigma_{10} = \left[\begin{array}{c|cc} I_4 & & \\ \hline & N & \\ & & N \end{array}\right],$$

$$F \simeq \mathbb{Z}_6 : \sigma_{11} = \left[\begin{array}{c|cc} I_4 & & \\ \hline & D & \\ & & D \end{array}\right].$$

We take $b_1 = b_2 = b_3 = b_4 = \frac{e_1}{2}$, $b_5 = b_6 = b_7 = \frac{e_1}{4}$, $b_8 = b_9 = b_{10} = \frac{e_1}{3}$ and $b_{11} = \frac{e_1}{6}$.

The next theorem gives the cohomology rings over $\mathbb{R}$ of the hyperkähler manifolds $\Gamma_i\backslash\mathbb{R}^8$, where $\Gamma_i = \langle \gamma_i, \Lambda_i\rangle$ with $\gamma_i = \sigma_i L_{b_i}$,$1 \le i \le 11$, and Γ_i is one of the 8-dimensional Bieberbach groups listed above.

Theorem 4.1. *Let M_Γ be an 8-dimensional hyperkähler manifold that is not a torus, where $\Gamma = \Gamma_i$, $1 \le i \le 11$, is one of the Bieberbach groups given above.*

Then the cohomology ring is an exterior algebra with generators e_i, η_j given as follows:

$$\begin{aligned}
&\textit{If} \quad 1 \le i \le 4 : \{e_i : 1 \le i \le 4,\ deg\, e_i = 1,\ \eta_j : \ 1 \le j \le 6;\ deg\, \eta_j = 2\}.\\
&\textit{If} \quad 5 \le i \le 11 : \{e_i : 1 \le i \le 4, \ \ deg\, e_i = 1,\ \eta_j : 1 \le j \le 4; deg\, \eta_j = 2\}.
\end{aligned}$$

The Poincaré polynomials of M_{Γ_i} are respectively given by

$$p(t) = \begin{cases} (t+1)^4(t^4 + 6\,t^2 + 1), & \textit{for } 1 \le i \le 4, \\ (t+1)^4(t^4 + 4\,t^2 + 1), & \textit{for } 5 \le i \le 11. \end{cases}$$

Proof. In this case we will not proceed as in Theorem 3.1, but, instead, we will diagonalize the induced holonomy action of F on $\mathbb{C}^8$.

In the case when $\Gamma = \Gamma_1$, clearly the F-invariants are an exterior algebra generated by the elements of the form $e_i, 1 \le i \le 4$ and $e_i \wedge e_j$ with $5 \le i < j \le 8$. If $\Gamma = \Gamma_2$, the answer is the same, with the same generators of degree 1; as elements of degree two we have to take $f_i \wedge f_j$, $5 \le i < j \le 8$, where $f_j = e_j$ for $j = 5, 7$ and $f_6 = e_6 + \frac{1}{2}e_1$, $f_8 = e_8 + \frac{1}{2}e_3$. The cases of Γ_3 and Γ_4 are identical to that of Γ_2.

In the next case, when $F \simeq \mathbb{Z}_4$, the holonomy action can be diagonalized over $\mathbb{C}$ in a suitable basis so that $\sigma f_j = f_j$ for $1 \le j \le 4$, and $\sigma f_j = i f_j$ (resp. $-i f_j$), for $i = 5, 7$ (resp. for $i = 6, 8$). Thus the algebra of complex F-invariants is an exterior algebra with generators f_j, $1 \le j \le 4$, and exterior products of the form $f_i \wedge f_j$ where $i = 5, 7$, $j = 6, 8$. Furthermore we have that $f_6 = \bar{f}_5$ and $f_8 = \bar{f}_7$.

We now determine the *real* F-invariants in degree two. We have, over $\mathbb{C}$, that the generators are $f_5 \wedge \bar{f}_5$, $f_7 \wedge \bar{f}_7$, $f_5 \wedge \bar{f}_7$ and $f_7 \wedge \bar{f}_5$.

