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ON THE VARIETY OF PLANAR NORMAL SECTIONS

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Abstract. In the present paper we present a survey of results concerning the variety $X[M]$ of planar normal sections associated to a natural embedding of a real flag manifold M^m. The results included are those that, we feel, better describe the nature of this algebraic variety of RP^{m-1}. In particular we present results concerning its Euler characteristic showing that it depends only on $\dim M$ and not on the nature of M itself. Furthermore, when M is the manifold of complete flags of a compact simple Lie group, we present what is, in some sense, its dimension and a large class of submanifolds of RP^{m-1} contained in $X[M]$.

1. Introduction

In Differential Geometry, the study of submanifolds is frequently associated to the to the theme of "normal sections". This was already present in works of Euler (1707-1783) when he studied surfaces embedded in R^3. Given a surface M embedded in R^3, we can obtain geometric information about the surface itself and the way it is contained in R^3 via properties of the curves that are obtained by cutting the surface with planes determined by unit tangent vectors and the normal vector to the surface at each point. This curves are called *normal sections* and they give information about the intrinsic and extrinsic geometry of the surface M.

Approximately in 1980, Bang Yen Chen generalized this notion for submanifolds of R^N of codimension larger than 1, in the following natural manner:

Let $j : M^m \to R^N$ be an isometric immersion and p a point in M. We identify a neighborhood of p with its image by j and consider, in the tangent space $T_p(M)$, a unit vector Y. If $T_p(M)^\perp$ denotes the normal space to M at p, we may define an affine subspace of R^N by

$$S(p,Y) = p + Span\left\{Y, T_p(M)^\perp\right\}.$$

If U is a small enough neighborhood of p in M, then the intersection $U \cap S(p,Y)$ can be considered the image of a C^∞ regular curve $\gamma(s)$, parametrized by arc-length, such that $\gamma(0) = p$, $\gamma'(0) = Y$. This curve is called a *normal section of M at p in the direction of Y*. In a strict sense, we ought to speak of the "germ" of a normal section at p determined by the unit vector Y. A change in the neighborhood U will change the curve; however, this new curve will coincide with γ in a neighborhood of zero. Since our computations with the curve γ are done at

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the point p, we may take any one of these curves. We may also assume that j is an embedding.

Since 1980, several authors, for instance Chen, Verheyen, Deprez ([2], [3], [8]) have studied geometric properties de submanifolds of Euclidean spaces in term of their normal sections. They obtained interesting results which characterize submanifolds of R^N where: the geodesic are planar; the normal sections are geodesics; the normal sections have the same constant curvature; etc.

Following B.Y. Chen, we say that the normal section γ of M at p in the direction of Y is *pointwise planar* at p if its first three derivatives $\gamma'(0), \gamma''(0)$ and $\gamma'''(0)$ are linearly dependent, i.e. if $\gamma'(0) \wedge \gamma''(0) \wedge \gamma'''(0) = 0$.

In 1982, Chen obtained the following interesting result.

Theorem 1. [2] *A spheric submanifold of R^N (i.e. contained in a sphere) has all its normal sections pointwise planar if and only if the second fundamental form is parallel.*

This fact was for us of the particular interest because, in 1980, Ferus had related symmetric R-spaces with properties of the second fundamental form through the following:

Theorem 2. [9] *A spheric submanifold of R^N is a symmetric R-space if and only if the second fundamental form is parallel.*

The previous theorems clearly yield the following:

Theorem 3. *A spheric submanifold of R^N is a symmetric R-spaces if and only if all its normal sections are pointwise planar.*

2. The variety of planar normal sections

As consequence of the Theorem 3, for a spheric submanifold M^m of R^N, the set of tangent vectors which define pointwise planar normal sections contains information about whether a given R-space is or not symmetric. So, if for each $p \in M$, we denote by $\widehat{X}_p[M]$ the set of $Y \in T_p(M)$ such that $\|Y\| = 1$ and define pointwise planar normal sections, we have that M is a symmetric R-space if and only if $\widehat{X}_p[M] = S^{m-1}$, $\forall p \in M$. Therefore, if M is a R-space which is not symmetric we have that $\widehat{X}_p[M] \subsetneq S^{m-1}$.

