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4-STEP CARNOT SPACES AND THE 2-STEIN CONDITION

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Dedicated to the memory of my goddaughter María Pía

ABSTRACT. We consider the 2-stein condition on k-step Carnot spaces S. These spaces are a subclass in the class of solvable Lie groups of Iwasawa type of algebraic rank one and contain the homogeneous Einstein spaces within this class. They are obtained as a semidirect product of a graded nilpotent Lie group N and the abelian group $\mathbf{R}$.

We show that the 2-stein condition is not satisfied on a proper 4-step Carnot spaces S.

A Riemannian manifold M is said to be a 2-stein space, if there exist functions μ_l, $l = 1, 2$, defined on M such that

$$\operatorname{tr}(R_X^l) = \mu_l \, |X|^{2l}, \; l = 1, 2, \text{ for all } X \in TM.$$

Here, R_X denote the Jacobi operator associated to X, defined by $R_X Y = R(Y, X)X$ for all $Y \in TM$, where R is the curvature tensor of M. Harmonic riemannian manifolds are necessarily 2-stein.

A k-step Carnot space ($k \geq 2$) is a simply connected solvable Lie group S, which is a semidirect product of a nilpotent Lie group N and the abelian group $\mathbf{R}$. Assume that S and N have associated Lie algebras $\mathfrak{s}$ and $\mathfrak{n}$, respectively. S has the left invariant metric induced by the one given on $\mathfrak{s}$, where $\mathfrak{s}$ is a solvable metric Lie algebra $\mathfrak{s}$ with inner product $\langle , \rangle$ such that:

(i) $\mathfrak{s} = \mathfrak{n} \oplus \mathbf{R}H$ with $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ and $H \perp \mathfrak{n}$, $|H| = 1$.

(ii) $\mathfrak{n}$ has an orthogonal decomposition $\mathfrak{n} = \sum_{i=1}^{k-1} \mathfrak{n}_i$ into $(k-1)$ subspaces given by

$$\mathfrak{n}_i = \{X \in \mathfrak{n} : \operatorname{ad}_H(X) = i\alpha X\}, \; i = 1, ..., k-1,$$

for some positive constant $\alpha \in \mathbf{R}$.

Note that, since the adjoint representation ad_H is a derivation of $\mathfrak{n}$ the above decomposition defines a graded Lie algebra structure of $\mathfrak{n}$, that is,

$$[\mathfrak{n}_i, \mathfrak{n}_j] \subset \mathfrak{n}_{i+j} \quad 1 \leq i, j \leq k-1$$

with the convention $\mathfrak{n}_i = \{0\}$ for $i > k-1$. In particular, $\mathfrak{n}$ is a $(k-1)$-step nilpotent Lie algebra, that is, the $(k-1)$-th derived algebra vanishes: $\mathfrak{n}^{k-1} = [\mathfrak{n}, ...[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]]...] = \{0\}$.

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Note, that up to a constant in the metric, we may suppose that $\alpha = 1$. In this terminology, the 3-Carnot spaces are the well known Carnot spaces and the case of a 2-step Carnot space corresponds to the hyperbolic symmetric space $\mathbf{R}H^n$. Moreover, due to a result of [6], any homogeneous Einstein space of Iwsawa type and rank one is a k- step Carnot space for some $k \geq 2$.

We remark that the 2-stein condition was studied in [1] and [3] on Lie groups S of Iwasawa type in the case of 2-step nilpotent $\mathfrak{n}$, and in particular, on Carnot spaces: in this class those which are 2-stein are exactly the Damek-Ricci spaces (also the harmonic ones). Furthermore, in [7] is shown that a simply connected homogenous harmonic space is a Carnot space, which in turn is equivalent to S be a Damek-Ricci space.

1. k-step Carnot spaces

Assume that S is a k-step Carnot space. Let $\mathfrak{s}$ be the Lie algebra of S: that is $\mathfrak{s}$ is a solvable metric Lie algebra $\mathfrak{s} = \mathfrak{n} \oplus \mathbf{R}H$ with $H \perp \mathfrak{n}$, $|H| = 1$, with $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ a graded nilpotent Lie algebra as described above.
Let $\mathfrak{z}$ denote the center of $\mathfrak{n}$ and let $\mathfrak{v}$ be the orthogonal complement of $\mathfrak{z}$ with respect to the metric $\langle , \rangle$ restricted to $\mathfrak{n}$. Thus $\mathfrak{n}$ decomposes $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ and note that $\mathrm{ad}_H : \mathfrak{z} \to \mathfrak{z}$ hence $\mathrm{ad}_H : \mathfrak{v} \to \mathfrak{v}$ since ad_H is symmetric.

For any $Z \in \mathfrak{z}$ the skew-symmetric linear operator $j_Z : \mathfrak{v} \to \mathfrak{v}$ is defined by

$$\langle j_Z X, Y \rangle = \langle [X, Y], Z \rangle \text{ for all } X,\ Y \in \mathfrak{v},\ \text{all } Z \in \mathfrak{z}.$$

The Levi Civita connection and the curvature tensor on S can be computed by,

$$\begin{aligned} 2\langle \nabla_X Y, Z \rangle &= \langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle \\ R(X, Y) &= [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \end{aligned}$$

for any X, Y, Z in $\mathfrak{s}$.

Recall that since $\mathfrak{z}$ and $\mathfrak{v}$ are ad_H-invariant, they also have decompositions into eigenspaces as $\mathfrak{z} = \sum_\lambda \mathfrak{z}_\lambda$, $\mathfrak{v} = \sum_\mu \mathfrak{v}_\mu$ with $\mathfrak{z}_\lambda = \mathfrak{n}_\lambda \cap \mathfrak{z}$, $\mathfrak{v}_\mu = \mathfrak{n}_\mu \cap \mathfrak{v}$ and the equality $\lambda = \mu$ may be possible. Moreover, $\mathfrak{n}$ is said to be 2-step nilpotent if $[\mathfrak{n}, \mathfrak{n}] = [\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}$. Here, $\{\lambda\}$ and $\{\mu\}$ denote the eigenvalues of $\mathrm{ad}_H|_{\mathfrak{z}}$ and $\mathrm{ad}_H|_{\mathfrak{v}}$, respectively, that we assume they are natural numbers.

1.1. **Properties of the operator j_Z, $Z \in \mathfrak{z}$.** We recall that for any $Z \in \mathfrak{z}_\lambda$ so that $j_Z \neq 0$, $j_Z : \mathfrak{v}_\mu \to \mathfrak{v}_{\lambda-\mu} \to \mathfrak{v}_\mu$ and $j_Z : \left(\ker j_Z|_{\mathfrak{v}_\mu}\right)^\perp \to \left(\ker j_Z|_{\mathfrak{v}_{\lambda-\mu}}\right)^\perp$ isomorphically. Consequently, as follows from [5, Lemma1.1])

$$\mathrm{tr}\left(j_Z^2|_{\mathfrak{v}_\mu}\right) = \mathrm{tr}\left(j_Z^2|_{\mathfrak{v}_{\lambda-\mu}}\right).$$

1.2. **The curvature formulas.** By applying the connection formula, one obtains $\nabla_H = 0$ and if $Z,\ Z^* \in \mathfrak{z}$ and $X,\ Y \in \mathfrak{v}$, then $\nabla_Z Z^* = \nabla_{Z^*} Z = \langle [H, Z], Z^* \rangle H$, $\nabla_X Z = \nabla_Z X = -\frac{1}{2} j_Z X$ and $\nabla_X Y = \frac{1}{2}[X, Y] + \langle [H, X], Y \rangle H$ in case of 2-step nilpotent $\mathfrak{n}$. Consequently, by a direct computation we obtain the following formulas involving curvatures (see [2, Section 2]).

