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CONNECTIONS COMPATIBLE WITH TENSORS. A CHARACTERIZATION OF LEFT-INVARIANT LEVI–CIVITA CONNECTIONS IN LIE GROUPS

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Abstract. Symmetric connections that are compatible with semi-Riemannian metrics can be characterized using an existence result for an integral leaf of a (possibly non integrable) distribution. In this paper we give necessary and sufficient conditions for a left-invariant connection on a Lie group to be the Levi–Civita connection of some semi-Riemannian metric on the group. As a special case, we will consider constant connections in $\mathbb{R}^n$.

1. Introduction

In this short note we address the following problem: given a (symmetric) connection ∇ on a smooth manifold M, under which conditions there exists a semi-Riemannian metric g in M which is ∇-parallel? This problem can be studied using holonomy theory (see [2]). Alternatively, the problem can be cast in the language of distributions and integral submanifolds, as follows. A connection ∇ on a manifold M induces naturally a connection in all tensor bundles over M (see for instance [4, § 2.7]), in particular, on the bundle of all (symmetric) (2,0)-tensors on M, say, $\nabla^{(2,0)}$. If g is a $(2,0)$-tensor on M, $p \in M$ and $v, w \in T_pM$, then the curvature $R^{(2,0)}(v,w)g$ is the bilinear form on T_pM given by:

$$\big(R^{(2,0)}(v,w)g\big)(\xi,\eta) = -g\big(R(v,w)\xi,\eta\big) - g\big(\xi,R(v,w)\eta\big), \tag{1}$$

where $\xi, \eta \in T_pM$ and R is the curvature tensor of ∇. A semi-Riemannian metric is a (globally defined) symmetric nondegenerate $(2,0)$-tensor on M, and compatibility with ∇ is equivalent to the property that the section is everywhere tangent to the horizontal distribution determined by the connection $\nabla^{(2,0)}$. However, such distribution is in general non integrable, namely, integrability of the horizontal distribution is equivalent to the vanishing of the curvature tensor $R^{(2,0)}$, which is equivalent to the vanishing of R. Hence, the classical Frobenius theorem cannot be employed in this situation. Nevertheless, the existence of simply *one* integral submanifold of a distribution, or, equivalently, of a parallel section of a vector bundle endowed with a connection, may occur even in the case of non integrable distributions. From (1), one sees immediately that if g is a $(2,0)$-tensor on M, then the condition that $R^{(2,0)}$ vanishes along g is equivalent to the condition of anti-symmetry of $g\big(R(v,w)\cdot,\cdot\big)$, for all $p \in M$ and all $v, w \in T_pM$.

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Let us consider the case that an open neighborhood V of a point m_0 of a manifold M is *ruled* by a family of curves issuing from m_0, parameterized by points of some manifold Λ. What this means is that it is given a smooth function $\psi : Z \subset \mathbb{R}\times\Lambda \to M$, defined on an open subset Z of $\mathbb{R}\times\Lambda$, with $\psi(0,\lambda) = m_0$ for all λ, and that admits a smooth right inverse $\alpha : V \subset M \to Z$. Assume that it is given a nondegenerate symmetric bilinear form $g_0 : T_{m_0}M \times T_{m_0}M \to \mathbb{R}$; one obtains a semi-Riemannian metric on V by *spreading* g_0 with parallel transport along the curves $t \mapsto \psi(t,\lambda)$. If the tensor g obtained in this way is such that $g\big(R(v,w)\cdot,\cdot\big)$ is an antisymmetric bilinear form on T_pM for all $p \in V$ and all $v, w \in T_pM$, then g is ∇-parallel. The precise statement of this fact is the following:

