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CARNOT MANIFOLDS

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SUBMATH:

Subriemannian Geometry, Subelliptic Operators, Subanalytic Varieties, Subsymplectic Geometry.
Only eventually (not originally) related.
Motivations: PDE, Control Theory, non-holonomic systems, ...

Subriemannian Geometry:

M = smooth or analytic manifold
V = distribution on M (subbundle of $T(M)$, smooth or analytic)
g = inner product on V, smooth or analytic

A submanifold $N \subset M$ is *horizontal* if $\forall p \in N$

$$T_p(N) \subset V_p.$$

• Must think that moving along non-horizontal directions is forbidden. Analogy with Parking Problem.

• Whatever the V, $\exists$ always horizontal *curves*: integral curves of vector fields in V.

• Maximal horizontal submanifolds? 2nd. half of talk.

• Horizontal curves $\gamma : [a, b] \to M$ have

$$Lenght(\gamma) = \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))}\ dt$$

Carnot-Carathéodory "distance" on M:

$$d(p, q) = \inf_\gamma \{Lenght(\gamma)\}$$

γ horizontal, $\gamma(a) = p$, $\gamma(b) = q$,

$$d(p, q) = \infty$$

if $\exists$ no such curve.

• Main problems since ~ 1985:
Regularity of $d(p,q)$
Recovering V from d (Gromov)
Admissible domains for Bdy. Value Problems for

Subelliptic Operators

Defined by regularity condition: $\exists \epsilon > 0$:

$$||f||_{H^\epsilon} \leq C(||Df||_{H^0} + ||f||_{H^0})$$

True for elliptic D, where $\epsilon = 1$, wherefrom

$$Df = g,\ g \text{ smooth } \Rightarrow f \text{ smooth}$$

(def. of hypoelliptic).

But false for $\frac{\partial}{\partial x} f(x,y) = 0$! (take $f(x,y) = \phi(y)$ arbitrary).

Important special case: sublaplacians

$$D = X_1^2 + ... + X_k^2$$

X_i vector fields on a manifold M

• Classical: if $\{X_i(p)\}_{i=1}^n$ is a basis of $T_p(M)$ $\forall p$, then D elliptic. This implies Existence, Uniqueness and Regularity for

$$Df = g$$

• Relation with previous situation:

$$V_p \leftrightarrow span\{X_i(p)\}$$

• Kohn's sublaplacian in $\mathbf{R}^3$ (bdy. value of the Bargmann laplacian of the 3-ball in C^2):

$$D = \Big(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\Big)^2 + \Big(\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\Big)^2$$

not elliptic, degenerates along z-axis. Still, hypoelliptic, even analytically so:

$$Df = g \in C^r \quad \Rightarrow \quad f \in C^r \qquad (r = \infty, \omega)$$

Ultimate reason: R^3 is spanned by

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \qquad Y = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \qquad [X,Y] = -2\frac{\partial}{\partial z}$$

Remark: Kohn's sublaplacian can be viewed as the boundary value of the laplacian of the 3-ball in C^2, with its Bargmann metric.

Classes of distributions

$\mathcal{T}$ = tangent sheaf of M (germs of vector fields)

$\mathcal{V}$ = subsheaf of fields $X_q \in V_q$.

Filtration of $\mathcal{T}$ generated by $\mathcal{V}$:

$$\begin{aligned} \mathcal{W}_1 &= \mathcal{V} \\ \mathcal{W}_2 &= \mathcal{V} + [\mathcal{V},\mathcal{V}] \\ \mathcal{W}_3 &= \mathcal{V} + [\mathcal{V},\mathcal{V}] + [\mathcal{V},[\mathcal{V},\mathcal{V}]] \\ &\vdots \end{aligned}$$

- V Involutive: $\mathcal{W}_j = \mathcal{V}\ \forall j \qquad ([\mathcal{V},\mathcal{V}] \subset \mathcal{V})$.

