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INTEGRABILITY OF F-STRUCTURES ON GENERALIZED FLAG MANIFOLDS

SOFÍA PINZÓN*

Abstract. Here we consider a generalized flag manifold $\mathbb{F} = U/K$, and a differential structure $\mathcal{F}$ which satisfy $\mathcal{F}^3 + \mathcal{F} = 0$; these structures are called f-structures. Such structure $\mathcal{F}$ determines in the tangent bundle of $\mathbb{F}$ some $ad(K)-$invariant distributions. Since flag manifolds are homogeneous reductive spaces, they certainly have combinatorial properties that allow us to make some easy calculations about integrability conditions for $\mathcal{F}$ itself and the distributions that it determines on $\mathbb{F}$. An special case corresponds to the case $U = U(n)$, the unitary group, this is the geometrical classical flag manifold and in fact tools coming from graph theory are very useful.

1. Introduction

A tensor field $\mathcal{F}$ of type (1,1) on a Riemannian manifold is called an *f-structure* if $\mathcal{F}^3 + \mathcal{F} =$, and *almost complex* if $\mathcal{F}^2 = -I$. Obviously, an almost complex structure is also an f-structure. Integrability of almost complex structures is equivalent to the associate Nijenhuis tensor being null. In [7] Ishihara and Yano present an analogous theorem in the case of f-structures. We use their results to study integrability conditions when one generalized flags manifold are consider.

Let G be a semisimple Lie group. A generalized flag manifold with Lie group G is a reductive homogeneous space $\mathbb{F} = G/C(S)$ where $C(S)$ is a centralizer of a torus. This manifold can be expressed as a $\mathbb{F} = U/K$, where U is the compact connected form of G and $K = C(S) \cap U$. This manifold and its tangent space $T_b(\mathbb{F}) = \mathfrak{q}$, have a characterization in terms of the corresponding root system terms. We will consider along this paper a generalized flag manifold together with an invariant metric Λ and an U-invariant f-structure $\mathcal{F}$, meaning that $\mathcal{F}$ commute with the adjoint action of U. Ww will denote by $\mathcal{F}$ the complexification of this f-structure, which is diagonalizable with eigenvalues $i, 0, -i$ and eigenspaces $\mathfrak{q}^+_\Theta, \mathfrak{q}^0_\Theta, \mathfrak{q}^-_\Theta$. In analogy with the almost complex case we will distinguish between

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vectors of types $(1,0,0)$, $(0,1,0)$ or $(0,0,1)$ corresponding to the eigenvalues of $0, i, -i$ respectively.

The integrability of $\mathcal{F}$, the distributions that it determines in $\mathfrak{q}$ and the properties of that distributions are our central topic. We characterize in root terms and in graph theoretical terms the integrability conditions.

In particular, we get that the only integrable f-structure, different from the null structure, in the maximal classical flag manifold is the structure which corresponds to the integrable almost complex structure, that is, in graph theoretic terms which corresponds to the canonical tournament.

Theorem A necessary and sufficient condition for $\mathcal{F}$ to be integrable is that $N \equiv 0$. Therefore $\mathcal{F}$ is integrable if in $\mathfrak{q}$ there are not exist triples of type $\{0,3,0\}, \{2,1,0\}, \{1,1,1\}$ or $\{1,2,0\}$. In $\mathbb{F}(n)$ this condition is equivalent to the associated digraph avoiding the subdigraphs (2), (3), (4), (5) or (6) in figure 2, that is, the digraph associated to $\mathcal{F}$ must be isomorphic to the null digraph or the canonical tournament.

The Theorem above will appear like Theorem 8.8 and with this result we generalized a Theorem from Burstall [4] given in the context of almost complex structures where he shows that one almost complex structure is integrable if and only if its associated tournament is isomorphic to the canonical tournament, that is, the tournament which does not have three-cycles in root terms it avoids {0,3,0}-triples.

2. Preliminaries

In this section we shall briefly review some general concepts involving generalized flag manifold and some operators and structures which we will use in all this paper. First we need to present a survey about some operators in differential geometry, then we will calculate them specifically on generalized flag manifolds, to this topic we used, specially Props I.3.2, I.3.4, I.3.5 in [8].

2.1. **Operators on a general differential manifold.**

- **Lie derivative.** This is the resulting derivative when a tensor field or a differential form is differentiated with respect to a vector field.

  1) Lie derivative on tensor fields: $Lie_X Y = [X, Y]$.

  2) Lie derivative on tensors: On tensors K of type (1,r) we get

$$(Lie_X K)(Y_1, \ldots, Y_r) = [X, K(Y_1, \ldots, Y_r)] - \sum_{i=1}^{r} K(Y_1, \ldots, [X, Y_i] \ldots, Y_r). \quad (1)$$

3) Lie derivative on forms: If ω is an r-form, then

$$(Lie_X\omega)(Y_1,\dots,Y_r) = X\omega(Y_1,\dots,Y_r) - \sum_{i=1}^{r} w(Y_1,\dots,[X,Y_i]\dots,Y_r). \quad (2)$$

- **The riemannian invariant connection.** Each Riemannian manifold admits a unique metric connection with vanishing torsion, called the Riemannian connection or Levi-Civita connection, and it satisfies $T(X,Y) = [X,Y] - \nabla_X Y + \nabla_Y X = 0$ and $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$, where g is the metric on the manifold and T is the torsion tensor [8].

1) The covariant derivative on tensors. Given a tensor field K of type (r,s), the covariant differential ∇K of K is a tensor field of type $(r,s{+}1)$ defined as follows. $(\nabla K)(X_1,\dots,X_s;Y) = (\nabla_Y K)(X_1,\dots,X_s)$.