Proposition 1.1. *Let M be a smooth manifold, ∇ be a symmetric connection on TM, $m_0 \in M$ and g_0 be a nondegenerate symmetric bilinear form on $T_{m_0}M$. Let $\psi : Z \subset \mathbb{R} \times \Lambda \to M$ be a Λ-parametric family of curves on M with a local right inverse $\alpha : V \subset M \to Z$; assume that $\psi(0,\lambda) = m_0$, for all $\lambda \in M$. For each $(t,\lambda) \in Z$, we denote by $P_{(t,\lambda)} : T_{m_0}M \to T_{\psi(t,\lambda)}M$ the parallel transport along $t \mapsto \psi(t,\lambda)$. Assume that for all $(t,\lambda) \in Z$ the linear operator:*

$$P_{(t,\lambda)}^{-1}\big[R_{\psi(t,\lambda)}(v,w)\big]P_{(t,\lambda)} : T_{m_0}M \longrightarrow T_{m_0}M \tag{2}$$

is anti-symmetric with respect to g_0, for all $v, w \in T_{\psi(t,\lambda)}M$, where

$$R_{\psi(t,\lambda)}(v,w) : T_{\psi(t,\lambda)}M \longrightarrow T_{\psi(t,\lambda)}M$$

denotes the linear operator corresponding to the curvature tensor of ∇. Then ∇ is the Levi-Civita connection of the semi-Riemannian metric g on $V \subset M$ defined by setting:

$$g_m(\cdot,\cdot) = g_0(P_{\alpha(m)}^{-1}\cdot, P_{\alpha(m)}^{-1}\cdot),$$

for all $m \in V$.

Proof. See [3] □

In the real analytic case, we have the following global result:

Proposition 1.2. *Let M be a simply-connected real-analytic manifold and let ∇ be a real-analytic symmetric connection on TM. If there exists a semi-Riemannian metric g on a nonempty open connected subset of M having ∇ as its Levi-Civita connection then g extends to a globally defined semi-Riemannian metric on M having ∇ as its Levi-Civita connection.* □

The two results above will be used in Sections 3, 4 and 5 to characterize symmetric connections in Lie groups that are constant in left invariant referentials. The case of $\mathbb{R}^n$ (Lemma 3.1 and Proposition 3.2), and more specifically the 2-dimensional case (Proposition 4.9), will be studied with some more detail.

It is an interesting problem to study conditions for the existence, uniqueness, multiplicity, etc., of (symmetric) connections that are compatible with arbitrarily given tensors. It is well known that semi-Riemannian metrics admit *exactly one* symmetric and compatible connection, called the *Levi–Civita connection* of the metric. Uniqueness can be deduced also by a curious combinatorial argument, see Corollary 2.2. The next interesting case is that of symplectic forms, in which case one has existence, but not uniqueness. We will start the paper with a short section containing a couple of simple results concerning compatible connections. First, we will show the combinatorial argument that shows the uniqueness of the Levi–Civita connection of a semi-Riemannian metric tensor (Corollary 2.2). Second, we will prove that the existence of a symmetric connection compatible with a nondegenerate two-form ω is equivalent to the fact that ω is closed, in which case there are infinitely many symmetric connections compatible with ω (Lemma 2.3).

2. Connections compatible with tensors

Let M be a smooth manifold and let τ be any tensor in M; we will be mostly interested in the case when $\tau = g$ is a *semi-Riemannian metric tensor* on M (i.e., τ is a nondegenerate symmetric $(2,0)$-tensor), or when $\tau = \omega$ is a *symplectic form* on M (i.e., τ is a nondegenerate closed 2-form). If ∇ is a connection in M, i.e., a connection on the tangent bundle TM, then we have naturally induced connections on all tensor bundles on M, all of which will be denoted by the same symbol ∇.

The *torsion* of ∇ is the anti-symmetric tensor

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

where $[X,Y]$ denotes the Lie brackets of the vector fields X and Y; ∇ is called *symmetric* if $T = 0$. The connection ∇ is said to be *compatible* with τ if τ is ∇-parallel, i.e., when $\nabla\tau = 0$.