If we set $f_5 = g_5 + ih_5$, $f_7 = g_7 + ih_7$, with g_5, g_7, h_5, h_7 real forms, we see that

$$\begin{aligned}
f_5 \wedge \bar{f}_5 &= & -2ig_5 \wedge h_5,\\
f_7 \wedge \bar{f}_7 &= & -2ig_7 \wedge h_7,\\
f_5 \wedge \bar{f}_7 &= & (g_5 \wedge g_7 + h_5 \wedge h_7) + i(h_5 \wedge g_7 - g_5 \wedge h_7),\\
f_7 \wedge \bar{f}_5 &= & -(g_5 \wedge g_7 + h_5 \wedge h_7) + i(h_5 \wedge g_7 - g_5 \wedge h_7)
\end{aligned}$$

Thus, we get that the real cohomology of M_{Γ_3} in degree two is spanned by

$$g_5 \wedge h_5,\ g_7 \wedge h_7,\ g_5 \wedge g_7 + h_5 \wedge h_7 \text{ and } \ h_5 \wedge g_7 - g_5 \wedge h_7.$$

Similarly, if $F \simeq \mathbb{Z}_3$, the holonomy action can be diagonalized over $\mathbb{C}$ in a basis f_j, such that $\sigma f_j = f_j$ for $1 \le j \le 4$, and $\sigma f_j = \omega f_j$ (resp. $\bar{\omega} f_j$), for $i = 5, 7$ (resp. for $i = 6, 8$). Here ω is a primitive root of 1 of order 3. Furthermore, again $f_6 = \bar{f}_5$ and $f_8 = \bar{f}_7$.

Thus, in this case, the algebra of F-invariants is an exterior algebra with generators f_j, $1 \le j \le 4$, and exterior products of degree two $f_i \wedge f_j$ where $i = 5, 7$, $j = 6, 8$. The real invariants are obtained in the same way as in the case $F \simeq \mathbb{Z}_4$.

The situation when $F \simeq \mathbb{Z}_6$ is entirely similar except that we must take ω to be a primitive root of 1 of order 6.

Using the above information we see that the Betti numbers of the manifolds M_{Γ_i}, $1 \le i \le 4$, are as follows:

$$\beta_1 = 4, \;\; \beta_2 = \binom{4}{2} + \binom{4}{2} = 12, \;\; \beta_3 = 28, \;\; \beta_4 = \binom{4}{2}^2 + 2 = 38.$$

whereas, for $5 \le i \le 11$:

$$\beta_1 = 4, \;\; \beta_2 = \binom{4}{2} + \binom{4}{1} = 10, \;\; \beta_3 = 4 + 4^2 = 20, \;\; \beta_4 = 2 + \binom{4}{2} 4 = 26.$$

As a verification, we note that in both cases we have that $\sum_{i=0}^{8} (-1)^i \beta_i = 0$ (as it should).

One finally easily checks that the corresponding Poincaré polynomials are respectively given by $(t+1)^4(t^4+6t^2+1)$ and $(t+1)^4(t^4+6t^2+1)$, as asserted. □

Remark 4.2. (i) In the case when M_Γ is the torus T^8, we have $\Gamma = \mathbb{Z}^8$, and the cohomology ring is just the exterior algebra $\bigwedge^* \mathbb{R}^n$; we have $\beta_i = \binom{n}{i}$ for $0 \le i \le 8$.

(ii) We note that all Poincaré polynomials are divisible by $(t+1)^4$, a fact valid for all hyperkähler manifolds ([**Sa**]). This fails to be true in the quaternionic Kähler case (see Example 5.2).

In [**Sa**] Salamon obtains a general identity for the Betti numbers of a $4m$-dimensional hyperkähler manifold. This identity reads, for $n = 4, 8, 12$:

$$4\beta_1 + \beta_2 = 22 \quad (n = 4) \tag{1}$$

$$25\beta_1 + \beta_3 + \beta_4 = 46 + 10\beta_2 \quad (n = 8) \tag{2}$$

$$48\beta_1 + 16\beta_3 + \beta_6 = 70 + 30\beta_2 + 6\beta_4 \quad (n = 12) \tag{3}$$

In the next section, we shall give examples showing these identities need not hold in the quaternionic Kähler case.

5. Quaternionic Kähler manifolds

The purpose of this section is to compute the cohomology ring of some quaternionic Kähler flat manifolds which are not hyperkähler. These examples will reveal several new features.

Example 5.1. We first look at a simple 4-dimensional manifold with holonomy group $\mathbb{Z}_2$.