2) The covariant derivative on forms. If ω is an r-form, then

$$(\nabla\omega)((X_1,\dots,X_r),Y) = Y\omega(X_1,\dots,X_s) - \sum_{i=1}^{r}\omega(X_1,\dots,\nabla_Y X_i,\dots,X_r).$$

- **Exterior derivative on forms.** Exterior differentiation d can be characterize as follow:
  - d is a degree-increasing $\mathbb{R}$-linear mapping, that is if ω is a p form, $d\omega$ is a $p+1$-form;
  - signed derivation w.r.t. the wedge product, that is, if ω is a p-form and θ is a q-form, then $d(\omega\wedge\theta) = d\omega\wedge\theta + (-1)^p\omega\wedge d\theta$;
  - $d^2 = 0$.
  - For 0-forms d is defined by $\langle df, X\rangle = Xf$.

  d extends to r-forms coefficient-wise, using basis expansions, resulting in

$$(r+1)d\omega(X_0,X_1,\dots,X_r) = \sum_{i=0}^{r}(-1)^i X_i(\omega(X_0,X_1,\dots,\hat{X}_i,\dots,X_r)) + \sum_{0\le i<j\le r}(-1)^{i+j}\omega([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_r).$$

Here $\hat{X}_i$ means that you not consider that component. On functions the 1-form df is defined by $df(Y) = Yf$. Otherwise we get

$$d\omega(\underline{X}) = -\sum \pm(\nabla\omega)(\hat{X},X_i). \quad (3)$$

On 1-forms we get

$$2d\omega(X,Y) = X(\omega(Y)) - Y\omega(X) - \omega([X,Y]). \quad (4)$$

On 2-forms,

$$3d\omega(X,Y,Z) = Cyclic\ \{X\omega(Y,Z) - \omega([X,Y],Z)\} \tag{5}$$

where $Cyclic$ is the cyclic symmetrization operator w.r.t. the vector fields involved. See [8] Prop. I.3.11. By duality we get $\langle \nabla f, \nabla g\rangle = \langle df, dg\rangle$.

3. Generalized Flag Manifold

A generalized flag manifold is and homogeneous space of the form $G/K = G/C(S)$, where G is a semisimple compact Lie group and $C(S)$ is the centralizer of some torus S in G. If the torus S is maximal, say T, then $\mathbb{F} = G/T$ is called a maximal (full) flag manifold, if S is not maximal, $G/C(S)$ is called a partial flag manifold. Lets us describe generalized flag manifolds associated with the Lie group G in terms of the root systems associated with the corresponding semisimple Lie algebra $\mathfrak{g}$.

Assume $\mathfrak{g}$ complex and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$; denote by Π and Π^+ the root system and the positive root system , respectively, (of $\mathfrak{g}$ with respect to $\mathfrak{h}$) and

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\Pi}$$

its root decomposition, where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} :\ \ [H,X] = \alpha(H)X, \qquad \forall H \in \mathfrak{h}\}$, $\alpha \in \Pi$, is the one-dimensional complex root space corresponding to α. As Π is the generator of $\mathfrak{h}^*$ (the dual of $\mathfrak{h}$), we have the elements H_α, defined by $\alpha(\cdot) = \langle H_\alpha, \cdot\rangle$. Denote by $\mathfrak{h}_\mathbb{R}$ the subspace of $\mathfrak{h}$ generated over $\mathbb{R}$ by $H_\alpha,\ \alpha \in \Pi$. Choose now $\Sigma = \{\alpha_1, \dots, \alpha_l\} \subset \Pi^+$ a simple root system, take $\Theta \subset \Sigma$ and denote by $\langle\Theta\rangle$ the set of roots generated by Θ. Each subset $\langle\Theta\rangle^\pm = \langle\Theta\rangle \cap \Pi^\pm$ splits $\mathfrak{g}$ as follows

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\langle\Theta\rangle^+} \mathfrak{g}_\alpha \oplus \sum_{\alpha\in\langle\Theta\rangle^+} \mathfrak{g}_{-\alpha} \oplus \sum_{\beta\in\Pi^+\setminus\langle\Theta\rangle^+} \mathfrak{g}_\beta \oplus \sum_{\beta\in\Pi^+\setminus\langle\Theta\rangle^+} \mathfrak{g}_{-\beta}. \tag{6}$$

Fix a Weyl basis of $\mathfrak{g}$, that is, a set of vectors $\{X_\alpha \in \mathfrak{g} \mid \alpha \in \Pi\}$ which satisfies $[X_\alpha, X_{-\alpha}] = H_\alpha$ or equivalently $\langle X_\alpha, X_{-\alpha}\rangle = 1$, since $[X_\alpha, X_{-\alpha}] = \langle X_\alpha, X_{-\alpha}\rangle H_\alpha$ and $[X_\alpha, X_\beta] = m_{\alpha,\beta} X_{\alpha+\beta}$ with $m_{\alpha,\beta} \in \mathbb{R}$, $m_{\alpha,\beta} = m_{-\alpha,-\beta}$ and $m_{\alpha,\beta} = 0$ if $\alpha + \beta \notin \Pi$.