Establishing whether a given tensor τ admits compatible connections is a *local* problem. Namely, one can use partition of unity to extend locally defined connections and observe that a convex combination of compatible connections is a compatible connection. In local coordinates, finding a connection compatible with a given tensor reduces to determining the existence of solutions for a non homogeneous linear system for the Christoffel symbols of the connection.

It is well known that semi-Riemannian metric tensors admit a *unique* compatible symmetric connection, called the Levi–Civita connection of the metric tensor, which can be given explicitly by Koszul formula (see for instance [1]). Uniqueness of the Levi–Civita connection can be obtained by a curious combinatorial argument, as follows.

Suppose that ∇ and $\widetilde{\nabla}$ are connections on M; their difference $\widetilde{\nabla} - \nabla$ is a *tensor*, that will be denoted by $\mathfrak{t}$:

$$\mathfrak{t}(X,Y) = \widetilde{\nabla}_X Y - \nabla_X Y,$$

where X and Y are smooth vector fields on M. If both ∇ and $\widetilde{\nabla}$ are symmetric connections, then $\mathfrak{t}$ is symmetric:

$$\mathfrak{t}(X,Y) - \mathfrak{t}(Y,X) = \widetilde{\nabla}_X Y - \nabla_X Y - \widetilde{\nabla}_Y X + \nabla_Y X = [X,Y] + [Y,X] = 0.$$

Lemma 2.1. *Let U be a set and $\rho : U \times U \times U \to \mathbb{R}$ be a map that is symmetric in its first two variables and anti-symmetric in its last two variables. Then ρ is identically zero.*

Proof. Let $u_1, u_2, u_3 \in U$ be fixed. We have:

$$\rho(u_1, u_2, u_3) = \rho(u_2, u_1, u_3) = -\rho(u_2, u_3, u_1) = -\rho(u_3, u_2, u_1),$$

so that ρ is anti-symmetric in the first and the third variables. On the other hand:

$$\rho(u_1, u_2, u_3) = -\rho(u_3, u_2, u_1) = -\rho(u_2, u_3, u_1) = \rho(u_1, u_3, u_2),$$

so that ρ is symmetric in the second and the third variables. This concludes the proof. □

Corollary 2.2. *There exists at most one symmetric connection which is compatible with a semi-Riemannian metric.*

Proof. Assume that g is a semi-Riemannian metric on M, and let ∇ and $\widetilde{\nabla}$ two symmetric connections such that $\nabla g = \widetilde{\nabla} g = 0$; for all $p \in M$ consider the map $\rho : T_pM \times T_pM \times T_pM \to \mathbb{R}$ given by:

$$\rho(X, Y, Z) = g\big(\mathfrak{t}(X, Y), Z\big),$$

where $\mathfrak{t}$ is the difference $\widetilde{\nabla} - \nabla$. Since $\mathfrak{t}$ is symmetric, then ρ is symmetric in the first two variables. On the other hand, ρ is anti-symmetric in the last two variables:

$$\begin{aligned}\rho(X, Y, Z) + \rho(X, Z, Y) \\ = g\big(\widetilde{\nabla}_X Y, Z\big) - g\big(\nabla_X Y, Z\big) + g\big(\widetilde{\nabla}_X Z, Y\big) - g\big(\nabla_X Z, Y\big) \\ = \widetilde{\nabla} g(X, Y, Z) - \nabla g(X, Y, Z) = 0.\end{aligned}$$

By Lemma 2.1, $\rho = 0$, hence $\mathfrak{t} = 0$, and thus $\widetilde{\nabla} = \nabla$. □

For symplectic forms, the situation changes radically. Among all nondegenerate two-forms, the existence of a symmetric compatible connection characterizes the symplectic ones:

Lemma 2.3. *Let ω be a nondegenerate 2-form on a (necessarily even dimensional) manifold M. There exists a symmetric connection in M compatible with ω if and only if ω is closed. In this case, there are infinitely many symmetric connections that are compatible with ω.*