Our first objective was to obtain information about $\widehat{X}_p[M]$ when M is a R-space (also called *real flag manifold*).

We recall that an R-space or a *real flag manifold* is an orbit of an s-representation. The reader is referred to [4, p. 225] and references therein, for basic information concerning R-spaces, canonical connections, etc.

For our study we need methods and techniques different from the known ones. The first result that we obtained in this direction was the following.

Theorem 4. [4, (2.5)] *If $j : M^m \to R^N$ is a natural embedding of a real flag manifold and p is a point in M, then the normal section γ with $\gamma(0) = p$ and $\gamma'(0) = Y$ is pointwise planar at p if and only if the unit tangent vector Y at p satisfies the equation*

$$\alpha(D(Y,Y), Y) = 0,$$

where α is the second fundamental form of the embedding j and $D = \nabla - \nabla^c$ denotes the difference tensor between the Riemannian connection ∇ (associated to the metric induced from the Euclidean metric) and the canonical connection ∇^c (associated to the "usual" reductive decomposition of the Lie algebra of the compact Lie group defining M).

Then, given a point p in the real flag manifold M^m

$$\widehat{X}_p[M] = \{Y \in T_p(M) : \|Y\| = 1, \alpha(D(Y,Y), Y) = 0\}.$$

Since $Y \in \widehat{X}_p[M]$ clearly implies $-Y \in \widehat{X}_p[M]$, we may take $X_p[M]$ as the image of this set in the real projective space RP^{m-1}. Since M is an orbit of a group of isometries of the ambient space R^N, it is clear that $X_p[M]$ does not depend on the point p and we may denote it by $X[M]$.

The last theorem allowed us to describe tangent vectors which define pointwise planar normal sections as solutions of an equation and furthermore to obtain the following interesting consequence.

Corollary 1. [4, (2.9)] *$X[M]$ is a real algebraic variety of RP^{m-1} and its natural complexification $X_c[M]$ is a complex algebraic variety of CP^{m-1}, defined both by homogeneous polynomials of degree 3.*

These varieties measure, in some sense, how far is the real flag manifold M from being a symmetric space (i.e. a symmetric real flag manifold). By Theorem 3, $X[M]$ and $X_c[M]$ differ respectively from RP^{m-1} and CP^{m-1}. However, surprisingly enough, they have the same Euler characteristic as we see in the following:

Theorem 5. [4], [14] *Let M^m be a real flag manifold and let $j : M \to \mathfrak{p}$ be its natural imbedding. Let $X[M] \subset RP^{m-1}$ be the variety of directions of pointwise planar normal sections at a point $p \in M$ and let $X_c[M] \subset CP^{m-1}$ be the natural complexification of $X[M]$. If χ denotes the Euler characteristic with respect to rational coefficients, then*

$$\begin{array}{llll} (i)\ \chi(X[M]) & = \chi(RP^{m-1}) & = \begin{cases} 0 & if\ \dim M\ is\ even \\ 1 & if\ \dim M\ is\ odd \end{cases} \\ (ii)\ \chi(X_c[M]) & = \chi(CP^{m-1}) & = m = \dim M. \end{array}$$

In [4] we gave a proof of this fact when M is a complex flag manifold. The methods used in that paper were not strong enough to tackle the general case. However, several years later we were able to obtain the proof for the general case (see [14]).

3. Submanifolds in the variety of planar normal sections

Looking for information about the "size" of $X[M]$, we studied the existence of a great deal of smooth subvarieties embedded into RP^{m-1} and contained in $X[M]$, when M^m is a manifold of complete flags of a compact simple Lie group.

In order to indicate our results we need introduce the following notation.