Let now

$$\mathfrak{p}_\Theta = \mathfrak{h} \oplus \sum_{\alpha\in\langle\Theta\rangle^+} \mathfrak{g}_\alpha \oplus \sum_{\alpha\in\langle\Theta\rangle^+} \mathfrak{g}_{-\alpha} \oplus \sum_{\beta\in\Pi^+\setminus\langle\Theta\rangle^+} \mathfrak{g}_\beta. \tag{7}$$

$\mathfrak{p}_\Theta$ is the parabolic subalgebra determined by Θ in $\mathfrak{g}$. Then equation ((6)) becomes

$$\mathfrak{g} = \mathfrak{p}_\Theta \oplus \sum_{\beta\in\Pi^+\setminus\langle\Theta\rangle^+} \mathfrak{g}_{-\beta}. \tag{8}$$

Then the generalized flag manifold $\mathbb{F}_\Theta$ associated to $\{\mathfrak{g}, \Theta\}$ corresponds to the homogeneous space $\mathbb{F}_\Theta = G/P_\Theta$, where P_Θ is the normalizer of $\mathfrak{p}_\Theta$ in G.

Denote by $\mathfrak{u}$ a real compact form of $\mathfrak{g}$, and by $U \subset G$ the connected subgroup associated to $\mathfrak{u}$. Assume $\mathfrak{u}$ the real subspace generated by $i\mathfrak{h}_\mathbb{R}, A_\alpha, S_\alpha$, with $\alpha \in \Pi \setminus \Theta$, where $A_\alpha = X_\alpha - X_{-\alpha}$ and $S_\alpha = i(X_\alpha + X_{-\alpha})$. Let $K_\Theta = P_\Theta \cap U$, which, by construction, is the centralizer of a torus. U acts in a transitively way on $\mathbb{F}_\Theta$, and we can write $\mathbb{F}_\Theta = U/K_\Theta$. If $\Theta = \emptyset$, $\mathbb{F}_\Theta = \mathbb{F}$ correspondes to the maximal flag manifold $\mathbb{F}$, otherwise it corresponds to a partial flag manifold.

The generalized flag manifold $\mathbb{F}_\Theta = U/K_\Theta$ is a reductive homogeneous space. In fact let $\mathfrak{u}_\beta = \mathfrak{u} \cap (\mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta})$, $\beta \in \Pi \setminus \langle\Theta\rangle$ and
$\mathfrak{q}_\Theta = \sum\limits_{\beta \in \Pi\setminus\langle\Theta\rangle} \mathfrak{u}_\beta$. Then,

(i) $\mathfrak{u} = \mathfrak{t}_\Theta \oplus \mathfrak{q}_\Theta, \ \mathfrak{t}_\Theta \cap \mathfrak{q}_\Theta = \emptyset$;
(ii) $Ad(K_\Theta)\mathfrak{q}_\Theta \subset \mathfrak{q}_\Theta$, that is, $[\mathfrak{t}_\Theta, \mathfrak{q}_\Theta] \subset \mathfrak{q}_\Theta$,

and $\mathbb{F}_\Theta$ satisfies the condition to be a a reductive homogeneous space (see [8]).

Denote by b the origin of $\mathbb{F}_\Theta = U/K_\Theta$. We identify with $\mathfrak{q}_\Theta = T_b(\mathbb{F}_\Theta)$. This identification is given by $X \in \mathfrak{q}_\Theta \to X_b \in T_b(\mathbb{F}_\Theta)$, that is, by evaluation of $X \in \mathfrak{q}_\Theta$ in b like a vector field in $T_b(\mathbb{F}_\Theta)$. The tangent space to $\mathbb{F}_\Theta$ in b is, naturally, identify with $\mathfrak{q}_\Theta = \mathfrak{u} \ominus \mathfrak{t} = \sum\limits_{\beta \in \Pi\setminus\langle\Theta\rangle} \mathfrak{u}_\beta$ generated by $A_\alpha, S_\alpha, \alpha \in \Pi \setminus \langle\Theta\rangle$, where $A_\alpha = X_\alpha - X_{-\alpha}$ and $S_\alpha = i(X_\alpha + X_{-\alpha})$. In the same way, the complexified tangent space of $\mathbb{F}_\Theta$ is identified with $\mathfrak{q}_\Theta^\mathbb{C} = \mathfrak{g} \ominus \mathfrak{h} = \oplus_{\alpha \in \Pi\setminus\langle\Theta\rangle} \mathfrak{g}_\alpha$.

One special case of flag manifold corresponds to the geometrical or classical flag manifold, in this case $U = U(n)$ the unitary group and $C(S)$ has to be conjugate to some subgroup $S(U_{n_1} \times U_{n_2} \times ... \times U_{n_k})$, with $n_1, n_2, ..., n_k$ positive integers and $n_1 + n_2 + ... + n_k = n$. If $m_i = n_1 + ... + n_i$, the homogeneous space $SU_n/S(U_{n_1} \times ... \times U_{n_k})$ can be identified with the set of "partial flags" $F(m_1, ..., m_k)$, that is, the manifold of the flags $\{0\} = E_0 \subset E_{m_1} \subset ... \subset E_{m_{k-1}} \subset E_{m_k} = \mathbb{C}^n$, where E_i is an i-dimensional subspace of $\mathbb{C}^n$. The flag manifold corresponding to the case $n_r = 1$ for all $1 \le r \le k$ will be denoted by $\mathbb{F}(n)$, the space of "full flags" or maximal flags in $\mathbb{C}^n$. Each flag consists of the sequence $\{0\} = E_0 \subset E_1 \subset ... \subset E_{n-1} \subset E_n = \mathbb{C}^n$. In particular, the vectors $E_{jk}, j \neq k$ y $E_{jj} - E_{kk}, j < k$, is a Weyl basis for $\mathfrak{sl}(n, \mathbb{C})$, and $\mathfrak{h}$ is the subalgebra of diagonal matrices in $\mathfrak{sl}(n, \mathbb{C})$, $\Theta = \emptyset$, then $T_b\mathbb{F} = \mathfrak{q} \subset \mathfrak{su}(n)$ is spanned by $A_{jk} = E_{jk} - E_{kj}$ and $S_{jk} = i(E_{jk} + E_{kj})$, where E_{jk} is the usual canonical basis in $\mathfrak{gl}(n, \mathbb{C})$.