Let $\Gamma = \langle EL_{\frac{e_1}{2}}, \Lambda \rangle$ where $E = \begin{bmatrix} I_2 & \\ & -I_2 \end{bmatrix}$ and Λ is the canonical lattice. Note that Γ is essentially the double of the Klein bottle group.

Consider the two anticommuting complex structures in $\mathbb{R}^4$ given by

$$J_1 = \begin{bmatrix} J & \\ & J \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix}. \tag{4}$$

It is easy to verify that $EJ_1 = J_1E$, $EJ_2 = -J_2E$, thus it follows that M_Γ is quaternionic Kähler.

Relative to the Betti numbers we have $\beta_0 = \beta_4 = 1$, $\beta_1 = \beta_3 = 2$, $\beta_2 = 2$, since the F-fixed vectors in degree 2 are $e_1 \wedge e_2, e_3 \wedge e_4$. Again, the algebra of invariants is an exterior algebra with generators of degree one and two: $e_1, e_2, e_3 \wedge e_4$.

Thus we see that $4\beta_1 + \beta_2 = 10 \neq 22$, so Salamon's identity (1) does not always hold in the 4-dimensional quaternionic Kähler case.

Furthermore the Poincaré polynomial is

$$p(t) = 1 + 2t + 2t^2 + 2t^3 + t^4 = (t+1)^2(t^2+1).$$

Thus we see that $p(t)$ is not divisible by $(t+1)^4$, but only by $(t+1)^2$. Note that this should be the case, since M_Γ is Kähler (the complex structure J_1 descends). Note also that the torus T^4 is a hyperkähler covering of M_Γ. It has Poincaré polynomial $(t+1)^4$.

Example 5.2. We now look at the cohomology ring for a quaternionic Kähler 8-manifold with holonomy group $\mathbb{Z}_4 \times \mathbb{Z}_2$. Let $\Gamma = \langle AL_{\frac{e_2}{2}}, CL_{\frac{e_1}{2}}, \Lambda \rangle$ where Λ is the canonical lattice and

$$C = \left[\begin{array}{c|cc} I_4 & & \\ \hline & J & \\ & & J \end{array}\right], \; A = \begin{bmatrix} I_2 & & & \\ & -I_2 & & \\ & & -I_2 & \\ & & & I_2 \end{bmatrix}.$$

Consider the two anticommuting complex structures on $\mathbb{R}^8$ given by

$$J_1 = \begin{bmatrix} J & & & \\ & -J & & \\ & & J & \\ & & & -J \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & I_2 & 0 & 0 \\ -I_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_2 \\ 0 & 0 & I_2 & 0 \end{bmatrix}. \qquad (5)$$

Recall that as usual, $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Here, note that C commutes with both J_1, J_2 and A commutes with J_1 and anticommutes with J_2. Thus M_Γ is a quaternion Kähler manifold. It is also Kähler since J_1 descends.

It is easy to see that, in degree 1, the F-invariants are generated by e_1, e_2 and in degree 2, by $e_1 \wedge e_2$, $e_3 \wedge e_4$, $e_5 \wedge e_6$, $e_7 \wedge e_8$. Thus $\beta_1 = 2$, $\beta_2 = 4$.

In degrees 3 and 4 we find that

$$\begin{aligned} \bigwedge^3 (\mathbb{R}^8)^F = \operatorname{span}\{ & e_1 \wedge e_3 \wedge e_4,\, e_2 \wedge e_3 \wedge e_4,\, e_1 \wedge e_5 \wedge e_6,\, e_2 \wedge e_5 \wedge e_6, \\ & e_1 \wedge e_7 \wedge e_8,\, e_2 \wedge e_7 \wedge e_8,\, e_3 \wedge e_5 \wedge e_7,\, e_3 \wedge e_5 \wedge e_8, \\ & e_4 \wedge e_5 \wedge e_7,\, e_4 \wedge e_6 \wedge e_8\}. \end{aligned}$$