(i) $R_H = -\mathrm{ad}_H^2$

(ii) If either $Z \in \mathfrak{z}_\lambda$, $|Z| = 1$, or $X \in \mathfrak{v}_\mu$, $|X| = 1$,

$$R_Z H = -\lambda^2 H \text{ and } R_X H = -\mu^2 H.$$

(iii) If $Z \in \mathfrak{z}_\lambda$, $|Z| = 1$, for any $Z^* \in \mathfrak{z}$ and $X \in \mathfrak{v}$,then

$$R_Z Z^* = \lambda\left(\langle Z, \mathrm{ad}_H Z^*\rangle Z - \mathrm{ad}_H Z^*\right) \text{ and } R_Z X = -\frac{1}{4} j_Z^2(X) - \lambda \mathrm{ad}_H X.$$

(iv) Let μ_1 denote the maximum eigenvalue of $\mathrm{ad}_H|_{\mathfrak{v}}$. If $X \in \mathfrak{v}_{\mu_1}$, $|X| = 1$, then

$$\begin{aligned} R_X Z &= \frac{1}{4}[X, j_Z X] - \mu_1 \mathrm{ad}_H Z - \frac{1}{4}(\mathrm{ad}_{j_Z X})^* X,\ Z \in \mathfrak{z} \\ R_X Y &= -\frac{1}{2}[X, \nabla^{\mathfrak{n}}_X Y] - \frac{3}{4} j_{[X,Y]} X - \nabla^{\mathfrak{n}}_X (\nabla^{\mathfrak{n}}_X Y)_{\mathfrak{v}} - \mu_1 \mathrm{ad}_H Y,\ Y \perp X \text{ in } \mathfrak{v}. \end{aligned}$$

1.3. **The Einstein condition.** Assume that S is an Einstein space, that is $\mathrm{Ric}(Y) = c\,|Y|^2$ for all $Y \in \mathfrak{s}$. Let $\{Y_i\}_{i=1}^m$ and $\{Z_i\}_{i=1}^k$ be orthonormal bases of $\mathfrak{v}$ and $\mathfrak{z}$, respectively, with $k = \dim \mathfrak{z}$ and $m = \dim \mathfrak{v}$. We use the well known formulas that hold in a solvable metric Lie algebra $\mathfrak{s} = \mathfrak{n} \oplus \mathbf{R}H$,

$$\begin{aligned} \mathrm{Ric}(Z) &= \frac{1}{4}\sum_{i=1}^m |j_Z X_i|^2 - \lambda \,\mathrm{tr}\,\mathrm{ad}_H \text{ for } Z \in \mathfrak{z}_\lambda, \\ \mathrm{Ric}(X) &= -\frac{1}{2}\sum_{i=1}^k |j_{Z_i} X|^2 - \mu \mathrm{tr}\ \mathrm{ad}_H \text{ for } X \in \mathfrak{v}_\mu, \text{ if } \mathfrak{n} \text{ is 2-step-nilpotent.} \end{aligned}$$

Note that the first Einstein condition becomes

$$\mathrm{tr}\left(-\frac{1}{4} j_Z^2\right) = -\mathrm{tr}\,\mathrm{ad}_H^2 + \lambda \mathrm{tr}\,\mathrm{ad}_H,\ \text{for } Z \in \mathfrak{z}_\lambda,\ |Z| = 1. \tag{1}$$

In general, for a unit vector $X \in \mathfrak{v}_{\mu_1}$, μ_1 the maximun eigenvalue of $\mathrm{ad}_H|_{\mathfrak{v}}$, we have

$$\mathrm{Ric}(X) = -\frac{1}{2}\sum_{i=1}^k |j_{Z_i} X|^2 - \mu_1 \mathrm{tr}\ \mathrm{ad}_H + \sum_{i=1}^m |(\nabla^{\mathfrak{n}}_X Y_i)_{\mathfrak{v}}|^2. \tag{2}$$

The last one is a direct computation by applying the above expressions of R_X in (iv) to

$$\mathrm{Ric}(X) = \mathrm{tr} R_X = \sum_{j=1}^k \langle R_X Z_j, Z_j\rangle + \sum_{i=1}^m \langle R_X Y_i, Y_i\rangle,$$

having into account that the following equality holds,

$$\sum_{i=1}^m |[X, Y_i]|^2 = \sum_{j=1}^k |j_{Z_i} X|^2.$$

In fact, this follows by computing

$$\begin{aligned} \sum_{i=1}^m |[X, Y_i]|^2 &= \sum_{i=1}^m \sum_{j=1}^k \langle [X, Y_i], Z_j\rangle^2 = \sum_{i=1}^m \sum_{j=1}^k \left\langle j_{Z_j} X, Y_i\right\rangle^2 \\ &= \sum_{j=1}^k \left(\sum_{i=1}^m \left\langle j_{Z_j} X, Y_i\right\rangle^2\right) = \sum_{j=1}^k \left|j_{Z_j} X\right|^2, \end{aligned}$$

since $[X, Y_i] \in \mathfrak{z}$ for any $X \in \mathfrak{v}_{\mu_1}$.

1.4. **The 2-stein condition.** For any $X \in \mathfrak{s}$, the Jacobi operator R_X associated to X is the symmetric endomorphism of $\mathfrak{s}$ defined by $R_X Y = R(Y, X)X$. We said that S is a 2-stein space, or equivalently S (or $\mathfrak{s}$) satisfies the 2-stein condition, if there exist

$$\operatorname{tr}(R_X^l) = \mu_l \, |X|^{2l}, \; l = 1, 2, \text{ for all } X \in \mathfrak{s}.$$

In particular, 2-stein spaces are Einstein: $\operatorname{Ric}(X) = \operatorname{tr} R_X = \mu_1 |X|^2$, μ_1 a constant, for all $X \in \mathfrak{s}$, and harmonic riemannian spaces are necessarily 2-stein. The 2-stein condition was studied on Carnot spaces in [1] and [3]: the 2-stein Carnot spaces are exactly the Damek-Ricci spaces (also the harmonic ones) in this class.

The proposition below express the 2-stein condition in an adequate form for our purposes. Let $\{\mu\}$ denote the set of eigenvalues of $\operatorname{ad}_H|_{\mathfrak{v}}$.