Example 1.1: Consider

$$\mathbb{F}(4) = U(4)/(U(1) \times U(1) \times U(1) \times U(1)) = U(4)/T,$$

$$\mathfrak{q} = T(\mathbb{F}(4))_{(b)} = \left\{ \begin{pmatrix} 0 & a & b & c \\ -\bar{a} & 0 & d & e \\ -\bar{b} & -\bar{d} & 0 & f \\ -\bar{c} & -\bar{e} & -\bar{f} & 0 \end{pmatrix} : a,b,c,d,e,f \in \mathbb{C} \right\},$$

here,

$$A_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad S_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix} : i = \sqrt{-1}.$$

For an interesting and complete review about homogeneous spaces we recommend [1].

4. Invariant Metrics

A U-invariant Riemannian metric ds^2_Λ in $\mathbb{F}_\Theta$ is completely determined by its values in b, that is, by an inner product $(\cdot,\cdot)$ in $\mathfrak{q}_\Theta$, invariant under the associated action of K_Θ ([3]). Any inner product in $\mathfrak{q}_\Theta$, invariant under the associated action of K_Θ, has the form $(X,Y)_{\Lambda^\Theta} = -\langle \Lambda^\Theta \circ X, Y\rangle$, with $\Lambda^\Theta : \mathfrak{q}_\Theta \to \mathfrak{q}_\Theta$ definite with respect to the Cartan-Killing form and $\circ$ is the Hadamard product or term by term product [3]. The inner product $(\cdot,\cdot)_{\Lambda^\Theta}$ admits a natural extension to a bilinear symmetric form on ${\mathfrak{q}_\Theta}^{\mathbb{C}}$. We use the same notation $(\cdot,\cdot)_{\Lambda^\Theta}$ for this form, as well as for the correspondent complexified form Λ^Θ. K_Θ-invariance of $(\cdot,\cdot)_{\Lambda^\Theta}$ amounts to the Weyl basis being a complex basis of eigenvectors for the action of Λ^Θ, in other words in $\mathfrak{q}^{\mathbb{C}}_\Theta$ we have

$$\Lambda^\Theta X_\alpha = {\lambda^\Theta}_\alpha X_\alpha, \tag{9}$$

with ${\lambda^\Theta}_\alpha = {\lambda^\Theta}_{-\alpha} > 0$. for $\alpha \in \Pi \setminus \langle\Theta\rangle$

for the real space $\mathfrak{q}_\Theta$, the elements of the canonical base $A_\alpha,\ S_\alpha$, with $\alpha \in \Pi \setminus \langle\Theta\rangle$, are eigenvectors for the same eigenvalue ${\lambda^\Theta}_\alpha$. We denote by $ds^2_{\Lambda^\Theta}$ the U-invariant metric associated with Λ^Θ. In what follows we will use Λ^Θ as synonymous of $ds^2_{\Lambda^\Theta}$.

5. Invariant f-estructures

K. Yano [20] in 1961 introduced f-structure for general manifolds; here we shall be interested in invariant structures. An U-invariant f-structure in $\mathbb{F}_\Theta$ is completely determined by an endomorphism $\mathcal{F}^\Theta : \mathfrak{q}_\Theta \to \mathfrak{q}_\Theta$, satisfying $(\mathcal{F}^\Theta)^3 + \mathcal{F}^\Theta = 0$, which commutes with the adjoint action of K_Θ. We also denote by $\mathcal{F}^\Theta$ its complexification $\mathcal{F}^\Theta : \mathfrak{q}^{\mathbb{C}}_\Theta \to \mathfrak{q}^{\mathbb{C}}_\Theta$ which is diagonalizable with eigenvalues $i, 0, -i$, and denote by $\mathfrak{q}^+_\Theta, \mathfrak{q}^0_\Theta, \mathfrak{q}^-_\Theta$ the corresponding eigenspaces. Then we have

$\mathfrak{q}_\Theta^{\mathbb{C}} = \mathfrak{q}_\Theta^+ + \mathfrak{q}_\Theta^0 + \mathfrak{q}_\Theta^-$ with $\overline{\mathfrak{q}_\Theta^+} = \mathfrak{q}_\Theta^-$. The U-invariance of $\mathcal{F}^\Theta$ guarantees that $\mathcal{F}^\Theta(\mathfrak{g}_\alpha) \subseteq \mathfrak{g}_\alpha$ for all $\alpha \in \Pi \setminus \langle\Theta\rangle$, with equality when $\mathcal{F}^\Theta$ is an invariant almost complex structure (see [17]). Thus $\mathcal{F}^\Theta$ is determined uniquely by the values $\varepsilon_\alpha^\Theta \in \{0, \pm 1\}$, $\alpha \in \Pi \setminus \langle\Theta\rangle$, defined by $\mathcal{F}^\Theta(X_\alpha) = i\varepsilon_\alpha^\Theta X_\alpha$. These values satisfy $\varepsilon_{-\alpha}^\Theta = -\varepsilon_\alpha^\Theta$, therefore $\mathcal{F}^\Theta$ is defined by its values in $\Pi^+ \setminus \langle\Theta\rangle^+$. In the sequel we allow some abuse of notation and identify the invariant f-structure $\mathcal{F}^\Theta$ on $\mathbb{F}_\Theta$ with $\{\varepsilon_\alpha^\Theta :\ \alpha \in \Pi \setminus \langle\Theta\rangle\}$. In our invariant context if the f-structure is an invariant almost complex structure this amounts to $\varepsilon_\alpha^\Theta \neq 0$ for all $\alpha \in \Pi \setminus \langle\Theta\rangle$.