$$\begin{aligned} \bigwedge^4 (\mathbb{R}^8)^F = \operatorname{span}\{ & e_1 \wedge e_2 \wedge e_3 \wedge e_4,\, e_1 \wedge e_2 \wedge e_5 \wedge e_6,\, e_1 \wedge e_2 \wedge e_7 \wedge e_8, \\ & e_1 \wedge e_3 \wedge e_5 \wedge e_7, e_1 \wedge e_3 \wedge e_6 \wedge e_8,\, e_1 \wedge e_4 \wedge e_6 \wedge e_8, \\ & e_1 \wedge e_4 \wedge e_5 \wedge e_7,\, e_5 \wedge e_6 \wedge e_7 \wedge e_8,\, e_3 \wedge e_4 \wedge e_7 \wedge e_8, \\ & e_3 \wedge e_4 \wedge e_5 \wedge e_6,\, e_2 \wedge e_3 \wedge e_6 \wedge e_8,\, e_2 \wedge e_4 \wedge e_5 \wedge e_7, \\ & e_2 \wedge e_3 \wedge e_5 \wedge e_7,\, e_2 \wedge e_4 \wedge e_6 \wedge e_8\}. \end{aligned}$$

The Poincaré polynomial is given by:

$$p(t) = 1+2t+4t^2+10t^3+14t^4+10t^5+4t^6+2t^7+t^8 = (t+1)^2(t^6+3t^4+4t^3+3t^2+1).$$

We see that this cannot be the polynomial of a hyperkähler manifold, since the odd Betti numbers are not a multiple of 4 and $p(t)$ is not divisible by $(t+1)^4$.

Note also that Salamon's identity (2) is not satisfied. Indeed

$$25\beta_1 + \beta_3 + \beta_4 = 25.2 + 10 + 14 = 74 \neq 46 + 10\beta_2 = 46 + 10.4 = 86. \tag{6}$$

Example 5.3. We now look at a quaternionic square double of the Klein bottle. As shown in [**DM**], there are several such manifolds non diffeomorphic to each other. Since they all have the same holonomy representation it will suffice to consider only one example of this type. Let $\Gamma = \langle EL_{\frac{e_1}{2}},\, A''L_{\frac{e_2}{2}},\, B''L_{\frac{e_3}{2}},\, \Lambda\rangle$ where Λ is the canonical lattice and

$$E = \begin{bmatrix} E_1 & & & \\ & E_1 & & \\ & & E_1 & \\ & & & E_1 \end{bmatrix},\ A'' = \begin{bmatrix} I_2 & & & \\ & -I_2 & & \\ & & -I_2 & \\ & & & I_2 \end{bmatrix},\ B'' = \begin{bmatrix} I_4 & \\ & -I_4 \end{bmatrix}.$$ Let

J_1, J_2 be the following anticommuting complex structures on $\mathbb{R}^8$:

$$J_1 = \begin{bmatrix} J & & & \\ & J & & \\ & & -J & \\ & & & -J \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & I_4 \\ -I_4 & 0 \end{bmatrix}. \tag{7}$$

Here, as usual, $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

In this case, each of the elements in the holonomy group, either commutes or anticommutes with J_1, J_2 but neither of these complex structures descends; M_Γ is a quaternion Kähler manifold.

It is easy to see that, in degree 1, the F-invariants are generated by e_1 and in degree 2, this space is zero.

In degrees 3 and 4 we find that

$$\bigwedge^3(\mathbb{R}^n)^F = \mathrm{span}\{e_2\wedge e_3\wedge e_4,\, e_2\wedge e_5\wedge e_6,\, e_2\wedge e_7\wedge e_8,\, e_3\wedge e_5\wedge e_7,\, e_3\wedge e_6\wedge e_8,\, e_4\wedge e_5\wedge e_8,\, e_4\wedge e_6\wedge e_7\}.$$

$$\begin{aligned}\bigwedge^4(\mathbb{R}^n)^F = \mathrm{span}\{&e_1\wedge e_2\wedge e_3\wedge e_4,\, e_1\wedge e_2\wedge e_5\wedge e_6,\, e_1\wedge e_2\wedge e_7\wedge e_8,\\ &e_1\wedge e_3\wedge e_5\wedge e_7, e_1\wedge e_3\wedge e_6\wedge e_8,\, e_1\wedge e_4\wedge e_5\wedge e_8,\\ &e_1\wedge e_4\wedge e_6\wedge e_7,\, e_5\wedge e_6\wedge e_7\wedge e_8,\, e_3\wedge e_4\wedge e_7\wedge e_8,\\ &e_3\wedge e_4\wedge e_5\wedge e_6,\, e_2\wedge e_4\wedge e_6\wedge e_8,\, e_2\wedge e_4\wedge e_5\wedge e_7,\\ &e_2\wedge e_3\wedge e_6\wedge e_7,\, e_2\wedge e_3\wedge e_5\wedge e_8\}\end{aligned}$$