Proposition 1.1. *Assume that $\mathfrak{s}$ satisfies the 2-stein condition. If $Z \in \mathfrak{z}_\lambda$ is a unit vector, then*

$$tr(\, ad_H^4)\big|_{\mathfrak{z} \cap \mathfrak{z}_\lambda^\perp \oplus \mathfrak{v}} = tr\,(-R_Z \circ ad_H^2)\big|_{\mathfrak{z} \cap \mathfrak{z}_\lambda^\perp \oplus \mathfrak{v}} + \frac{1}{2} \sum_\mu (\lambda - 2\mu)^2 tr\left(-\frac{1}{4}\, j_Z^2\big|_{\mathfrak{v}_\mu}\right).$$

In particular, if $ad_H|_{\mathfrak{z}} = \lambda Id$

$$tr\left(ad_H^4\big|_{\mathfrak{v}}\right) = tr\left(-R_Z \circ ad_H^2\big|_{\mathfrak{v}}\right) + \frac{1}{2} \sum_\mu (\lambda - 2\mu)^2 tr\left(-\frac{1}{4}\, j_Z^2\big|_{\mathfrak{v}_\mu}\right). \qquad (3)$$

Proof. It follows from the same argument as those used in Theorem 4.1 of [4] for $k = 2$ (See also [5, Proposition 1.2]). □

2. The 2-stein condition on 4-step Carnot spaces

We consider the case of a 4-step Carnot space; that is, $\mathfrak{s} = \mathfrak{n} \oplus \mathbf{R}H$ as above with $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3$. Hence, by the propeties of $\mathfrak{n}_i$, $[\mathfrak{n}_1, \mathfrak{n}_1] \subset \mathfrak{n}_2$, $[\mathfrak{n}_1, \mathfrak{n}_2] \subset \mathfrak{n}_3$, $[\mathfrak{n}_1, \mathfrak{n}_3] = 0$, $[\mathfrak{n}_2, \mathfrak{n}_2] = 0 = [\mathfrak{n}_2, \mathfrak{n}_3]$, $[\mathfrak{n}_3, \mathfrak{n}_3] = 0$ and $\mathfrak{n}_3 \subset \mathfrak{z}$. We observe that:
$[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{n}_2 \oplus \mathfrak{n}_3$ and $[[\mathfrak{n}, \mathfrak{n}], \mathfrak{n}] \subset [\mathfrak{n}_2 \oplus \mathfrak{n}_3, \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3] \subset \mathfrak{n}_3$. Thus, $\mathfrak{n}^3 = 0$ since $[[[\mathfrak{n}, \mathfrak{n}], \mathfrak{n}], \mathfrak{n}] \subset [\mathfrak{n}_3, \mathfrak{n}] = 0$; that is $\mathfrak{n}$ is 3-step nilpotent and $\mathfrak{n}_3 \subset \mathfrak{z}$. Since $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ we have that $\mathfrak{v} \subset \mathfrak{n}_1 \oplus \mathfrak{n}_2$. Moreover, from the facts that $\mathfrak{z}$ and $\mathfrak{v}$ are ad_H-invariant, it follows that

$$\mathfrak{z} = \mathfrak{n}_1 \cap \mathfrak{z} \oplus \mathfrak{n}_2 \cap \mathfrak{z} \oplus \mathfrak{n}_3 \text{ and } \mathfrak{v} = \mathfrak{n}_1 \cap \mathfrak{v} \oplus \mathfrak{n}_2 \cap \mathfrak{v}.$$

We set $\dim \mathfrak{n}_i = n_i$, $i = 1, 2, 3$. Following with the notation introduced in the previous section we have,

Lemma 2.1. *If S is a 4-step Carnot space that satisfies the 2-stein condition, then S is either a Damek-Ricci space or, up to scaling, the symmetric hyperbolic space $S = \mathbf{R}H^n$ ($\mathfrak{n}_2 = 0$ or $\mathfrak{n}_3 = 0$) with $n = \dim S$. If $\mathfrak{n}_3 \neq 0$ the equalities $\mathfrak{z} = \mathfrak{n}_3$, $\mathfrak{v} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ hold in the decomposition of $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$.*

Proof. (i) $\mathfrak{n}_1 \cap \mathfrak{z} = 0$, if the Einstein condition is satisfied.
If $0 \neq Z \in \mathfrak{n}_1 \cap \mathfrak{z}$, $\langle j_Z X, Y\rangle = \langle [X,Y], Z\rangle = 0$ for all $X,\ Y \in \mathfrak{v}$ since $[X,Y] \in \sum_{i\geq 2} \mathfrak{n}_i$ and $\langle \mathfrak{n}_i, \mathfrak{n}_j\rangle = 0$ if $i \neq j$. Hence, $j_Z = 0$ and $\operatorname{tr}(j_Z^2) = 0$, and the Einstein condition (1) implies that

$$\begin{aligned} \operatorname{tr}\left(-\frac{1}{4} j_Z^2\right) &= -\operatorname{tr}\operatorname{ad}_H^2 + \operatorname{tr}\operatorname{ad}_H = -(n_1 + 4n_2 + 9n_3) + (n_1 + 2n_2 + 3n_3) \\ &= -(2n_2 + 6n_3). \end{aligned}$$

Thus, $n_2 = n_3 = 0$, and consequently $\mathfrak{n} = \mathfrak{n}_1$. In this case $\mathfrak{n}$ is abelian and S corresponds to the symmetric hyperbolic space $\mathbf{R}H^n$ with $n = n_1 + 1$.

The previous fact implies that the eigenvalue $\mu = 1$ is not achieved in $\mathfrak{z}$, when $\mathfrak{n}$ is 3-step properly. Thus,

$$\mathfrak{z} = \mathfrak{n}_3 \oplus \mathfrak{n}_2 \cap \mathfrak{z} \quad \text{and } \mathfrak{v} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \cap \mathfrak{v}.$$

(ii) $\mathfrak{n}_2 \cap \mathfrak{z} = 0$ if the 2-stein condition holds.

In fact, assume that there exists $0 \neq Z \in \mathfrak{n}_2 \cap \mathfrak{z}$. In this case, if $|Z| = 1$ condition (1) gives

$$\begin{aligned} \operatorname{tr}\left(-\frac{1}{4} j_Z^2\right) &= -\operatorname{tr}\operatorname{ad}_H^2 + 2\operatorname{tr}\operatorname{ad}_H = -(n_1 + 4n_2 + 9n_3) + 2(n_1 + 2n_2 + 3n_3) \\ &= n_1 - 3n_3. \end{aligned}$$