In what follows we shall simplify notation by suppressing the subscript Θ in the context of partial flag manifolds.

Denote by p and l the complementary projections onto the spaces $\mathfrak{q}_0$ and $\mathfrak{q}_+ + \mathfrak{q}_-$ denoted as p and l, respectively and defined as follow

$$I = -\mathcal{F}^2, \qquad p = \mathcal{F}^2 + I. \tag{10}$$

Since $\mathcal{F}$ is an f-structure we have

$$l + p = 1, \qquad l^2 = l, \qquad p^2 = p, \qquad lp = pl = 0, \tag{11}$$

where I denote the identity. In other words l and p are complementary projection operators in $\mathfrak{q}$.

Consider an f-estructure $\mathcal{F}$ and l, p like below. Then:

$$\mathcal{F}l = l\mathcal{F} = \mathcal{F}, \quad \mathcal{F}p = p\mathcal{F} = 0, \quad \mathcal{F}^2 l = -l,$$

that is, $\mathcal{F}$ act in $\mathfrak{q}_+ + \mathfrak{q}_-$ like an almost complex structure and in $\mathfrak{q}_0$ like the null operator.

We are interested in studying integrability conditions for $\mathcal{F}$, the distributions $\mathfrak{q}_0$ and $\mathfrak{q}_+ + \mathfrak{q}_-$. For this purpose, we need the **Nijenhuis tensor** which describes the torsion of $\mathcal{F}$. It is given by

$$N(X,Y) = 2([\mathcal{F}(X), \mathcal{F}(Y)] - \mathcal{F}([\mathcal{F}(X), Y]) - \mathcal{F}([X, \mathcal{F}(Y)]) - L([X,Y]).$$

Using Weyl basis properties we get

$$\begin{aligned} &1/2N(X_\alpha, X_\beta) = [\mathcal{F}(X_\alpha), \mathcal{F}(X_\beta)] - \mathcal{F}([\mathcal{F}(X_\alpha), X_\beta]) + \\ &-\mathcal{F}([X_\alpha, \mathcal{F}(X_\beta)]) - L([X_\alpha, X_\beta] \\ &= (-\varepsilon_\alpha\varepsilon_\beta + \varepsilon_\alpha\varepsilon_{\alpha+\beta} + \varepsilon_\beta\varepsilon_{\alpha+\beta} + \varepsilon^2_{\alpha+\beta})[X_\alpha, X_\beta]. \end{aligned}$$

Because $l + p = 1$,

$$\begin{aligned} &N(X_\alpha, X_\beta) = lN(lX_\alpha, lX_\beta) + pN(lX_\alpha, lX_\beta) + \\ &+N(lX_\alpha, pX_\beta) + N(pX_\alpha, lX_\beta) + N(pX_\alpha, pX_\beta). \end{aligned}$$

With some simple calculations and using, again, Weyl basis properties we obtain this other identities:

$$lN(lX_\alpha, lX_\beta) = \varepsilon_\alpha\varepsilon_\beta\varepsilon_{\alpha+\beta}(\varepsilon_\alpha + \varepsilon_\beta - \varepsilon_{\alpha+\beta} - \varepsilon_\alpha\varepsilon_\beta\varepsilon_{\alpha+\beta})[X_\alpha, X_\beta]; \tag{12}$$

$$pN(lX_\alpha, lX_\beta) = p[\mathcal{F}X_\alpha, \mathcal{F}X_\beta] = \varepsilon_\alpha\varepsilon_\beta(\varepsilon_{\alpha+\beta}^2 - 1)[X_\alpha, X_\beta]; \tag{13}$$

$$N(lX_\alpha, lX_\beta) = \mathcal{F}\{l(Lie_{pX_\beta}\mathcal{F})lX_\alpha\}; \tag{14}$$

$$N(pX_\alpha, pX_\beta) = lN(pX_\alpha, pX_\beta) = (\varepsilon_\alpha\varepsilon_\beta^2 - \varepsilon_\alpha^2 - \varepsilon_\beta^2 + 1)\varepsilon_{\alpha+\beta}^2[X_\alpha, X_\beta]. \tag{15}$$

6. Forms on $\mathbb{F}$

• **Riemannian Connection.** Since $\mathbb{F}_\Theta$ is a naturally reductive homogeneous space its Riemannian connection is given by

$$2\nabla_X Y \quad = \quad [X, Y]_{\mathfrak{q}_\Theta} \quad + \quad 2U(X, Y), \tag{16}$$

where U is a symmetric bilinear application $U : \mathfrak{q}_\Theta \times \mathfrak{q}_\Theta \to \mathfrak{q}_\Theta$ defined by

$$2\Lambda^\Theta(U(X, Y), Z) = \Lambda^\Theta(X, [Y, Z]_{\mathfrak{q}_\Theta}) + \Lambda^\Theta([Z, X]_{\mathfrak{q}_\Theta}, Y),$$

for all $X, Y, Z \in \mathfrak{q}_\Theta$.