The Poincaré polynomial is given by:

$$p(t) = 1 + t + 7t^3 + 14t^4 + 7t^5 + t^7 + t^8 = (t+1)^4(t^4 - 4t^3 + 6t^2 - 3t + 1).$$

We see that this cannot be the polynomial of a hyperkähler manifold, since the odd Betti numbers are odd and they should be a multiple of 4. Actually M_Γ cannot even have a Kähler structure since odd Betti numbers are odd and $\beta_2 = 0$.

On the other hand, we observe that the Poincaré polynomial is divisible by $(t+1)^4$ and Salamon's identity (2) is satisfied. Indeed

$$25\beta_1 + \beta_3 + \beta_4 = 25 + 7 + 14 = 46 = 46 + 10\beta_2. \tag{8}$$

By inspection, we see that in this (non hyperkähler) case, the algebra of invariants is not an exterior algebra. The invariants in degree 1 and 3 do not suffice to generate the invariants in degree 4 unless we include the action of the star operator.

6. *Doubling Hantzsche-Wendt groups.*

In this section we shall compute the cohomology ring of $M^6 = d\Gamma\backslash\mathbb{R}^6$, and $M^{12} = d^2\Gamma\backslash\mathbb{R}^{12}$ where Γ is the classical Hantzsche-Wendt Bieberbach group in dimension 3 (see [**Wo**]). We shall see that the cohomology ring of M^{12} is far from having the structure of an exterior algebra in this case.

Let $\Gamma = \{AL_a, BL_b, L_{e_i} : 1 \le i \le 3\}$, where $A = \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}$ $a = \frac{e_3}{2}, b = \frac{e_1+e_2}{2}$.

It turns out that $\Gamma\backslash\mathbb{R}^3$ is the only 3-dimensional compact flat manifold with $\beta_1 = 0$. It is called the Hantzsche-Wendt manifold ([Wo]). The Poincaré polynomial is given by $p(t) = 1 + t^3$ and the holonomy group is $\mathbb{Z}_2^2$. We shall next study the cohomology ring of $M_{d\Gamma}$ and $M_{d^2\Gamma}$.

Theorem 6.1. *The cohomology ring* $\bigwedge^*(\mathbb{R}^6)^F$ *of* $M_{d\Gamma}$ *is a graded algebra of dimension 16 generated by the elements of degree 2 :* $\{\eta_i : 1 \le i \le 3\}$ *and of degree 3:* $\{\delta_j : 1 \le j \le 8\}$ *subject to the relations* $\eta_i^2 = \eta_i\delta_j = \delta_i.\delta_j = 0$.

The Poincaré polynomial of $M_{d\Gamma}$ *is given by*

$$p(t) = 1 + 3t^2 + 8t^3 + 3t^4 + t^6 = (1+t)^2(t^4 - 2t^3 + 6t^2 - 2t + 1).$$

Proof. The generators of the holonomy group F of $d\Gamma$ are

$$A' = \begin{bmatrix} -1 &&&&& \\ & -1 &&&& \\ && 1 &&& \\ &&& -1 && \\ &&&& -1 & \\ &&&&& 1 \end{bmatrix} \quad B' = \begin{bmatrix} 1 &&&&& \\ & -1 &&&& \\ && -1 &&& \\ &&& 1 && \\ &&&& -1 & \\ &&&&& -1 \end{bmatrix}$$

The holonomy action on the exterior algebra diagonalizes in the canonical basis with eigenvalues ± 1. Clearly there are no vectors fixed by both A', B'. Thus $\beta_1(M_{d\Gamma}) = 0$. On the other hand, the fixed vectors in degree $d > 1$ are:

$\bigwedge^2(\mathbb{R}^6)^F = \text{span}\{e_2 \wedge e_5, e_3 \wedge e_6, e_1 \wedge e_4\}$, thus $\beta_2 = 3$.