Proof. If ω is closed, i.e., if ω is a symplectic form on M, Darboux theorem tells us that one can find coordinates (q, p) around every point of M such that $\omega = \sum_i \mathrm{d}q^i \wedge \mathrm{d}p_i$, which means that ω is constant in such coordinate system. The (locally defined) symmetric connection which has vanishing Christoffel symbols in such coordinates is clearly compatible with ω. As observed above, using partitions of unity one can find a globally defined symmetric connection compatible with ω. Conversely, if ∇ is any symmetric connection in M, then $\mathrm{d}\omega$ is given by $\frac{1}{2}\mathrm{Alt}(\nabla\omega)$, where Alt denotes the alternator; in particular, if there exists a compatible symmetric connection it must be $\mathrm{d}\omega = 0$. □

3. Constant connections in $\mathbb{R}^n$

Let $\Gamma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a symmetric bilinear map and consider the symmetric connection ∇ on $\mathbb{R}^n$ defined by:

$$\nabla_X Y = \mathrm{d}Y(X) + \Gamma(X, Y), \tag{3}$$

for any smooth vector fields X, Y on $\mathbb{R}^n$. We now apply the result of Proposition 1.1 to determine when ∇ is the Levi-Civita connection of a semi-Riemannian metric on $\mathbb{R}^n$. Given $v \in \mathbb{R}^n$ then the parallel transport $P_{(t,v)} : \mathbb{R}^n \to \mathbb{R}^n$ along the curve $t \mapsto tv$ is given by:

$$P_{(t,v)} = \exp\big(-t\,\Gamma(v)\big),$$

where we identify Γ with the linear map $\mathbb{R}^n \ni v \mapsto \Gamma(v, \cdot) \in \mathrm{Lin}(\mathbb{R}^n)$. For any $v, w \in \mathbb{R}^n$, the curvature tensor R of ∇ is given by:

$$R_x(v, w) = \Gamma(v)\Gamma(w) - \Gamma(w)\Gamma(v) = [\Gamma(v), \Gamma(w)] \in \mathrm{Lin}(\mathbb{R}^n),$$

for all $x \in \mathbb{R}^n$. Applying Proposition 1.1 to the $\mathbb{R}^n$-parametric family of curves $\psi(t, \lambda) = t\lambda \in \mathbb{R}^n$ with right inverse $\alpha : \mathbb{R}^n \ni v \mapsto (1, v) \in \mathbb{R} \times \mathbb{R}^n$ we obtain the following:

Lemma 3.1. *Let g_0 be a nondegenerate symmetric bilinear form on $T_0\mathbb{R}^n \cong \mathbb{R}^n$. Then g_0 extends to a semi-Riemannian metric g on $\mathbb{R}^n$ having* (3) *as its Levi-Civita connection if and only if the linear operator:*

$$\exp\big(\Gamma(v)\big)[\Gamma(w_1), \Gamma(w_2)]\exp\big(-\Gamma(v)\big) : \mathbb{R}^n \to \mathbb{R}^n$$

is anti-symmetric with respect to g_0, for all $v, w_1, w_2 \in \mathbb{R}^n$. □

Given a nondegenerate symmetric bilinear form g_0 on $\mathbb{R}^n$ we denote by $\mathfrak{so}(g_0)$ the Lie algebra of all g_0-anti-symmetric endomorphisms of $\mathbb{R}^n$. Given a linear endomorphism X of $\mathbb{R}^n$ we write:

$$\mathrm{ad}_X(Y) = [X, Y] = XY - YX,$$

for all $Y \in \mathrm{Lin}(\mathbb{R}^n)$.