Then the concrete action of the Riemannian connection on the elements of the Weyl basis is given by:

Proposition 6.1. *For* $(\mathbb{F}_\Theta, \Lambda^\Theta)$ *let* $\alpha, \beta, \alpha+\beta \in \Pi\backslash\langle\Theta\rangle$, *and* $X_\alpha, X_\beta, X_{\alpha+\beta} \in \mathfrak{q}_\Theta$, *elements in the Weyl base. Then*

$$\nabla_{X_\alpha} X_\beta = m_{\alpha,\beta}\frac{\lambda_{\alpha+\beta}^\Theta + \lambda_\beta^\Theta - \lambda_\alpha^\Theta}{2\lambda_{\alpha+\beta}^\Theta}X_{\alpha+\beta}. \tag{17}$$

•**Kähler form.** $\sigma(X_\alpha, X_\beta) = (X_\alpha, \mathcal{F}(X_\beta))_\Lambda = -\langle\ \Lambda(X_\alpha), \mathcal{F}(X_\beta)\ \rangle$,

$$\sigma(X_\alpha, X_\beta) = \begin{cases} i\varepsilon_\alpha\lambda_\alpha & \beta = -\alpha, \quad \varepsilon_\alpha \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

for all $\alpha, \beta \in \Pi'$.

• **Derivative in the connection.**

$$\begin{aligned} (d^\nabla\mathcal{F})(X_\alpha, X_\beta) &= \nabla_{X_\alpha}\mathcal{F}X_\beta - \nabla_{X_\beta}\mathcal{F}X_\alpha - \mathcal{F}[X_\alpha, X_\beta] \\ &= i\tfrac{(\lambda_\alpha-\lambda_\beta)(\varepsilon_\alpha-\varepsilon_\beta)+\lambda_{\alpha+\beta}(\varepsilon_\alpha+\varepsilon_\beta-2\varepsilon_{\alpha+\beta})}{2(\lambda_{\alpha+\beta})}[X_\alpha, X_\beta]. \end{aligned}$$

• **Lie derivative**:

$$\begin{aligned} (Lie_{X_\beta}\mathcal{F})X_\alpha &= \mathcal{F}[X_\alpha, X_\beta] - [\mathcal{F}X_\alpha, X_\beta] \\ &= i(\varepsilon_{\alpha+\beta} - \varepsilon_\beta)[X_\alpha, X_\beta]. \end{aligned}$$

7. Graph theoretic description of $(\mathbb{F}(n), \mathcal{F}, \Lambda)$

On $\mathbb{F}(n)$ invariant f-structures are in 1:1 correspondence with digraphs $\mathcal{G} = (V, E)$. The correspondence is given by associating with the f-structure $\mathcal{F}(E_{jk}) = i\varepsilon_{jk}E_{jk}$ a digraph $\mathcal{G}$ whose vertices are $\{1, \cdots, n\}$ and whose arrows are given by the following rules: For $j < k$

$$j \to k \quad \Longleftrightarrow \quad \varepsilon_{jk} = 1,$$

$$j \leftarrow k \quad \Longleftrightarrow \quad \varepsilon_{jk} = -1,$$

$$j \not\leftrightarrow k \quad \Longleftrightarrow \quad \varepsilon_{jk} = 0.$$

Similarly, through the matrix $\Lambda = \{\lambda_{jk}\}$ we may identify an invariant metric ds^2 on $(\mathbb{F}(n), \mathcal{F})$ with a positive weighting on the edge set E of the digraph.

Example 7.1. *Again in $\mathbb{F}(4)$ let the f-structure*

$$\begin{pmatrix} 0 & a & b & c \\ -\bar{a} & 0 & d & e \\ -\bar{b} & -\bar{d} & 0 & f \\ -\bar{c} & -\bar{e} & -\bar{f} & 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & ia & -ib & -ic \\ -i\bar{a} & 0 & 0 & ie \\ i\bar{b} & 0 & 0 & 0 \\ i\bar{c} & -i\bar{e} & 0 & 0 \end{pmatrix}$$

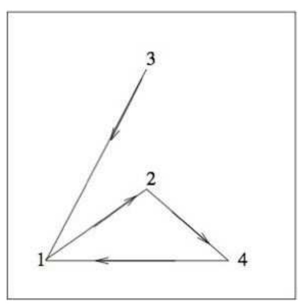

Figure 1. Digraph associated to the f-structure in Example 7.1.

There exists a complete classification for invariant f-structures on $\mathbb{F}(n)$ (see [5]). Here we present, up to isomorphism, the invariant f-structures in the case $\mathbb{F}(3)$, this graphs are the most relevant in our present work.

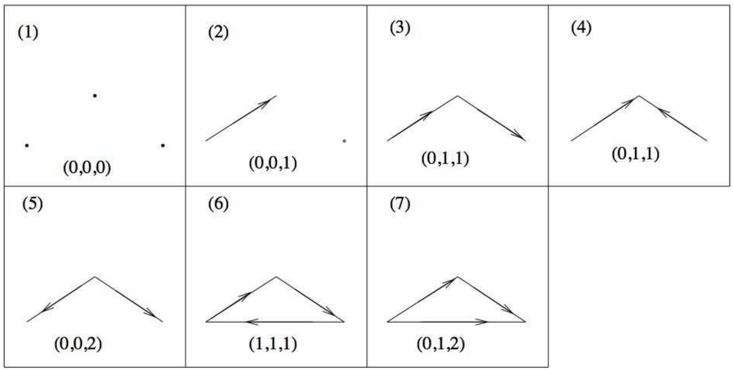

FIGURE 2. Isomorphism classes for $n = 3$

We are now ready to present our characterization of integrability in graph theoretical terms (classical case) or in root terms (general case).

8. Integrability via projections

In this section we will use all the formulas given before to establish integrability conditions for $\mathcal{F}$ and for the associated distributions $\mathfrak{q}_0$ and $\mathfrak{q}_+ + \mathfrak{q}_-$.

First, we will call the root triple $\{\alpha, \beta, \gamma\}$ with $\alpha, \beta, \gamma \in \Pi$ whenever $\alpha+\beta+\gamma = 0$, $\{\alpha, \beta, \gamma\}$ is called a zero-sum triple. Given an invariant f-structure $\mathcal{F}$, each root assumes a sign in $\{0, \pm 1\}$.