- V Outvolutive, bracket-generating, fat: $\exists k \mathcal{W}_k = \mathcal{T}$

  Recall *Involutive* case:

- Frobenius: *Involutive* ⇒ *completely integrable:*

$$M = \biguplus_\alpha N_\alpha, \qquad V_p = T_p(N_{\alpha(p)})$$

with the leaves N_α maximal.

Involutive subriemannian geometry? Not very interesting: each N_α is riemannian. d_{CC} distance between leaves is ∞. $\sum X_i^2$ is far from regular. Still, Lie algebras of vector fields are fundamental in Control. If L is such, then $p \mapsto L(p)$ not a distribution: $\dim L(p)$ may jump. It is a "Distribution with singularities", but involutive, so one can ask

- Is every point of M contained in a unique maximal integral submanifold of L?

  Answer: NO for smooth, YES for analytic.

(Hermann-Nagano Theorem). This is why real analyticity – and, eventually, subanalyticity – eventually come in [S].

Outvolutive distributions

• Example: $M = \mathbf{R}^3$

$$\mathcal{V} = span\{X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z},\ \ Y = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\}$$

$$[X,Y] = -2\frac{\partial}{\partial z} \notin \mathcal{V}$$

so $X, Y, [X,Y]$, span $T_p(\mathbf{R}^3)$ everywhere.

From now on, V will be outvolutive.

THEOREMS. Assume M can be connected with smooth arcs.

• Chow (anti-Frobenius). *Any two points can be joined by a smooth horizontal curve.*

$\Rightarrow d(p,q) < \infty$ for any subriemannian structure on (M,V).

• Regularity of $d(p,q)$ is critical. For example, unless $V \neq T(M)$, *the function* $p \mapsto d(p_o,p)$ *is not continuously differentiable in any punctured neighborhood of* p_o*!*

Instead,

• Agrachev:

(M,V,g) *analytic* $\Rightarrow$ $d(p_o,p)$ *subanalytic*

i.e., d-balls are subanalytic.

Subanalytic sets: locally projections of semianalytic. Equivalently, locally of the form

$$\{\mathbf{x} :\ f_i(\mathbf{x},\mathbf{y}) = 0,\ g_j(\mathbf{x},\mathbf{y}) \geq 0\}$$

with f_i, g_j, real analytic.

Lojasiewicz, Sussmann, ...

As to subellipticity,

• Hormander:

If X_i *is a local basis of* $\mathcal{V}$*, then* $\sum X_j^2$ *is hypoelliptic*

Horizontal Submanifolds

$$N \hookrightarrow M : \qquad T_p(N) \subset V_p$$

• $\exists$ many horizontal curves. Higher dimension?

Integral objects of non-integrable things are likely interesting. Also occur spontaneously in minimal surfaces, Jets of Maps, Control, ... But are hard to find, no general pattern.

Models:

Carnot Groups

$$Lie(G) = \mathbf{g} = \mathbf{v}_1 \oplus \mathbf{v}_2 \oplus ... \oplus \mathbf{v}_k$$

satisfying

$$[\mathbf{v}_1, \mathbf{v}_j] = \mathbf{v}_{j+1}$$

Canonical distribution on G:

$$V = \mathbf{v}_1$$

• Origin: Gromov's Theorems on growth of discrete groups

• Not just examples: any outvolutive distribution filters $\mathcal{T}_M$. The associated graded

$$Gr^W = \bigoplus_j \mathcal{W}_j/\mathcal{W}_{j+1}$$

is a sheaf of Carnot algebras.

• How many? Even step 2

$$\mathbf{g} = \mathbf{v} \oplus \mathbf{z} \qquad \mathbf{z} = [\mathbf{v}, \mathbf{v}]$$

no classification is possible for $\dim \mathbf{z} > 3$. (Bernstein-Gelfand-Ponomarev-Gabriel-Coxeter-Dynkin Diagram has $\dim \mathbf{z}$ edges joining 2 vertices)

Models of models?

Groups of Heisenberg type

But "as role models go, they are hard to emulate": only Carnot groups with abundant domains admissible for the Dirichlet problem and/or explicit fundamental solutions for $\sum X_i^2$, weakly convex gauge ...