Proposition 3.2. *Let $\Gamma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a symmetric bilinear map and let $\mathcal{S} \subset \mathrm{Lin}(\mathbb{R}^n)$ be the range of the linear map $v \mapsto \Gamma(v, \cdot)$. A nondegenerate symmetric bilinear form g_0 on $T_0\mathbb{R}^n \cong \mathbb{R}^n$ extends to a semi-Riemannian metric g on $\mathbb{R}^n$ having* (3) *as its Levi-Civita connection if and only if:*

$$(\mathrm{ad}_X)^k[Y, Z] \in \mathfrak{so}(g_0),$$

for all $X, Y, Z \in \mathcal{S}$ and all $k \geq 0$.

Proof. By Lemma 3.1, g_0 extends to a semi-Riemannian metric g on $\mathbb{R}^n$ having (3) as its Levi-Civita connection if and only if:

$$\exp(tX)[Y, Z]\exp(-tX) \in \mathfrak{so}(g_0),$$

for all $X, Y, Z \in \mathcal{S}$ and all $t \in \mathbb{R}$. The conclusion follows by observing that:

$$\exp(tX)[Y, Z]\exp(-tX) = \sum_{k=0}^{+\infty} \frac{t^k}{k!}(\mathrm{ad}_X)^k[Y, Z]. \qquad \square$$

Corollary 3.3. *Let $\Gamma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a symmetric bilinear map and let $\mathcal{S} \subset \mathrm{Lin}(\mathbb{R}^n)$ be the range of the linear map $v \mapsto \Gamma(v, \cdot)$. Denote by $\mathfrak{g}$ the Lie algebra spanned by $\mathcal{S}$ and by $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ the commutator subalgebra of $\mathfrak{g}$. If $\mathfrak{g}'$ is contained in $\mathfrak{so}(g_0)$ for some nondegenerate symmetric bilinear form g_0 on $T_0\mathbb{R}^n \cong \mathbb{R}^n$ then g_0 extends to a semi-Riemannian metric g on $\mathbb{R}^n$ having* (3) *as its Levi-Civita connection.* □

4. The two-dimensional case

If $n = 2$, the Lie algebra $\mathfrak{so}(g_0)$ is one-dimensional. This observation allows us to show that, for $n = 2$, the condition $\mathfrak{g}' \subset \mathfrak{so}(g_0)$ in the statement of Corollary 3.3 is also necessary for g_0 to extend to a semi-Riemannian metric g on $\mathbb{R}^2$ having (3) as its Levi-Civita connection.

Lemma 4.1. *Let $\Gamma : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be a symmetric bilinear map and let $\mathcal{S} \subset \mathrm{Lin}(\mathbb{R}^2)$ be the range of the linear map $v \mapsto \Gamma(v, \cdot)$. Denote by $\mathfrak{g}$ the Lie algebra spanned by $\mathcal{S}$. Then a nondegenerate symmetric bilinear form g_0 on $T_0\mathbb{R}^2 \cong \mathbb{R}^2$ extends to a semi-Riemannian metric g on $\mathbb{R}^2$ having* (3) *as its Levi-Civita connection if and only if $\mathfrak{g}' \subset \mathfrak{so}(g_0)$.*

Proof. Define a sequence $\mathcal{S}_k$ of subspaces of $\mathfrak{g}$ inductively by setting $\mathcal{S}_1 = \mathcal{S}$ and by taking $\mathcal{S}_{k+1} = [\mathcal{S}, \mathcal{S}_k]$ to be the linear span of all commutators $[X, Y]$, with $X \in \mathcal{S}$, $Y \in \mathcal{S}_k$. Using the Jacobi identity it is easy to show that $[\mathcal{S}_k, \mathcal{S}_l] \subset \mathcal{S}_{k+l}$ and therefore:

$$\mathfrak{g} = \sum_{k=1}^{\infty} \mathcal{S}_k, \quad \mathfrak{g}' = \sum_{k=2}^{\infty} \mathcal{S}_k.$$