Moreover, since $Z \in \mathfrak{n}_2$ and $\langle j_Z X, Y\rangle = \langle [X,Y], Z\rangle = 0$ for all $X \in \mathfrak{n}_2 \cap \mathfrak{v}$, $Y \in \mathfrak{v}$, we have that $j_Z|_{\mathfrak{n}_2 \cap \mathfrak{v}} = 0$ and $j_Z : \mathfrak{n}_1 \to \mathfrak{n}_1$. Now, we apply the formula given in Proposition 1.1 for $\lambda = 2$, and $\mu = 1, 2$. We first note that,

$$\frac{1}{2} \sum_{\mu=1,2} (\lambda - 2\mu)^2 \operatorname{tr}\left(-\frac{1}{4}\ j_Z^2|_{\mathfrak{v}_\mu}\right) = 0.$$

since $j_Z|_{\mathfrak{n}_2 \cap \mathfrak{v}} = 0$ and $\lambda - 2\mu = 0$ for $\mu = 1$. This implies that $\operatorname{tr}(\operatorname{ad}_H^4)|_{\mathfrak{z} \cap \mathfrak{z}_\lambda^\perp \oplus \mathfrak{v}} = \operatorname{tr}(-R_Z \circ \operatorname{ad}_H^2)|_{\mathfrak{z} \cap \mathfrak{z}_\lambda^\perp \oplus \mathfrak{v}}$ and consequently, equality holds in the Cauchy-Schwartz Inequality

$$\begin{aligned} \left(\operatorname{tr}\ (-R_Z \circ \operatorname{ad}_H^2)|_{\mathfrak{z} \cap \mathfrak{z}_\lambda^\perp \oplus \mathfrak{v}}\right)^2 &\leq \operatorname{tr}\ (-R_Z)^2|_{\mathfrak{z} \cap \mathfrak{z}_\lambda^\perp \oplus \mathfrak{v}} \operatorname{tr}\ (\operatorname{ad}_H^2)|_{\mathfrak{z} \cap \mathfrak{z}_\lambda^\perp \oplus \mathfrak{v}} \\ &= \operatorname{tr}\ \operatorname{ad}_H^4|_{\mathfrak{z} \cap \mathfrak{z}_\lambda^\perp \oplus \mathfrak{v}} \operatorname{tr}\ \operatorname{ad}_H^4|_{\mathfrak{z} \cap \mathfrak{z}_\lambda^\perp \oplus \mathfrak{v}} = \left(\operatorname{tr}\ \operatorname{ad}_H^4|_{\mathfrak{z} \cap \mathfrak{z}_\lambda^\perp \oplus \mathfrak{v}}\right)^2. \end{aligned}$$

Therefore, $-R_Z = c\operatorname{ad}_H^2$ in $\mathfrak{z} \cap \mathfrak{z}_\lambda^\perp \oplus \mathfrak{v}$ with $c = 1$, by the Einstein condition. The equality $R_Z|_{\mathfrak{n}_3} = -\operatorname{ad}_H^2|_{\mathfrak{n}_3}$, implies that $\mathfrak{n}_3 = 0$, since $Z \in \mathfrak{n}_2 \cap \mathfrak{z}$ and $R_Z|_{\mathfrak{n}_3} = -6\mathrm{Id}$. Hence, $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ with $\mathfrak{n}_1 = \mathfrak{v}$, $\mathfrak{n}_2 = \mathfrak{z}$ and S is a Carnot space. In this case S is a Damek-Ricci space (see [3, Theorem 3.1]). ☐

Assume that $\mathfrak{n}_3 \neq 0$. If $\mathfrak{s}$ satisfies the 2-stein condition then $\mathfrak{z} = \mathfrak{n}_3$, and $\mathfrak{v} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$.
The following proposition is basic for our purposes,

Proposition 2.2. *If S is a 4-step Carnot space that is 2-stein, then $n_1 = 2n_2$ and $-\frac{1}{4}\, j_Z^2\big|_{\mathfrak{n}_2} = 3Id$ for any unit vector $Z \in \mathfrak{z}$. Equivalently, $|j_Z X|^2 = 12$ for all unit vectors $Z \in \mathfrak{z}$ and $X \in \mathfrak{n}_2$.*

Proof. Next we will show that $n_1 = 2n_2$,

$$\operatorname{tr}\left(-\frac{1}{4}\, j_Z^2\big|_{\mathfrak{n}_2}\right) = 3n_2 \text{ and } \operatorname{tr}\left(\frac{1}{16}\, j_Z^4\big|_{\mathfrak{n}_2}\right) = 9n_2$$

for any unit vector $Z \in \mathfrak{n}_3$.
Let $Z \in \mathfrak{z} = \mathfrak{n}_3$ be a unit vector. The Einstein condition (1) applied to Z gives,

$$\begin{aligned} \operatorname{tr}\left(-\frac{1}{4} j_Z^2\right) &= -(1n_1 + 4n_2 + 9n_3) + 3(n_1 + 2n_2 + 3n_3) \qquad (4) \\ &= 2(n_1 + n_2). \end{aligned}$$

We compute separately the two terms in (3). First, using the definition of $R_Z\big|_{\mathfrak{v}} = -\frac{1}{4} j_Z^2 - 3\, \mathrm{ad}_H\big|_{\mathfrak{v}}$, it follows that

$$\begin{aligned} \operatorname{tr}\left(-R_Z \circ \mathrm{ad}_H^2\right)\big|_{\mathfrak{v}} &= \operatorname{tr}\left(-R_Z \circ \mathrm{ad}_H^2\big|_{\mathfrak{n}_1}\right) + \operatorname{tr}\left(-R_Z \circ \mathrm{ad}_H^2\big|_{\mathfrak{n}_2}\right) \\ &= \operatorname{tr}\left(\frac{1}{4}\, j_Z^2\big|_{\mathfrak{n}_1}\right) + 3n_1 + 4\left(\operatorname{tr}\left(\frac{1}{4}\, j_Z^2\big|_{\mathfrak{n}_2}\right) + 6n_2\right) \\ &= \operatorname{tr}\left(\frac{1}{4}\, j_Z^2\big|_{\mathfrak{n}_1}\right) + 4\operatorname{tr}\left(\frac{1}{4}\, j_Z^2\big|_{\mathfrak{n}_2}\right) + 3n_1 + 24n_2, \end{aligned}$$

and applying (4),

$$\begin{aligned} \frac{1}{2}\sum_{\mu=1}^{2}(\lambda - 2\mu)^2 \operatorname{tr}\left(-\frac{1}{4}\, j_Z^2\big|_{\mathfrak{v}_\mu}\right) &= \frac{1}{2}\operatorname{tr}\left(-\frac{1}{4}\, j_Z^2\big|_{\mathfrak{n}_1}\right) + \frac{1}{2}\operatorname{tr}\left(-\frac{1}{4}\, j_Z^2\big|_{\mathfrak{n}_2}\right) \\ &= \frac{1}{2}\operatorname{tr}\left(-\frac{1}{4} j_Z^2\Big|_{\mathfrak{v}}\right) = n_1 + n_2 \;\; (m = n_1 + n_2). \end{aligned}$$

Thus, (3) gives

$$\begin{aligned} n_1 + 2^4 n_2 &= \operatorname{tr}(\mathrm{ad}_H^4\big|_{\mathfrak{v}}) \\ &= \operatorname{tr}\left(\frac{1}{4}\, j_Z^2\big|_{\mathfrak{n}_1}\right) + 4\operatorname{tr}\left(\frac{1}{4}\, j_Z^2\big|_{\mathfrak{n}_2}\right) + 3n_1 + 24n_2 + n_1 + n_2 \end{aligned}$$

and consequently,

$$\operatorname{tr}\left(-\frac{1}{4}\, j_Z^2\big|_{\mathfrak{n}_1}\right) + 4\operatorname{tr}\left(-\frac{1}{4}\, j_Z^2\big|_{\mathfrak{n}_2}\right) = 3n_1 + 9n_2 = 3(n_1 + 3n_2). \qquad (5)$$