Let G be a simply connected, complex, simple Lie group and let $\mathfrak{g}$ be its Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$. We may write $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\gamma \in \Delta^+} (\mathfrak{g}_\gamma \oplus \mathfrak{g}_{-\gamma})$, where Δ^+ indicates the set of positive roots with respect to some order.

Let us consider in $\mathfrak{g}$ the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \sum_{\gamma \in \Delta^+} \mathfrak{g}_{-\gamma}$. Let B be the analytic subgroup of G corresponding to the subalgebra $\mathfrak{b}$. B is closed and its own normalizer in G. The quotient space $M = G/B$ is a complex homogeneous space called the manifold of complete flags of G.

Let $\pi = \{\alpha_1, ..., \alpha_n\} \subset \Delta^+$ be a system of simple roots. We may take in $\mathfrak{g}$ a Weyl basis [12, III, 5] $\{X_\gamma : \gamma \in \Delta\}$ and $\{H_\beta : \beta \in \pi\}$. The following set of vectors provides a basis of a compact real form $\mathfrak{g}_u$ of $\mathfrak{g}$.

$$\begin{cases} U_\gamma = \frac{1}{\sqrt{2}} (X_\gamma - X_{-\gamma}) & \gamma \in \Delta^+ \\ U_{-\gamma} = \frac{i}{\sqrt{2}} (X_\gamma + X_{-\gamma}) & \gamma \in \Delta^+ \\ iH_\beta & \beta \in \pi. \end{cases} \tag{1}$$

We shall denote by $\mathfrak{h}_u$ the real vector space generated by $\{iH_\beta : \beta \in \pi\}$ and by $\mathfrak{m}_\gamma$ that of $\{U_\gamma, U_{-\gamma}\}$. Then we may write $\mathfrak{g}_u = \mathfrak{h}_u \oplus \sum_{\gamma \in \Delta^+} \mathfrak{m}_\gamma = \mathfrak{h}_u \oplus \mathfrak{m}$.

Let G_u be the analytic subgroup of G corresponding to $\mathfrak{g}_u$. G_u is compact and acts transitively on M which can be written as $M = G_u/T$, where the subgroup $T = G_u \cap B = \exp \mathfrak{h}_u$ is a maximal torus in G_u. The manifold M is then a compact simply connected complex manifold. This is *the manifold of complete flags* for the given compact connected simple Lie group G_u. In the rest of this paper we shall restrict our attention to this case.

It is well known that M is the orbit of a regular element $E \in \mathfrak{g}_u$ by the adjoint action of G_u on $\mathfrak{g}_u$. Then we have a natural embedding j of M on $\mathfrak{g}_u$ which we may assume isometric by taking in $\mathfrak{g}_u$ the inner product given by the opposite of the Killing form.

Then the tangent and normal space to M at E are $T_E(M) = [\mathfrak{g}_u, E] = [\mathfrak{m}, E] = \mathfrak{m}$ and $T_E(M)^\perp = \mathfrak{h}_u$.

If $Y = \sum_{\gamma \in \Delta^+} (y_\gamma U_\gamma + y_{-\gamma} U_{-\gamma}) \in \mathfrak{m}$ then $\overline{Y} = [Y, E] \in T_E(M)$ and for the second fundamental form of the embedding j we may write

$$\alpha([Y,E], D([Y,E],[Y,E])) = [Y,[Y,[Y,E]]_{\mathfrak{m}}]_{\mathfrak{h}_u} = \sum\nolimits_{1 \le r \le n} p_r i H_{\gamma_r}. \tag{2}$$

The coefficients p_r are homogeneous polynomials of degree 3 in the variables $y_\gamma, y_{-\gamma}$ ($\gamma \in \Delta^+$). Then, by Theorem 4, $\overline{Y}$ defines a pointwise planar normal section if and only if $p_r(\overline{Y}) = 0$ for $1 \le r \le n$.

3.1. **Fat Submanifold.** We obtained explicit enough expressions for the polynomials p_r defined by (2), which allowed us to prove that they are R-linearly dependent but this is not the case for any subset of them with $n-1$ elements.