By Proposition 3.2, if extends to a semi-Riemannian metric g on $\mathbb{R}^2$ having (3) as its Levi-Civita connection then $\mathcal{S}_2$ and $\mathcal{S}_3$ are contained in $\mathfrak{so}(g_0)$. Since $\mathfrak{so}(g_0)$ is one dimensional, we have either $\mathcal{S}_3 = 0$ or $\mathcal{S}_3 = \mathcal{S}_2$; in the first case, $\mathcal{S}_k = 0$ for all $k \geq 3$ and in the latter case $\mathcal{S}_k = \mathcal{S}_2$ for all $k \geq 3$. In any case, $\mathfrak{g}' = \mathcal{S}_2$ and the conclusion follows. □

Lemma 4.2. *Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be a nonzero linear map. There exists a nondegenerate symmetric bilinear form g_0 on $\mathbb{R}^2$ with $A \in \mathfrak{so}(g_0)$ if and only if* $\mathrm{tr}\, A = 0$ *and* $\det A \neq 0$*; moreover, g_0 is positive definite (resp., has index 1) if and only if* $\det A > 0$ *(resp.,* $\det A < 0$*).*

Proof. Assume that $\mathrm{tr}\, A = 0$ and $\det A \neq 0$. Write $\det A = -\epsilon a^2$, with $\epsilon = \pm 1$ and $a > 0$. It is easy to see that A is represented by the matrix $\left(\begin{smallmatrix} 0 & \epsilon a \\ a & 0 \end{smallmatrix}\right)$ in some basis (b_1, b_2) of $\mathbb{R}^2$. We define g_0 by setting:

$$g_0(b_1, b_2) = 0, \quad g_0(b_1, b_1) = 1, \quad g_0(b_2, b_2) = -\epsilon. \tag{4}$$

Conversely, if $A \in \mathfrak{so}(g_0)$ for some g_0 then we can choose a basis (b_1, b_2) of $\mathbb{R}^2$ such that (4) holds and the matrix of A on such basis is of the form $\left(\begin{smallmatrix} 0 & \epsilon a \\ a & 0 \end{smallmatrix}\right)$. □

Corollary 4.3. *Let $\Gamma : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be a symmetric bilinear map and let $\mathcal{S} \subset \mathrm{Lin}(\mathbb{R}^2)$ be the range of the linear map $v \mapsto \Gamma(v, \cdot)$. Denote by $\mathfrak{g}$ the Lie algebra spanned by $\mathcal{S}$. There exists a semi-Riemannian metric on $\mathbb{R}^2$ having* (3) *as its Levi-Civita connection if and only if either* $\mathfrak{g}' = 0$ *or* $\mathfrak{g}'$ *is one-dimensional and it is spanned by an invertible* 2×2 *matrix.*

Proof. Follows from Lemmas 4.1 and 4.2, observing that the elements of $\mathfrak{g}'$ have null trace. □

Lemma 4.4. *Let* $\mathfrak{g}$ *be a three-dimensional real Lie algebra with* $\mathfrak{g}'$ *one-dimensional. Then the center* $\mathfrak{z}(\mathfrak{g})$ *of* $\mathfrak{g}$ *is one-dimensional.*

Proof. Let Z denote a generator of $\mathfrak{g}'$, so that $[X, Y] = \alpha(X, Y)Z$, for all $X, Y \in \mathfrak{g}$, where α is an antisymmetric bilinear form on $\mathfrak{g}$; clearly, the kernel of α is the center of $\mathfrak{g}$. Since $\mathfrak{g}$ is three-dimensional, the kernel of α is either $\mathfrak{g}$ or it is one-dimensional; the first possibility does not occur, since $\mathfrak{g}'$ is nonzero. □

Corollary 4.5. *Let* $\mathfrak{g}$ *be a three-dimensional real Lie algebra with* $\mathfrak{g}'$ *one-dimensional. Then there exists a basis* (X, Y, Z) *of* $\mathfrak{g}$ *such that one the following commutation relations holds:*