Roots triples may then be classified by their sign characteristic, which is a triple (p, q, r) $(p + q + r = 3)$ where p corresponds to the quantity of roots in triple who has 0 like its eigenvalue, q corresponds to the quantity of roots in triple who has i like its eigenvalue and r corresponds to the quantity of roots in triple who has $-i$ like its eigenvalue. There are six possible sign characteristics: $(3, 0, 0)$, $\{2, 1, 0\} = \{(2, 1, 0), (2, 0, 1)\}$, $\{1, 2, 0\} = \{(1, 2, 0), (1, 0, 2)\}$, $(1, 1, 1)$, $\{0, 2, 1\} = \{(0, 2, 1), (0, 1, 2)\}$, $\{0, 3, 0\} = \{(0, 3, 0), (0, 0, 3)\}$.

By the Frobenious Theorem we know that a distribution is integrable if and only if it is involutive. Thus $\mathfrak{q}_0$ is integrable if and only if $l[pX_\alpha, pX_\beta] = 0$, that is,

$$l[pX_\alpha, pX_\beta] = (\varepsilon_\alpha^2 - 1)(\varepsilon_\beta^2 - 1)\varepsilon_{\alpha+\beta}^2 [X_\alpha, X_\beta] = 0. \tag{18}$$

Theorem 8.1. *A necessary and sufficient condition for the distribution* $\mathfrak{q}_0$ *to be integrable is that in* $\mathfrak{q}$ *does not admit triples of type* $\{2, 1, 0\}$*. In case of* $\mathbb{F}(n)$ *this* *condition is equivalent to the associated digraph avoiding the subdigraph (2) in Figure 2.*

Proof. By equation (18) $\mathfrak{q}_0$ is not integrable in case where $(\varepsilon_\alpha^2-1)(\varepsilon_\beta^2-1)\varepsilon_{\alpha+\beta}^2 \neq 0$. This can only occur when the triple $\alpha, \beta, \alpha+\beta$ is a $\{2,1,0\}$-triple. In the classical case, this corresponds the configuration (2) in Figure 2. □

Now $\mathfrak{q}_+ + \mathfrak{q}_-$ is integrable if and only if $p[lX_\alpha, lX_\beta] = 0$ but

$$\begin{aligned} p[lX_\alpha, lX_\beta] &= (\mathcal{F}^2+1)[-\mathcal{F}^2 X_\alpha, -\mathcal{F}^2 X_\beta] & (19)\\ &= \varepsilon_\alpha^2\varepsilon_\beta^2(\mathcal{F}^2+1)[X_\alpha, X_\beta] & (20)\\ &= \varepsilon_\alpha^2\varepsilon_\beta^2(1-\varepsilon_{\alpha+\beta}^2)[X_\alpha, X_\beta]. & (21)\end{aligned}$$

Thus we have the following.

Theorem 8.2. *A necessary and sufficient condition for distribution $\mathfrak{q}_+ + \mathfrak{q}_-$ to be integrable is that in $\mathfrak{q}$ does not amit triples of type $\{1,2,0\}$ and $\{1,1,1\}$. In case of $\mathbb{F}(n)$ this condition is equivalent to the associated digraph avoiding the subdigraph (3),(4) and (5) in Figure 2.*

Proof. By equation (21) $\mathfrak{q}_+ + \mathfrak{q}_-$ is not integrable in case $\varepsilon_\alpha^2\varepsilon_\beta^2(1-\varepsilon_{\alpha+\beta}^2) \neq 0$. This can only occur when the triple $\alpha, \beta, \alpha+\beta$ is a $\{1,2,0\}$-triple or a $\{1,1,1\}$-triple. In the classical case, this corresponds to configurations (3), (4) and (5) in Figure 2. □

When $\mathfrak{q}_0$ and $\mathfrak{q}_+ + \mathfrak{q}_-$ are integrable, the structure of the submanifolds defined by these distributions and its properties is of interest. We hope to be able to report on this structure in a future communication.

Looking for the integrability of $\mathcal{F}$ we need another definition from [7].

Definition 8.3. *Assume $\mathfrak{q}_+ + \mathfrak{q}_-$ integrable and let X' be an arbitrary vector field which is tangent to an integrable manifold of $\mathfrak{q}_+ + \mathfrak{q}_-$. It is defined $\mathcal{F}'X' = \mathcal{F}X'$. Then $\mathcal{F}'$ is an almost-complex structure on each integral manifold of $\mathfrak{q}_+ + \mathfrak{q}_-$. $\mathcal{F}$ is called partially integrable if both, $\mathfrak{q}_+ + \mathfrak{q}_-$ and $\mathcal{F}'$ are integrable.*

Theorem 8.4. *A necessary and sufficient condition for $\mathcal{F}$ be partially integrable is that one of the following equivalent conditions be satisfied:*

$$N(lX_\alpha, lX_\beta) = 0, \text{ or } N(\mathcal{F}X_\alpha, \mathcal{F}X_\beta) = 0.$$

Using Weyl basis properties the Theorem 8.4 is equivalent to the following.