Theorem 6. [7] *The polynomials p_r ($1 \le r \le n$) defined in (2) satisfy:*

(i)$\sum_{1 \le r \le n} \gamma_r(iE) p_r = 0$.

(ii) For any j such that $1 \le j \le n$ the set $\{p_r : 1 \le r \le n, r \ne j\}$ is $\mathbb{R}-$linearly independent.

With this fact, we can get certain information about the size of $X[M]$.

Theorem 7. [7] *There is an open set in the variety* $X[M]$ *which is an embedded submanifold in* RP^{m-1} *of dimension* $m-n$*, where* $m = \dim M$ *and* $n = rank\mathfrak{g}$.

(The topology of $X[M]$ is the induced one from the usual topology of RP^{m-1}.)

To get this result it was necessary to find points Y in S^{m-1}, the unit sphere of $\mathfrak{m}$, such that they are regular points of the function $S^{m-1} \to R^{n-1}$ whose coordinates are the polynomials $p_1, p_2, ..., p_{n-1}$ and that satisfy $[Y] \in X[M]$.

3.2. **Projective subspaces in** $X[M]$**.** Another way to get information about the "size" of $X[M]$ is to know a sufficient amount of projective subspaces in it.

We shall denote by $RP(\mathfrak{q})$ the real projective space associated to a real vector space $\mathfrak{q}$.

Associated to the simple group G_u, defining the complex flag manifold $M = G_u/T$, we have its family of symmetric spaces of type I [12, p. 518] and among them, we want to consider those which are *inner,* i.e. the spaces in which the symmetry at each point belongs to the group G_u. Among all compact symmetric spaces, these are the only ones strongly related with the algebraic variety $X[M]$. It is well known that each one of the simple groups gives rise to at least one of these symmetric spaces. They are those of the form G_u/K, where K is a subgroup of maximal rank in G_u. The ones which are not inner in the list in [12, p. 518] are AI, AII, BDI ($p+q=2n$, p *odd*, $1 \leq p \leq n$), EI and EIV.

By conjugating K if necessary, we may assume that K contains T.

Let $\mathfrak{k}$ be the Lie algebra of K and write $\mathfrak{g}_u = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{p}$ is the orthogonal complement to $\mathfrak{k}$ with respect to the Killing form. Then $\mathfrak{h}_u \subset \mathfrak{k}$ and $\mathfrak{p} \subset \mathfrak{m}$.

The motivation to consider the tangent space $\mathfrak{p}$ to the inner symmetric space G_u/K, in our study of the algebraic variety $X[M]$, arises from the following simple fact which provides the first examples of projective subspaces included in $X[M]$.

Proposition 1. [5, Prop. 4.1] *Let* $\mathfrak{p}$ *be the tangent space of the inner symmetric space* G_u/K *at* $[K]$*. Then* $RP(\mathfrak{p}) \subset X[M]$.

Remark 1. [5, Rem. 4.1] *For the subspace* $\mathfrak{p}$ *mentioned in the last proposition, there exists a root* $\gamma^* \in \pi$ *such that* $\mathfrak{p}$ *is of the form*

$$\mathfrak{p} = \sum_{\gamma \in \Delta^*} \mathfrak{m}_\gamma, \quad \textit{where} \quad \Delta^* = \{\gamma \in \Delta^+ : k_\gamma(\gamma^*) = 1\}$$

and $k_\gamma(\beta)$ *is defined by* $\gamma = \sum_{\beta \in \pi} k_\gamma(\beta)\beta$.

For this, it was natural to start by studying those subspaces of the tangent space to M at $o = [T]$ of the form

$$\widetilde{\mathfrak{p}} = \sum_{\gamma \in \widetilde{\Delta}} \mathfrak{m}_\gamma, \quad \textit{where} \quad \widetilde{\Delta} \subset \Delta$$

and such that $RP(\widetilde{\mathfrak{p}}) \subset X[M]$.