• Definition: $\mathbf{g} = \mathbf{v} \oplus \mathbf{z}$ with inner products such that

$$J_z : \mathbf{v} \to \mathbf{v}$$

defined by

$$(J_z u, u')_{\mathbf{v}} = (z, [u, u'])_{\mathbf{z}}$$

satisfies

$$J_z^2 = -2|z|^2 I.$$

Equivalently: J_z defines unitary representation of Cliff($\mathbf{z}$) on $\mathbf{v}$.

Groups of H type $\approx$ Clifford modules

Parametrized by 2 or 3 natural numbers

$$\mathbf{g} = (n\mathbf{S}_m) \oplus \mathbf{R}^m$$

or

$$\mathbf{g} = (n_+\mathbf{S}_m^+ \oplus n_-\mathbf{S}_m^-) \oplus \mathbf{R}^m$$

$\mathbf{S}_m, \mathbf{S}_m^+, \mathbf{S}_m^-$, spinor spaces.

• Analogy with symplectic.

• History: fundamental solution for $D = \sum X_j^2$:

$$D\Phi = \delta \qquad \text{for} \qquad \Phi(\exp(v+z)) = \frac{C}{(|v|^4 + 16|z|^2)^N}$$

Since then keep yielding interesting riemannian examples (K., Willmore-Damek-Ricci, Selberg-Lauret, Gordon, Szabo, ...).

• $Aut(G)$ is largest. As to subriemannian:

"Manifolds of Heisenberg type are to subriemannian Geometry, as Euclidean spaces, or symmetric spaces, are to riemannian geometry"

• The search for maximal horizontal submanifolds in groups of Heisenberg type is joint work with Levstein, Saal, Tiraboschi.

• *In any Carnot group, for any horizontal submanifold N and any point $p \in N$, there exists a unique horizontal subgroup $G_{N,p}$ such that*

$$T_p(N) = (L_p)_*(T_e(G_{N,p})).$$

Any horizontal subgroup is abelian.

Hope to find all the latter. The following points to other reasons

$$\begin{array}{cc} \textit{Maximal isotropic subspaces of } (\mathbf{v}, [\ ,\ ]) & U \\ \updownarrow & \\ \textbf{Maximal horizontal subgroups of } G & \exp(U) \\ \updownarrow & \\ \textit{Maximal abelian subgroups of } G & \exp(U+Z) \end{array}$$

• Classical examples to keep in mind:

On the 3-dimensional Heisenberg group, the distribution is 2-dimensional and the maximal horizontal submanifolds are 1-dimensional. But too many. Subgroups then.

On the $2n+1$ dimensional Heisenberg, the distribution is 2n-dimensional and the maximal horizontal submanifolds are n-dimensional. Distinguished class: $\exp(U)$ with $U \subset \mathbf{v}$ totally isotropic in the usual sense.

• In general, maximal possible dimension is $\dim \mathbf{v}/2$: "Lagrangian" subspaces. Not always achieved. $Lag(G)$ is a variety. Have a description (to be presented by Levstein in Colonia).

• Relation with Schroedinger Representations (?), Deligne's "Reality and the Heisenberg group".

SOME CONCLUSIONS

• Sometimes $Lag(G) = \emptyset$

• Sometimes any two Lagragians are conjugate by an automorphism of G, sometimes not.

• Sometimes $Lag(G)$ is a group. For example, if $m \equiv 7 \bmod 8$ and

$$Lie(G) = (\mathbf{S}_m^+)^n \oplus (\mathbf{S}_m^-)^n \oplus \mathbf{R}^m$$

then

$$Lag(G) = O(n)$$

• Always $Lag(G)$ = finite $\bigcup$ of $Aut(G)$-orbits, of the form K/K', with K, K', reductive

• NEXT: Maximal, but dim$< n/2$?

Examples! In G = Quaternionic 7-dimensional or Octonionic 15-dimensional Heisenberg there are no horizontal submanifolds of dimension > 1. The distribution has dimension 4 and 8, respectively.

Bibliography

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