Now, using that $(\operatorname{tr} j_Z^2\big|_{\mathfrak{n}_1}) = \operatorname{tr} j_Z^2\big|_{\mathfrak{n}_2})$ (see 1.1), (4) and (5), that is

$$\begin{aligned} 2\operatorname{tr}\left(-\frac{1}{4} j_Z^2\Big|_{\mathfrak{n}_1}\right) &= \operatorname{tr}\left(-\frac{1}{4}\, j_Z^2\big|_{\mathfrak{n}_1}\right) + \operatorname{tr}\left(-\frac{1}{4}\, j_Z^2\big|_{\mathfrak{n}_2}\right) = 2(n_1 + n_2) \\ 5\operatorname{tr}\left(-\frac{1}{4}\, j_Z^2\big|_{\mathfrak{n}_1}\right) &= \operatorname{tr}\left(-\frac{1}{4}\, j_Z^2\big|_{\mathfrak{n}_1}\right) + 4\operatorname{tr}\left(-\frac{1}{4}\, j_Z^2\big|_{\mathfrak{n}_2}\right) = 3(n_1 + 3n_2), \end{aligned}$$

$n_1 = 2n_2$ is obtained, since

$$3(n_1 + 3n_2) = 5(n_1 + n_2) \Leftrightarrow 2n_1 = 4n_2 \Leftrightarrow n_1 = 2n_2.$$

Finally, we will show the last assertion of the proposition. Using that $R_Z|_{\mathfrak{v}} = -\frac{1}{4}j_Z^2 - 3\,\mathrm{ad}_H|_{\mathfrak{v}}$ again, we compute

$$\mathrm{tr}(R_Z|_{\mathfrak{v}})^2 = \mathrm{tr}\left(\frac{1}{4}j_Z^2 + 3\,\mathrm{ad}_H|_{\mathfrak{v}}\right)^2$$

by developing $R_Z^2\big|_{\mathfrak{v}} = \left(-\frac{1}{4}j_Z^2 - 3\,\mathrm{ad}_H|_{\mathfrak{v}}\right)^2$. Thus,

$$\begin{aligned}
\mathrm{tr}\left(R_Z^2\big|_{\mathfrak{v}}\right) &= \mathrm{tr}\left(\frac{1}{16}j_Z^4\right) + 9\mathrm{tr}\,\mathrm{ad}_H^2\big|_{\mathfrak{v}} + \frac{3}{2}\mathrm{tr}\left(j_Z^2 \circ \mathrm{ad}_H|_{\mathfrak{v}}\right) \\
&= \mathrm{tr}\left(\frac{1}{16}j_Z^4\right) + 9\left(\mathrm{tr}(\mathrm{ad}_H^2\big|_{\mathfrak{n}_1}) + \mathrm{tr}(\mathrm{ad}_H^2\big|_{\mathfrak{n}_2})\right) \\
&\quad + \frac{3}{2}\left(\mathrm{tr}(j_Z^2\big|_{\mathfrak{n}_1} \circ \mathrm{ad}_H|_{\mathfrak{n}_1}) + \mathrm{tr}(j_Z^2\big|_{\mathfrak{n}_2} \circ \mathrm{ad}_H|_{\mathfrak{n}_2})\right)
\end{aligned}$$

and substituting the values of $\mathrm{ad}_H|_{\mathfrak{n}_i} = i\mathrm{Id}$, we have

$$\begin{aligned}
\mathrm{tr}\left(R_Z^2\big|_{\mathfrak{v}}\right) &= \mathrm{tr}\left(\frac{1}{16}j_Z^4\right) + 9(n_1 + 4n_2) + \frac{3}{2}\left(\mathrm{tr}(j_Z^2\big|_{\mathfrak{n}_1}) + 2\mathrm{tr}(j_Z^2\big|_{\mathfrak{n}_2})\right) \\
&= \mathrm{tr}\left(\frac{1}{16}j_Z^4\right) + \frac{9}{2}\mathrm{tr}\left(j_Z^2\big|_{\mathfrak{n}_1}\right) + 9(n_1 + 4n_2),
\end{aligned}$$

since $\mathrm{tr}\Big(j_Z^2\big|_{\mathfrak{n}_1}\Big) = \mathrm{tr}\Big(j_Z^2\big|_{\mathfrak{n}_2}\Big)$. Note, that the same argument used to show this equality, implies

$$\mathrm{tr}\left(j_Z^4\big|_{\mathfrak{n}_1}\right) = \mathrm{tr}\left(j_Z^4\big|_{\mathfrak{n}_2}\right). \tag{6}$$

Therefore, the 2-stein condition, $\mathrm{tr}\big(R_Z^2\big|_{\mathfrak{v}}\big) = \mathrm{tr}\big(\mathrm{ad}_H^4\big|_{\mathfrak{v}}\big)$, gives

$$\begin{aligned}
2\mathrm{tr}\left(\frac{1}{16}\,j_Z^4\big|_{\mathfrak{n}_1}\right) + \frac{9}{2}\mathrm{tr}\left(j_Z^2\big|_{\mathfrak{n}_1}\right) + 9(n_1 + 4n_2) &= n_1 + 16n_2 \quad \text{or} \\
2\mathrm{tr}\left(\frac{1}{16}\,j_Z^4\big|_{\mathfrak{n}_1}\right) + \frac{9}{2}\mathrm{tr}\left(j_Z^2\big|_{\mathfrak{n}_1}\right) &= -(8n_1 + 20n_2).
\end{aligned}$$

By applying (4), the last equality is equivalent to

$$\begin{aligned}
&2\mathrm{tr}\left(\frac{1}{16}\,j_Z^4\big|_{\mathfrak{n}_1}\right) - 18\mathrm{tr}\left(-\frac{1}{4}\,j_Z^2\big|_{\mathfrak{n}_1}\right) = -4(2n_1 + 5n_2), \text{ or} \\
= \;&2\mathrm{tr}\left(\frac{1}{16}\,j_Z^4\big|_{\mathfrak{n}_1}\right) - 18(n_1 + n_2) = -4(2n_1 + 5n_2).
\end{aligned}$$

It follows from (6) that

$$2\mathrm{tr}\left(\frac{1}{16}\,j_Z^4\big|_{\mathfrak{n}_1}\right) = 10n_1 - 2n_2 = 2(5n_1 - n_2),$$

and from the fact $n_1 = 2n_2$,

$$\mathrm{tr}\left(\frac{1}{16}\,j_Z^4\big|_{\mathfrak{n}_2}\right) = \mathrm{tr}\left(\frac{1}{16}\,j_Z^4\big|_{\mathfrak{n}_1}\right) = 5n_1 - n_2 = 9n_2.$$