The subspaces $\widetilde{\mathfrak{p}}$ mentioned above, are exactly those subspaces of the tangent space $\mathfrak{m}$ which are $Ad(T)$-invariant (see for instance [11]).

The first important result to our objective, was the following characterization, in terms of the structure of the Lie algebra of the simple group G_u. This characterization was a very useful tool for the proof of many of the results obtained.

Theorem 8. [5, Th. 4.2] *Set* $\widetilde{\mathfrak{p}} = \sum_{\gamma \in \widetilde{\Delta}} \mathfrak{m}_\gamma$ *with* $\widetilde{\Delta} \subset \Delta^+$*. Then*

$$RP(\widetilde{\mathfrak{p}}) \subset X\,[M] \Longleftrightarrow \left(\varepsilon,\ \rho \in \widetilde{\Delta} \Rightarrow \varepsilon + \rho \notin \widetilde{\Delta}\right).$$

The tangent spaces of the inner symmetric space associated to G_u play an important part among the subspaces $Ad(T)$-invariant of the tangent space $\mathfrak{m}$. This can be seen in the following two theorems.

Theorem 9. [5, Th. 4.3] *Let* $\mathfrak{p}$ *be the tangent space of the inner symmetric space* G_u/K *at* $[K]$*. Then* $RP\,(\mathfrak{p})$ *is maximal among the projective spaces* $RP\left(\widetilde{\mathfrak{p}}\right)$ *contained in* $X\,[M]$*, with* $\widetilde{\mathfrak{p}}$ *of the form* $\widetilde{\mathfrak{p}} = \sum_{\gamma \in \widetilde{\Delta}} \mathfrak{m}_\gamma$ *for* $\widetilde{\Delta} \subset \Delta^+$*.*

This theorem is the best we can hope to get for projective subspaces $RP(\mathfrak{p}) \subset X\,[M]$ arising from tangent spaces $\mathfrak{p}$, at the base point $[K]$, of irreducible inner symmetric spaces G_u/K. We were able to show that if $\pi_2\,(G_u/K) = 0$, the projective spaces generated by those $\mathfrak{p}$ are not maximal among all the projective spaces contained in $X\,[M]$ (see [5, section 5]).

The irreducible inner symmetric spaces G_u/K for which $\pi_2\,(G_u/K) = 0$, are the following

BDI	$S^{2n} = SO\,(2n+1)\,/SO\,(2n)$	
CII	$Sp\,(p+q)\,/Sp\,(p) \times Sp\,(q)$	$p \geq q \geq 1$
FII	$F_4/Spin(9)$.	

These are those whose tangent spaces, at the basic point, are of the form

$$\mathfrak{p} = \sum_{\gamma \in \Delta^*} \mathfrak{m}_\gamma$$

where Δ^* is the set of all short roots. We proved that for the spaces of the families BDI, CII and the single space FII, the tangent space $\mathfrak{p}$ does not generate a maximal projective space in $X[M]$. However these are the only ones with this property as the following result indicates.

Theorem 10. [5, Th. 4.4] *Let* $\mathfrak{p}$ *be the tangent space of the inner symmetric space* G_u/K *at* $[K]$*. Then* $RP\,(\mathfrak{p})$ *is maximal in* $X\,[M]$ *if and only if* $\pi_2\,(G_u/K)$ *does not vanish.*

Another question arises quite naturally. How large can a subspace $Ad(T)$ - invariant defining projective spaces contained in $X\,[M]$ be?. Clearly an answer to this question yields information about the “size” of the variety $X\,[M]$.

The tangent spaces of the irreducible symmetric spaces are deeply related to this question and, as we expected, they provide the $Ad(T)$-invariant subspaces $\widetilde{\mathfrak{p}}$ of larger dimension such that $RP\left(\widetilde{\mathfrak{p}}\right)$ is contained in the variety of planar normal sections.

The list of irreducible symmetric spaces [12, p. 518] indicates that the irreducible inner symmetric spaces *of maximal dimension* for given groups G_u are those included in the following table with their respective dimensions. We denote them by G_u/H and $d(G_u) = \dim G_u/H$.