Theorem 8.5. *A necessary and sufficient condition for $\mathcal{F}$ to be partially integrable is that in $\mathfrak{q}$ does not admit triples of type {0,3,0}, {1,2,0} and $\{1,1,1\}$. In the* *case of $\mathbb{F}(n)$ this condition is equivalent to the associated digraph avoiding the subdigraph (3), (4), (5) and (6) in Figure 2.*

Proof. By Theorem 8.4 is enough to see the conditions $N(LX_\alpha, LX_\beta) = 0$. Doing the respective calculations we have

$$N(LX_\alpha, LX_\beta) = \varepsilon_\alpha\varepsilon_\beta(-1 - \varepsilon_\beta\varepsilon_{\alpha+\beta} - \varepsilon_\alpha\varepsilon_{\alpha+\beta} + \varepsilon_\alpha\varepsilon_\beta\varepsilon^2_{\alpha+\beta})[X_\alpha, X_\beta].$$

Then $\mathcal{F}$ is not integrable in case $\varepsilon_\alpha\varepsilon_\beta(-1 - \varepsilon_\beta\varepsilon_{\alpha+\beta} - \varepsilon_\alpha\varepsilon_{\alpha+\beta} + \varepsilon_\alpha\varepsilon_\beta\varepsilon^2_{\alpha+\beta}) \neq 0$, this can only occur when the triples are the type in the Theorem. In the classical case this corresponds to configurations (3), (4), (5) and (6) in Figure 2. □

When $\mathfrak{q}_0$ and $\mathfrak{q}_+ + \mathfrak{q}_-$ are both integrables, it is possible to choose a local coordinates system such that the operators l and p can be supposed to have the components of the form:

$$l = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \quad p = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix},$$

where n is the dimension of the manifold, r is the rank of $\mathcal{F}$ and 1_s means the s-identity matrix. This coordinate system is called "adapted." Since $\mathcal{F}$ satisfy $\mathcal{F}l = l\mathcal{F} = \mathcal{F}$ and $\mathcal{F}p = p\mathcal{F} = 0$ then in an adapted coordinate system, we can express $\mathcal{F}$ in the following way:

$$\mathcal{F} = \begin{pmatrix} \mathcal{F}_r & 0 \\ 0 & 0 \end{pmatrix}.$$

But $Lie_{pX_\alpha}\mathcal{F} = 0$ means that the components of $\mathcal{F}$ are independent of the coordinates, thus in the next theorem we are interested on this condition in root terms.

Theorem 8.6. *Suppose that $\mathfrak{q}_0$ and $\mathfrak{q}_+ + \mathfrak{q}_-$ are both integrable and that an adapted coordinate system has been chosen. A necessary and sufficient condition for the local components of $\mathcal{F}$ to be functions independent of the coordinates is that $N(lX_\alpha, pX_\beta) = 0$, in combinatorial terms a necessary and sufficient condition is that in $\mathfrak{q}$ does not admit triples of type $\{1, 1, 1\}$. In case of $\mathbb{F}(n)$ this condition is equivalent to the associated digraph avoiding the subdigraphs (4) and (5) in Figure 2.*

Proof. By equation (14) we have the first affirmation and with some calculus we have

$$N(lX_\alpha, pX_\beta) = \varepsilon_\alpha\varepsilon_{\alpha+\beta}(1 - {\varepsilon_\beta}^2)(\varepsilon_\alpha\varepsilon_{\alpha+\beta} - 1)[X_\alpha, X_\beta].$$

Thus $N(lX_\alpha, pX_\beta)$ will be different from zero in case $\varepsilon_\alpha\varepsilon_{\alpha+\beta}(1 - {\varepsilon_\beta}^2)(\varepsilon_\alpha\varepsilon_{\alpha+\beta} - 1) \neq 0$.

This can only occur when the triple is $\{1,1,1\}$-triple. In the classical case this corresponds to configurations mentioned in the theorem. □

When $\mathcal{F}$ is an almost complex structure integrability is associated with the existence of canonical coordinate systems, which allows us to consider the manifold as a complex manifold as well is known, integrability is equivalent to $N \equiv 0$. The following definition in the context of general differential manifolds appears in [7]. Here we present it in the case of flag manifolds.

Definition 8.7. *The f-structure $\mathcal{F}$, is called integrable if it satisfies the following three conditions:*

(i) *$\mathcal{F}$ is partially integrable.*
(ii) *$\mathfrak{q}_0$ is integrable.*
(iii) *The components of $\mathcal{F}$ are independent of the coordinates.*

With Definition 8.7, we arrive at a general integrability theorem for f-structures.

Theorem 8.8. *A necessary and sufficient condition for $\mathcal{F}$ to be integrable is that $N \equiv 0$. Therefore $\mathcal{F}$ is integrable if in $\mathfrak{q}$ does not admit triples of type $\{0,3,0\},\{2,1,0\},\{1,1,1\}$ and $\{1,2,0\}$. In $\mathbb{F}(n)$ this condition is equivalent to the associated digraph avoiding the subdigraphs (2), (3), (4), (5) and (6) in figure 2, that is, the associated digraph to $\mathcal{F}$ must be isomorphic to the null digraph or canonical tournament.*

Proof. It is immediate by Definition 8.7 and Theorems 8.1, 8.6 and 8.5. □

Theorem 8.8 is a generalization of the results obtained by Burstall [4] in the case of almost complex structures for classical maximal flag manifolds.

At this point we would like to point out an important connection between integrability and complex structures. It is well known that for a general almost complex structure J on a differential manifold is parallel, namely $d^{\nabla}J = 0$ if the manifold with that structure is Kähler. That is, parallelism means that J is integrable and the manifold is Kähler.

Now a manifold with an f-structure $\mathcal{F}$ will be called Kähler if $d^{\nabla}\mathcal{F} = 0$.

Observe that in the case of generalized flag manifolds integrability condition is stronger than Kähler condition, for example, in $\mathbb{F}(3)$ the invariant f-structures to the digraphs (4) and (5) are Kähler but not integrable.

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Sofía Pinzón
Escuela de Matemáticas,
Universidad Industrial de Santander,
A.A. 678, Bucaramanga, Colombia.

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