The final assertion follows as claimed, since

$$\operatorname{tr}\left(-\frac{1}{4}j_Z^2\Big|_{\mathfrak{n}_2}\right) = 3n_2 \ \text{ and } \operatorname{tr}\left(\frac{1}{16}\, j_Z^4\big|_{\mathfrak{n}_2}\right) = 9n_2$$

implies that equality holds in the Cauchy-Schwartz Inequality

$$\left(\operatorname{tr}\left(-\frac{1}{4}\, j_Z^2\big|_{\mathfrak{n}_2}\right)\right)^2 \le n_2 \operatorname{tr}\left(\frac{1}{16}\, j_Z^4\big|_{\mathfrak{n}_2}\right),$$

which gives $-\frac{1}{4}\, j_Z^2\big|_{\mathfrak{n}_2} = 3\mathrm{Id}$, or equivalently, $|j_Z X|^2 = 12$ for all unit vector $X \in \mathfrak{n}_2$. □

Let $\mathfrak{s} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3 \oplus \mathbf{R}H$ be a 4-step Lie algebra so that $\mathfrak{z} = \mathfrak{n}_3$, and $\mathfrak{v} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$. In what follows let $\{Z_i\}_{i=1}^k$, $\{Y_i\}_{i=1}^{n_1}$ and $\{X_j\}_{j=1}^{n_2}$ be any orthonormal bases of $\mathfrak{z}$, $\mathfrak{n}_1$ and $\mathfrak{n}_2$, respectively. The following basic formulas are deduced from the hypothesis $\mathfrak{n}_3 \neq 0$ and not assuming that the 2-stein condition is satisfied. They are shown in [5, Proposition 2.3].

If $X \in \mathfrak{n}_2$ and $Y \in \mathfrak{n}_1$ are unit vectors, then

$$\sum_{j=1}^{n_2} |[Y, X_j]|^2 = \sum_{k=1}^{n_3} |j_{Z_k} Y|^2 \tag{7}$$

$$\sum_{j=1}^{n_2} \left|\left(\nabla^{\mathfrak{n}}_{X_j} Y\right)_{\mathfrak{v}}\right|^2 = \frac{1}{4}\sum_{i=1}^{n_1} |[Y, Y_i]|^2 \tag{8}$$

In order to show that the 2-stein condition is not satisfied by $\mathfrak{s}$, in case that $\mathfrak{s}$ is 4-step properly, we need to compute $\mathrm{Ric}(Y)$ and $\mathrm{Ric}(X)$ for unit vectors $Y \in \mathfrak{n}_1$ and $X \in \mathfrak{n}_2$.

Lemma 2.3. *If $Y \in \mathfrak{n}_1$ is a unit vector then,*

$$tr\ R_Y = -n_1 - 2n_2 - 3n_3 - \frac{1}{2}\sum_{j=1}^{n_2} |[Y, X_j]|^2 - \frac{1}{2}\sum_{i=1}^{n_1} |[Y, Y_i]|^2 .$$

Proof. Let $Y \in \mathfrak{n}_1$ be a unit vector. We first note, tha it is a direct computation to see that

$$\begin{aligned} \nabla_Y W &= \frac{1}{2}[Y, W] \text{ for } W \in \mathfrak{n}_1,\ W \perp Y \\ \nabla_Y X &= \frac{1}{2}[Y, X] + (\nabla^{\mathfrak{n}}_Y X)_{\mathfrak{v}} \ \text{ for } X \in \mathfrak{n}_2. \end{aligned}$$

Therefore, for unit vectors $Z \in \mathfrak{z}$, $W \in \mathfrak{n}_1$ and $X \in \mathfrak{n}_2$ using the definition of R_Y we obtain

$$\begin{aligned} R_Y Z &= -3Z + \frac{1}{2}\nabla_Y j_Z Y, \\ R_Y W &= -W - \frac{1}{2}\nabla_Y [W, Y] - \nabla_{[W,Y]} Y, \\ R_Y X &= -2X - \nabla_Y \nabla_X Y + \frac{1}{2} j_{[X,Y]} Y \ ,\ R_Y H = -H. \end{aligned}$$

Now, it is a strighforward computation using the definition of $\nabla_{(.)}$ to see that,

$$\langle R_Y Z, Z\rangle = -3 + \frac{1}{2}\langle \nabla_Y j_Z Y, Z\rangle = -3 + \frac{1}{4}\left|j_Z Y\right|^2,$$

$$\begin{aligned}\langle R_Y W, W\rangle &= -1 - \frac{1}{2}\langle \nabla_Y [W,Y], W\rangle - \left\langle \nabla_{[W,Y]} Y, W\right\rangle \\ &= -1 - \frac{1}{4}\left|[Y,W]\right|^2 + \frac{1}{2}\langle [Y,W],[W,Y]\rangle \\ &= -1 - \frac{3}{4}\left|[Y,W]\right|^2\end{aligned}$$

and

$$\begin{aligned}\langle R_Y X, X\rangle &= -2 - \langle \nabla_Y \nabla_X Y, X\rangle + \frac{1}{2}\left\langle j_{[X,Y]} Y, X\right\rangle \\ &= -2 + \langle \nabla_X Y, \nabla_Y X\rangle + \frac{1}{2}\langle [Y,X],[X,Y]\rangle \\ &= -2 + \left|\nabla_X Y\right|^2 + \langle \nabla_X Y, [Y,X]\rangle - \frac{1}{2}\left|[X,Y]\right|^2 \\ &= -2 + \left|\nabla_X Y\right|^2 - \left|[X,Y]\right|^2 \\ &= -2 - \frac{3}{4}\left|[X,Y]\right|^2 + \left|(\nabla^{\mathfrak{n}}_X Y)_{\mathfrak{v}}\right|^2.\end{aligned}$$

Next, by computing

$$\begin{aligned}\operatorname{tr} R_Y\big|_{\mathfrak{z}\oplus\mathbf{R}H} &= \sum_{k=1}^{n_3} \langle R_Y Z_k, Z_k\rangle - 1 = -3n_3 + \frac{1}{4}\sum_{k=1}^{n_3}\left|j_{Z_k} Y\right|^2 - 1 \\ \operatorname{tr} R_Y\big|_{\mathfrak{n}_1} &= \sum_{i=1}^{n_1} \langle R_Y Y_i, Y_i\rangle = -(n_1 - 1) - \frac{3}{4}\sum_{i=1}^{n_1}\left|[Y,Y_i]\right|^2 \\ \operatorname{tr} R_Y\big|_{\mathfrak{n}_2} &= \sum_{j=1}^{n2} \langle R_Y X_j, X_j\rangle = -2n_2 + \sum_{j=1}^{n_2}\left|\left(\nabla_{X_j} Y\right)_{\mathfrak{v}}\right|^2 - \frac{3}{4}\sum_{j=1}^{n_2}\left|[X_j,Y]\right|^2,\end{aligned}$$

it is immediate that

$$\operatorname{tr} R_Y = -n_1 - 2n_2 - 3n_3 + \frac{1}{4}\sum_{k=1}^{n_3}\left|j_{Z_k} Y\right|^2 - \frac{3}{4}\sum_{i=1}^{n_1}\left|[Y,Y_i]\right|^2$$
$$+\sum_{j=1}^{n_2}\left|\left(\nabla_{X_j} Y\right)_{\mathfrak{v}}\right|^2 - \frac{3}{4}\sum_{j=1}^{n_2}\left|[Y,X_j]\right|^2.$$