$\bigwedge^3(\mathbb{R}^6)^F = \text{span}\{e_i \wedge e_j \wedge e_k : \text{exactly one of } i,j,k \text{ is 1 or 4 and one is 3 or 6}\}$. That is:

$\bigwedge^3(\mathbb{R}^6)^F = \text{span}\{e_1 \wedge e_3 \wedge e_2,\, e_1 \wedge e_3 \wedge e_5,\, e_1 \wedge e_6 \wedge e_2,\, e_1 \wedge e_6 \wedge e_5,\, e_4 \wedge e_3 \wedge e_2,\, e_4 \wedge e_3 \wedge e_5,\, e_4 \wedge e_6 \wedge e_2,\, e_4 \wedge e_6 \wedge e_5\}$. Hence $\beta_3 = 8$.

$\bigwedge^4(\mathbb{R}^6)^F = \bigwedge^2(\mathbb{R}^6)^F \wedge \bigwedge^2(\mathbb{R}^6)^F = \text{span}\{e_2 \wedge e_5 \wedge e_3 \wedge e_6, e_2 \wedge e_5 \wedge e_1 \wedge e_4, e_3 \wedge e_6 \wedge e_1 \wedge e_4\}$. Thus $\beta_4 = 3$.

$\bigwedge^5(\mathbb{R}^6)^F = 0,\ \bigwedge^6(\mathbb{R}^6)^F = \mathbb{R}\, e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5 \wedge e_6$.

The remaining assertions in the theorem follow immediately from the above description of the F-invariants. □

We now look at the case of $d^2\Gamma$. By Proposition 2.2, $M^{12}_{d^2\Gamma} = d^2\Gamma\backslash\mathbb{R}^{12}$ is a 12-dimensional hyperkähler manifold with holonomy group $F = \mathbb{Z}_2^2$.

Theorem 6.2. *The even cohomology ring $\bigwedge^{ev}(M^{12})^F$ is an exterior algebra generated by $\eta_1, \dots, \eta_{18} \in \bigwedge^2(\mathbb{R}^{12})^F$ i.e., the $\mathbb{R}$-algebra generated by $\{\eta_i : 1 \le i \le 18\}$ subject to the relations $\eta_i^2 = 0, \eta_i\eta_j = \eta_j\eta_i$. The full cohomology ring $\bigwedge^*(\mathbb{R}^{12})^F$ is generated by elements of degree two $\{\eta_i : 1 \le i \le 18\}$ and degree 3 $\{\delta_j : 1 \le i \le 64\}$, subject to the relations:*

$$\eta_i^2 = 0,\ \eta_i\eta_j = \eta_j\eta_i,\ \delta_i^2 = 0,\ \ \delta_i\delta_j = -\delta_j\delta_i,\ \eta_k\delta_m = \delta_m\eta_k$$

and furthermore $\eta_k\delta_m$ may (or may not) be equal to zero. This vanishing can be explicitly given in terms of k, m (but is complicated).

Proof. The holonomy group is generated by:

$$A'' = \begin{bmatrix} A' & \\ & A' \end{bmatrix} \qquad B'' = \begin{bmatrix} B' & \\ & B' \end{bmatrix}$$

Again, we need to compute the F-invariants in $\bigwedge^*(\mathbb{R}^{12})$. Clearly, we have no fixed vectors of degree one, hence $\beta_1 = 0$. Now consider the three complementary sets:

$S_1 = \{e_3, e_6, e_9, e_{12}\} =$ set of basis vectors fixed by A'',
$S_2 = \{e_1, e_4, e_7, e_{10}\} =$ set of basis vectors fixed by B'',
$S_3 = \{e_2, e_5, e_8, e_{11}\} =$ basis vectors fixed by neither of A'', B''.

In degree 2, we note that $e_i \wedge e_j$ is F-invariant if and only if e_i, e_j lie both in one of S_1, S_2 or S_3. Thus $\bigwedge^2(\mathbb{R}^{12})^F = \text{span}\{e_i \wedge e_j : \{e_i, e_j\} \subset S_k,\ k = 1, 2, 3\}$ has dimension $3\binom{4}{2} = 18$.