$\mathfrak{g}$	name	G_u/H		$d(G_u)$
$\mathfrak{a}_l$	AIII	$SU(l+1)/S(U(k+1)\times U(k))$	$l=2k$	$\frac{1}{2}l(l+2)$
		$SU(l+1)/S(U(k+1)\times U(k+1))$	$l=2k+1$	$\frac{1}{2}(l+1)^2$
$\mathfrak{b}_l$	BDI	$SO(2l+1)/SO(l+1)\times SO(l)$		$l(l+1)$
$\mathfrak{c}_l$	CI	$Sp(l)/U(l)$		$l(l+1)$
$\mathfrak{d}_l$	BDI	$SO(2l)/SO(l)\times SO(l)$	l *even*	l^2
		$SO(2l)/SO(l+1)\times SO(l-1)$	l *odd*	l^2-1
$\mathfrak{e}_6$	EII	$E_6/SU(6)Sp(1)$		40
$\mathfrak{e}_7$	EV	$E_7/(SU(8)/Z_2)$		70
$\mathfrak{e}_8$	EVIII	$E_8/(Spin(16)/Z_{2)}$		128
$\mathfrak{f}_4$	FI	$F_4/Sp(3)Sp(1)$		28
$\mathfrak{g}_2$	G	$G_2/SO(4)$		8

Theorem 11. [6, Th. 1.1, Th. 1.2] *If $\widetilde{\mathfrak{p}}\subset\mathfrak{m}$ is a subspace $Ad(T)$-invariant defining a projective subspace in $X[G_u/T]$, then*

(i) $\dim\widetilde{\mathfrak{p}}\leq d(G_u)$;

(ii) If $\dim\widetilde{\mathfrak{p}}=d(G_u)$ then $\widetilde{\mathfrak{p}}$ is tangent to the symmetric space G_u/H at a fixed point of the action of the torus T.

4. The case $G_u = SU(n+1)$

The existence of projective subspaces in the variety of planar normal sections makes it rather special.

In the previous section we gave information about families of projective subspaces in $X[G_u/T]$ which have deep relation with the tangent spaces of the inner symmetric spaces associated to the simple group G_u. This subspaces originate in some $Ad(T)$- invariant subspaces of the tangent space of G_u/T.

In [11], related to the study of extrinsic symmetric CR-structures on the manifold of complete flags $M=G_u/T$, it was observed that there is a strong connection between the holomorphic tangent spaces of these structures and those subspaces of the tangent space to M which are $Ad(T)$- invariant and also give rise to projective subspaces in $X[M]$. This particular fact throws new light on the interest of the study of these subspaces in $X[M]$.

In the previous theorems we characterize, for the manifolds of the form $M=G_u/T$, those subspaces $\widetilde{\mathfrak{p}}$ of the tangent space to M which are $Ad(T)$-invariant and define projective subspaces of *maximal* dimension in $X[M]$. Making a deeper analysis in this direction, for the manifolds $M_n=SU(n+1)/T^n$, we continued studying those subspaces $\widetilde{\mathfrak{p}}$ that are $Ad(T^n)$-invariant and define projective subspaces in $X[M_n]$ but that, in some "interesting" sense, are of "*minimal*" dimension. These subspaces are those $\widetilde{\mathfrak{p}}$, which are of *minimal dimension and not properly contained in any other $Ad(T^n)$-invariant subspace, defining projective subspaces in $X[M_n]$*. For these subspaces we have obtained the following:

Theorem 12. [7] *Let $n\geq 2$ and $M_n=SU(n+1)/T^n$ be embedded in $\mathfrak{su}(n+1)$ as the orbit of any regular element E. Let $\widetilde{\mathfrak{p}}=\sum_{\beta\in\widetilde{\Delta}}\mathfrak{m}_{\beta;n}$ ($\widetilde{\Delta}\subset\Delta_n^+$) be a* *subspace of* $\mathfrak{m}_n = T_E(M_n) \subset \mathfrak{su}(n+1)$ *which is maximal among the subspaces* $Ad(T^n)$*-invariant of* $\mathfrak{m}_n$ *defining projective subspaces in* $X[M_n]$. *Then*

(i) $\dim \widetilde{\mathfrak{p}} \geq 2n$.