By applying the equalities given by (7) and (8) we have,

$$\operatorname{tr} R_Y = -n_1 - 2n_2 - 3n_3 - \frac{1}{2}\sum_{j=1}^{n_2}\left|[Y,X_j]\right|^2 - \frac{1}{2}\sum_{i=1}^{n_1}\left|[Y,Y_i]\right|^2,$$

the expression stated in the lemma. □

Corollary 2.4. *If $\mathfrak{s} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3 \oplus \mathbf{R}H$ satisfies the Einstein condition, then for all unit vectors $Y \in \mathfrak{n}_1$ and $X \in \mathfrak{n}_2$ we have,*

$$\begin{aligned}
(i) \qquad & \sum_{i=1}^{n_1} \left|(\nabla^{\mathfrak{n}}_X Y_i)_{\mathfrak{v}}\right|^2 - \frac{1}{2}\sum_{i=1}^{k} \left|j_{Z_i} X\right|^2 &=& \quad n_1 - 3n_3 \\
(ii) \qquad & \sum_{j=1}^{n_2} \left|[Y, X_j]\right|^2 + \sum_{i=1}^{n_1} \left|[Y, Y_i]\right|^2 &=& \quad 4\,(n_2 + 3n_3)\,.
\end{aligned}$$

Proof. (i) and (ii) are obtained by applying the Einstein condition $\mathrm{tr}(R_X) = -\mathrm{tr}(\mathrm{ad}_H^2)$ (2) to $X \in \mathfrak{n}_2$, and $\mathrm{tr}(R_Y) = -\mathrm{tr}(\mathrm{ad}_H^2)$ to $Y \in \mathfrak{n}_1$ (Lemma 2.3), respectively. In fact, they are immediate since

$$\begin{aligned}
-\frac{1}{2}\sum_{i=1}^{k} \left|j_{Z_i} X\right|^2 - 2\,(n_1 + 2n_2 + 3n_3) + \sum_{i=1}^{m} \left|(\nabla^{\mathfrak{n}}_X Y_i)_{\mathfrak{v}}\right|^2 &=& -\mathrm{tr}(\mathrm{ad}_H^2) \\
&& -n_1 - 4n_2 - 9n_3
\end{aligned}$$

and

$$\begin{aligned}
-n_1 - 2n_2 - 3n_3 - \frac{1}{2}\sum_{j=1}^{n_2} \left|[Y, X_j]\right|^2 - \frac{1}{2}\sum_{i=1}^{n_1} \left|[Y, Y_i]\right|^2 &=& -\mathrm{tr}(\mathrm{ad}_H^2) \\
&& -n_1 - 4n_2 - 9n_3,
\end{aligned}$$

implying the required formulas. □

Theorem 2.5. *If S is a 4-step Carnot space that satisfies the 2-stein condition, then S is either, up to scaling, the hyperbolic space $\mathbf{R}H^n$ ($n = \dim S$) or a Damek-Ricci space.*

Proof. Let S be a 4-step Carnot space with associated Lie algebra $\mathfrak{s}$. If $\mathfrak{n}_3 = 0$ then either S is, up to scaling, the hyperbolic symmetric space $\mathbf{R}H^n$ with $n = n_1 + 1$ or $n = n_2 + 1$ according to $\mathfrak{n}_2 = 0$ or $\mathfrak{n}_1 = 0$, or S is a Damek-Ricci space by [3, Theorem 3.1].
Assume that $\mathfrak{n}_3 \neq 0$; we will show that S cannot be 2-stein unless $\mathfrak{n}_1 = 0 = \mathfrak{n}_2$.

(i) Let $Y \in \mathfrak{n}_1$ a unit vector. If U is the operator on $\mathfrak{s}$ defined by $U(.) = R(.,Y)H + R(.,H)Y$ it follows that

$$\mathrm{tr}\left(\mathrm{ad}_H^4 + \left(R_Y \circ \mathrm{ad}_H^2\right) - \frac{1}{2}\,U^2\right)\Big|_{\mathfrak{s}_0^{\perp}} = 0. \tag{9}$$

where $\mathfrak{s}_0 = \mathrm{span}\{Y, H\}$ is the totally geodesic subalgebra of $\mathfrak{s}$ which corresponds to the symmetric hyperbolic space $\mathbf{R}H^2$ of constant curvature -1. We show this expression (9) following the same argument developed in [5, Proposition 1.2]. In order to do that we need to revise some properties fulfilled by R_Y and U. First, note that

$$R_Y : \mathfrak{z} \to \mathfrak{z} \oplus \mathfrak{n}_1 \quad \mathfrak{n}_1 \to \mathfrak{z} \oplus \mathfrak{n}_1 \ \text{ and } R_Y H = -H,$$

hence $R_Y : \mathfrak{n}_2 \to \mathfrak{n}_2$, since it is symmetric.

Next, by using the definition of U, for any unit vectors $Z \in \mathfrak{z}$, $W \perp Y \in \mathfrak{n}_1$ and $X \in \mathfrak{n}_2$ we compute:

$$\begin{aligned} U(Z) &= \nabla_Z \nabla_Y H - \nabla_Y \nabla_Z H - \nabla_{[Z,H]} Y = -\nabla_Z Y + 3\nabla_Y Z + 3\nabla_Z Y \\ &= 5\nabla_Z Y = -\frac{5}{2} j_Z Y, \end{aligned}$$

$$\begin{aligned} U(W) &= \nabla_W \nabla_Y H - \nabla_Y \nabla_W H - \nabla_{[W,Y]} H - \nabla_{[W,H]} Y \\ &= -\nabla_W Y + \nabla_Y W + 2[W,Y] + \nabla_W Y \\ &= \frac{1}{2}[Y,W] - 2[Y,W] = -\frac{3}{2}[Y,W] \quad \text{and} \end{aligned}$$

$$\begin{aligned} U(X) &= \nabla_X \nabla_Y H - \nabla_Y \nabla_X H - \nabla_{[X,Y]} H - \nabla_{[X,H]} Y \\ &= -\nabla_X Y + 2\nabla_Y X + 3[X,Y] + 2\nabla_X Y \\ &= \nabla_X Y + 2\nabla_Y X + 3[X,Y] = 3\nabla_X Y + [X,Y] \\ &= \frac{3}{2}[X,Y] + 3(\nabla^{\mathfrak{n}}_X Y)_{\mathfrak{v}} + [X,Y] = \frac{5}{2}[X,Y] + 3(\nabla^{\mathfrak{n}}_X Y)_{\mathfrak{v}}. \end{aligned}$$

Thus, $U : \mathfrak{n}_2 \to \mathfrak{z} \oplus \mathfrak{n}_1$ and $\mathfrak{z} \oplus \mathfrak{n}_1 \to \mathfrak{n}_2$. This property, togheter with the fact $R_Y : \mathfrak{z} \oplus \mathfrak{n}_1 \to \mathfrak{z} \oplus \mathfrak{n}_1$ and $\mathfrak{n}_2 \to \mathfrak{n}_2$ ($R_Y H = -H$) imply that (9) holds, since

$$\operatorname{tr}\left(r^2 R_Y - s^2 \operatorname{ad}_H^2\right) \circ U\big|_{\mathfrak{s}_0^\perp} = 0.$$