Similarly, we see that the fixed vectors in higher degrees can be expressed in terms of the sets S_i:

$\bigwedge^3(\mathbb{R}^{12})^F = \text{span}\{e_i \wedge e_j \wedge e_k :$ exactly one of e_i, e_j, e_k lies in S_1, one in S_2 and one in $S_3\}$. Thus $\beta_3 = \binom{4}{1}^3 = 64$.

$\bigwedge^4(\mathbb{R}^{12})^F = \text{span}\{e_i \wedge e_j \wedge e_k \wedge e_m :$ an even number of the e_i, e_j, e_k, e_m lie in each one of $S_1, S_2, S_3\}$. Thus, $\beta_4 = 3\,\binom{4}{4} + 3\binom{4}{2}^2 = 3 + 108 = 111$.

$\bigwedge^5(\mathbb{R}^{12})^F = \text{span}\{e_i \wedge e_j \wedge e_k \wedge e_m \wedge e_h :$ an odd number of the e_i's lie in each one of S_1, S_2, $S_3\}$. Hence $\beta_5 = 3\binom{4}{3}\binom{4}{1}^2 = 3 \times 64 = 192$.

$\bigwedge^6(\mathbb{R}^{12})^F = \text{span}\{e_i \wedge e_j \wedge e_k \wedge e_m \wedge e_h \wedge e_m$: an even number of the e_i's lie in each $S_i,\ i = 1, 2, 3\}$. Hence $\beta_6 = 6\binom{4}{2} + \binom{4}{2}^3 = 36 + 216 = 252$.

Thus, the Poincaré polynomial $p(t)$ is given by:

$$\begin{aligned}1 &+ 18t^2 + 64t^3 + 111t^4 + 192t^5 + 252t^6 + 192t^7 + 111t^8 + 64t^9 + 18t^{10} + t^{12} \\ &= (t+1)^4(t^8 - 4t^7 + 28t^6 - 28t^5 + 10t^4 - 28t^3 + 28t^2 - 4t + 1).\end{aligned}$$

As a verification, note that Salamon's identity (3) holds:

$$\begin{aligned}48.0 + 16.64 + 252 = &\quad 1024 + 252 = 1276 \\ 70 + 30.18 + 6.111 = &\quad 70 + 540 + 666 = 1276.\end{aligned}$$

We now look at the cohomology ring. By the description of the invariants it is clear that $\bigwedge^2(\mathbb{R}^{12})^F$ generates $\bigwedge^{2k}(\mathbb{R}^{12})^F$ for any $0 < k \leq 6$, while it is not hard to check that $\bigwedge^{2k+1}(\mathbb{R}^{12})^F$ can be generated by $\bigwedge^2(\mathbb{R}^{12})^F$ and $\bigwedge^3(\mathbb{R}^{12})^F$. The relations in the statement can also be easily verified.
Thus, the cohomology ring is generated as an algebra by $\bigwedge^2(\mathbb{R}^{12})^F$ and $\bigwedge^3(\mathbb{R}^{12})^F$, as claimed. □

Remark 6.3. There is a natural generalization of the previous example. It is known that for any n odd, there exists a large family of n-dimensional Bieberbach groups with holonomy group $\mathbb{Z}_2^{n-1}$, and such that the corresponding flat manifold M_Γ is a rational homology sphere, i.e. all Betti numbers except β_0, β_n are equal to zero (see [**MR**]). These manifolds generalize the classical 3-dimensional Hantzsche-Wendt manifold ([**Wo**]) and are called HW-manifolds, for short. The argument in the proof of Theorem 6.1 can be adapted to any odd dimension n and gives a similar result on the cohomology ring of $M_{d^2\Gamma}$, for any HW-group Γ.

Indeed, for any n odd, one shows that the cohomology ring of any HW-manifold $\bigwedge^{ev}(\mathbb{R}^{4n})^F$ is generated by $\bigwedge^2(\mathbb{R}^{4n})^F$. Actually, it is an exterior algebra generated by $\{\eta_i : 1 \leq i \leq 6n\}$, subject to the relations $\eta_i^2 = 0,\ \eta_i\eta_j = \eta_j\eta_i,\ i \neq j$.