(ii) If $\dim \widetilde{\mathfrak{p}} = 2n$ *then* $\widetilde{\mathfrak{p}}$ *is the tangent space to the projective space* $CP^n = SU(n+1)/S(U(n) \times U(1))$ *at a point* $E_1 = \widetilde{\sigma}(iv_n)$ *where* $\{v_j\}_{j=1}^n$ *is defined in* [15, p. 80] *and* $\widetilde{\sigma}$ *is an element in* W_{n+1}*, the Weyl group of the pair* $(SU(n+1), T^n)$.

Due to the fact that the converse statement of (ii) above is obviously true, this theorem gives a geometric characterization of the subspaces $\widetilde{\mathfrak{p}}$ of $\mathfrak{m}_n$ which are $2n$-dimensional, defining projective subspaces in $X[M_n]$ and maximal among the subspaces $Ad(T^n)$-invariant of $\mathfrak{m}_n$.

Joining Theorems 11 and 12, the subspaces $\widetilde{\mathfrak{p}}$ of $\mathfrak{m}_n$ which are maximal among the subspaces $Ad(T^n)$-invariant of $\mathfrak{m}_n$ defining projective subspaces in $X[M_n]$, satisfy

$$2n \leq \dim \widetilde{\mathfrak{p}} \leq d_n$$

and also, when $\dim \widetilde{\mathfrak{p}}$ is one of the two ends of the above inequality, the subspace $\widetilde{\mathfrak{p}}$ is tangent to the inner symmetric space of minimal and maximal dimension associated to the group $SU(n+1)$.

When the subspace $\widetilde{\mathfrak{p}}$ is such that $2n < \dim \widetilde{\mathfrak{p}} < d_n$, if we pose no restriction on n and $\dim \widetilde{\mathfrak{p}}$, we cannot assure that $\widetilde{\mathfrak{p}}$ is tangent to some inner symmetric space of the group $SU(n+1)$. Furthermore, we give examples in [7] to show that we cannot even assure that $\widetilde{\mathfrak{p}}$ is tangent to a homogeneous manifold $SU(n+1)/K$ with $T^n \subset K$.

The obtained results allow us to mention the following consequences which we feel are interesting and that in some sense motivated our interest in having a deeper understanding of the projective subspaces in the variety of planar normal section.

Keeping the notation of [11, Th.8] and calling

$$K_n = \begin{cases} S(U(\frac{n+2}{2}) \times U(\frac{n}{2})) & \text{if } n \text{ is even} \\ S(U(\frac{n+1}{2}) \times U(\frac{n+1}{2})) & \text{if } n \text{ is odd} \end{cases}$$

$$d_n = \begin{cases} \frac{n(n+2)}{2} & \text{if } n \text{ is even} \\ \frac{(n+1)^2}{2} & \text{if } n \text{ is odd} \end{cases}$$

we may write :

Corollary 2. [7] *Let* $\mathfrak{w}$ *be maximal among the holomorphic tangent spaces at the base point of* $SU(n+1)$*-invariant minimal almost Hermitian extrinsic symmetric CR-structure on* $M_n = SU(n+1)/T^n$. *Then*

(i) $2n \leq \dim_R \mathfrak{w} \leq d_n$.

(ii) If $\dim_R \mathfrak{w} = 2n$ *or* $\dim_R \mathfrak{w} = d_n$ *then* $\mathfrak{w}$ *is the tangent space, at some point, to the projective space* $CP^n = SU(n+1)/S(U(n) \times U(1))$ *or the symmetric space* $SU(n+1)/K_n$ *respectively.*

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