(ii) Next, (9) it will be applied, computing separately each term and using (7) and (8). First, from the above computations we obtain

$$\begin{aligned} \left\langle U^2 Z, Z \right\rangle &= |U(Z)|^2 = \frac{25}{4} |j_Z Y|^2 \text{ for all } Z \in \mathfrak{z}, \ |Z| = 1 \\ \left\langle U^2 W, W \right\rangle &= |U(W)|^2 = \frac{9}{4} |[Y,W]|^2 \ , \ |W| = 1, \text{ and} \\ \left\langle U^2 X, X \right\rangle &= |U(X)|^2 = \frac{25}{4} |[X,Y]|^2 + 9 \left|(\nabla^{\mathfrak{n}}_X Y)_{\mathfrak{v}}\right|^2, \text{ for } X \in \mathfrak{n}_2, \ |X| = 1. \end{aligned}$$

Hence,

$$\begin{aligned} \operatorname{tr}\left(U^2\big|_{\mathfrak{s}_{0^\perp}}\right) &= \sum_{k=1}^{n_3} |U(Z_k)|^2 + \sum_{i=1}^{n_1} |U(Y_i)|^2 + \sum_{j=1}^{n_2} |U(X_j)|^2 \\ &= \frac{25}{4} \sum_{k=1}^{n_3} \left|j_{Z_k} Y\right|^2 + \frac{9}{4} \sum_{i=1}^{n_1} |[Y,Y_i]|^2 + \frac{25}{4} \sum_{j=1}^{n_2} |[Y,X_j]|^2 + 9 \sum_{j=1}^{n_2} \left|(\nabla^{\mathfrak{n}}_{X_j} Y)_{\mathfrak{v}}\right|^2 \\ &= 2\frac{9}{4} \sum_{i=1}^{n_1} |[Y,Y_i]|^2 + 2\frac{25}{4} \sum_{j=1}^{n_2} |[Y,X_j]|^2, \end{aligned}$$

which implies that

$$-\frac{1}{2}\operatorname{tr}\left(U^2\big|_{\mathfrak{s}_{0^\perp}}\right) = -\frac{9}{4} \sum_{i=1}^{n_1} |[Y,Y_i]|^2 - \frac{25}{4} \sum_{j=1}^{n_2} |[Y,X_j]|^2.$$

It is a straightforward computation to see that,

$$\begin{aligned}
\operatorname{tr}\,(R_Y\circ \operatorname{ad}_H^2)\big|_{\mathfrak{s}_0^\perp} &= 9\operatorname{tr}\,R_Y\big|_{\mathfrak{z}}+\operatorname{tr}\,(R_Y\big|_{\mathfrak{n}_1\cap Y^\perp})+4\operatorname{tr}(R_Y\big|_{\mathfrak{n}_2})\\
&= 9\left(-3n_3+\sum_{k=1}^{n_3}\frac{1}{4}\left|j_{Z_k}Y\right|^2\right)-(n_1-1)-\frac{3}{4}\sum_{i=1}^{n_1}\left|[Y,Y_i]\right|^2\\
&\quad +4\left(-2n_2+\sum_{j=1}^{n_2}\left|\nabla_{X_j}Y\right|^2-\frac{3}{4}\sum_{j=1}^{n_2}\left|[Y,X_j]\right|^2\right)\\
&= -27n_3-(n_1-1)-8n_2-\frac{3}{4}\sum_{j=1}^{n_2}\left|[Y,X_j]\right|^2+\frac{1}{4}\sum_{i=1}^{n_1}\left|[Y,Y_i]\right|^2,
\end{aligned}$$

and (9) becomes

$$\begin{aligned}
0 &= \operatorname{tr}\left(\operatorname{ad}_H^4+\left(R_Y\circ\operatorname{ad}_H^2\right)-\frac{1}{2}\,U^2\right)\Big|_{\mathfrak{s}_0^\perp}=(n_1-1)+16n_2+81n_3\\
&\quad -27n_3-(n_1-1)-8n_2-\frac{3}{4}\sum_{j=1}^{n_2}\left|[Y,X_j]\right|^2+\frac{1}{4}\sum_{i=1}^{n_1}\left|[Y,Y_i]\right|^2\\
&\quad -\frac{9}{4}\sum_{i=1}^{n_1}\left|[Y,Y_i]\right|^2-\frac{25}{4}\sum_{j=1}^{n_2}\left|[Y,X_j]\right|^2\\
&= 54n_3+8n_2-28\frac{1}{4}\sum_{j=1}^{n_2}\left|[Y,X_j]\right|^2-8\frac{1}{4}\sum_{i=1}^{n_1}\left|[Y,Y_i]\right|^2.
\end{aligned}$$

Thus,

$$7\sum_{j=1}^{n_2}\left|[Y,X_j]\right|^2+2\sum_{i=1}^{n_1}\left|[Y,Y_i]\right|^2=54n_3+8n_2. \tag{10}$$

(iii) Now, from the equality given by Corollary 2.4 (ii), we obtain

$$2\sum_{j=1}^{n_2}\left|[Y,X_j]\right|^2+2\sum_{i=1}^{n_1}\left|[Y,Y_i]\right|^2=24n_3+8n_2=2\left(12n_3+4n_2\right) \tag{11}$$

and the expression (10)−(11) gives

$$5\sum_{j=1}^{n_2}\left|[Y,X_j]\right|^2=30n_3\ \Leftrightarrow \sum_{j=1}^{n_2}\left|[Y,X_j]\right|^2=6n_3 \text{ for a unit vector } Y\in\mathfrak{n}_1.$$

Finally, it follows from (7) that

$$\sum_{k=1}^{n_3}\left|j_{Z_k}Y\right|^2=6n_3$$

and we get a contradiction, since $\left|j_ZY\right|^2=12$ for any unit vectors $Z\in\mathfrak{z}$ and $Y\in\mathfrak{n}_1$. In order to show this last assertion, note that $j_Z:\mathfrak{n}_2\to\ker(j_Z\big|_{\mathfrak{n}_1})^\perp\subset\mathfrak{n}_1$ isomorphically (1.1) and by Proposition 2.2, a unit vector $Y\in\mathfrak{n}_1$ is expressed as $Y=\frac{1}{\sqrt{12}}j_ZX$ with $X\in\mathfrak{n}_2$, $|X|=1$ ($Y=\sum_{i=1}^m a_i\frac{1}{\sqrt{12}}j_ZX_i$ where $j_Z^2X_i=$

$-|j_Z X_i|^2 X_i$, $|X_i| = 1$, in terms of the orthonormal basis $\{X_i\}$ and $\{\frac{1}{\sqrt{12}} j_Z X_i\}$: $i = 1, ..., m$). Hence $|j_Z Y|^2 = 12$ and we have that $\sum_{k=1}^{n_3} |j_{Z_k} Y|^2 = 12n_3$. □

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