The full cohomology ring $\bigwedge(\mathbb{R}^{4n})^F$ is generated by $\bigwedge^2(\mathbb{R}^{4n})^F$ and $\bigwedge^3(\mathbb{R}^{4n})^F$. It has generators $\{\eta_i : 1 \leq i \leq 6n\}$ of degree 2, and $\{\delta_j : 1 \leq j \leq 4^n\}$ of degree 3. They satisfy the following relations

$$\eta_i^2 = 0,\ \eta_i\eta_j = \eta_j\eta_i,\ \delta_j^2 = 0,\ \delta_j\delta_i = -\delta_i\delta_j,\ \eta_i\delta_i = \delta_j\eta_i.$$

Furthermore, we note that there are more relations, linking $\eta_i\delta_j$ with $\eta_k\delta_m$, for different i, j, k, m.

The basis $\{e_i\}_{i=1}^{4n}$ is split into n complementary sets $S_1, S_2, \ldots, S_n$, with $S_i = \{e_i, e_{i+3}, e_{i+6}, e_{i+9}\}$, and where S_i is the fixed set of B_i'', for $i = 1, \ldots n$.

By arguing as in the case $n = 3$ we see that
$\bigwedge^2(\mathbb{R}^{4n})^F = \text{span}\{e_r \wedge e_s :$ both e_r, e_s lie in the same set $S_i\}$. Hence we obtain $\beta_2 = n\binom{4}{2} = 6n$.

Similarly
$\bigwedge^{2k}(\mathbb{R}^{4n})^F = \text{span}\{e_{r_1} \wedge \ldots \wedge e_{r_{2k}}$: an even number of e_{r_j}'s lie in each $S_i\}$ for each i.

$\bigwedge^{2k+1}(\mathbb{R}^{4n})^F = \text{span}\{e_{r_1} \wedge ... \wedge e_{r_{2k+1}}$: an odd number of e_{i_j}'s lie in each $S_i\}$ for each i.

We note that this implies in particular that $\beta_i = 0$, for i odd, $i < n$.

Let us illustrate the previous discussion by computing the cohomology for $n = 5$.

Clearly we have $\beta_0 = 1$, $\beta_1 = 0$. Furthermore

$\bigwedge^2(\mathbb{R}^{20})^F \simeq \text{span}\{e_r \wedge e_s$: both e_r, e_s lie in one of the S_i $1 \leq i \leq 5\}$. Thus $\beta_2 = 5\binom{4}{2} = 30$.

$\bigwedge^3(\mathbb{R}^{20})^F \simeq \text{span}\{e_{r_1} \wedge e_{r_2} \wedge e_{r_3}$ such that each S_i $(1 \leq i \leq 5)$ contains one of the $e_{r_j}\}$. That is, $\beta_3 = 0$.

Similarly:
$\beta_4 = 5\binom{4}{4} + \binom{5}{2}\binom{4}{2}^2 = 365,$
$\beta_5 = \binom{4}{1}^5 = 1024,$
$\beta_6 = 20\binom{4}{4}\binom{4}{2} + \binom{5}{3}\binom{4}{2}^3 = 2280,$
$\beta_7 = \binom{5}{1}\binom{4}{3}\binom{4}{1}^4 = 20 \times 256 = 5120,$
$\beta_8 = \binom{5}{2}\binom{4}{4}\binom{4}{4} + 5\binom{4}{2}^3\binom{4}{4} + \binom{5}{4}\binom{4}{2}^4 = = 10 + 5 \times 216 + 30 \times 216 = 7570,$
$\beta_9 = \binom{5}{2}\binom{4}{3}^2\binom{4}{1}^3 = 10240,$
$\beta_{10} = \binom{5}{2}3 \times \binom{4}{2} + \binom{5}{1}\binom{4}{4}\binom{4}{3}\binom{4}{2}^3 + \binom{4}{2}^5 = 10 \times 18 + 20 \times 216 + 36 \times 216 = 122276.$

Therefore, we finally obtain:

$$p(t) = \quad 1 + t^{20} + 30(t^2 + t^{18}) + 365(t^4 + t^{16}) + 1024(t^5 + t^{15}) + 2280(t^6 + t^{14}) + 5120(t^7 + t^{13}) + 7570(t^8 + t^{12}) + 10240(t^9 + t^{11}) + 12276t^{10}.$$

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