1. $[X, Y] = [X, Z] = 0$*,* $[Y, Z] = X$*;*
2. $[X, Y] = [Y, Z] = 0$*,* $[Z, X] = X$*.*

Proof. Choose a basis (X, Y, Z) of $\mathfrak{g}$ with X in $\mathfrak{g}'$. If $\mathfrak{z}(\mathfrak{g}) = \mathfrak{g}'$ then $[Y, Z] \neq 0$, otherwise $\mathfrak{g}' = 0$; thus, we can replace Z with a scalar multiple of Z so that $[Y, Z] = X$ and relations (1) hold. If $\mathfrak{z}(\mathfrak{g}) \neq \mathfrak{g}'$, we may assume that $Y \in \mathfrak{z}(\mathfrak{g})$ and $[Z, X] \neq 0$; again, replacing Z with a scalar multiple of Z gives $[Z, X] = X$ and relations (2) hold. □

In what follows we denote by $\mathrm{gl}(n, \mathbb{R})$ the Lie algebra of linear endomorphisms of $\mathbb{R}^n$.

Lemma 4.6. *Let* $\mathfrak{g}$ *be a three-dimensional Lie subalgebra of* $\mathrm{gl}(2, \mathbb{R})$ *with* $\mathfrak{g}'$ *one-dimensional. There exists a basis* (X, Y, Z) *of* $\mathfrak{g}$ *with* $Y = \mathrm{Id}$ *and* $[Z, X] = X$*.*

Proof. We show that Id is in $\mathfrak{g}$. Assume not. By Lemma 4.4, there exists a nonzero element W in $\mathfrak{z}(\mathfrak{g})$. Then W commutes with $\mathfrak{g}$ and with Id, which implies that W is in the center of $\mathrm{gl}(2, \mathbb{R})$; thus W is a nonzero multiple of Id, contradicting our assumption.

Now $\mathrm{Id} \in \mathfrak{g}$ implies that Id spans $\mathfrak{z}(\mathfrak{g})$; thus possibility (1) in the statement of Corollary 4.5 does not occur for it would imply that $[Y, Z]$ is a nonzero multiple of the identity. Hence possibility (2) occurs and we can assume that $Y = \mathrm{Id}$. □

Lemma 4.7. *If* $\mathfrak{g}$ *is a two-dimensional real Lie algebra with* $\mathfrak{g}' \neq 0$ *then there exists a basis* (X, Z) *of* $\mathfrak{g}$ *with* $[Z, X] = X$*.*

Proof. Let X be a nonzero element in $\mathfrak{g}'$; clearly, $\mathfrak{g}'$ is one-dimensional. We can choose $Z \in \mathfrak{g}$, $Z \notin \mathfrak{g}'$, with $[Z, X] = X$. □

Lemma 4.8. *If* $X, Z \in \mathrm{gl}(n, \mathbb{R})$ *and* $[Z, X] = X$ *then* X *is not invertible.*

Proof. If X were invertible then $[Z, X] = X$ would imply $Z - XZX^{-1} = \mathrm{Id}$. A contradiction is obtained by taking traces on both sides. □

Proposition 4.9. *Let $\Gamma : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be a symmetric bilinear map and let $\mathcal{S} \subset \mathrm{Lin}(\mathbb{R}^2)$ be the range of the linear map $v \mapsto \Gamma(v, \cdot)$. Then there exists a semi-Riemannian metric g on $\mathbb{R}^2$ having* (3) *as its Levi-Civita connection if and only if $[X, Y] = 0$, for all $X, Y \in \mathcal{S}$. In this case, a semi-Riemannian metric g on $\mathbb{R}^2$ having* (3) *as its Levi-Civita connection can be chosen with an arbitrary value g_0 at the origin.*

Proof. If $[X, Y] = 0$ for all $X, Y \in \mathcal{S}$ then, by Lemma 4.1, any nondegenerate symmetric bilinear form g_0 on $T_0\mathbb{R}^2 \cong \mathbb{R}^2$ extends to semi-Riemannian metric g on $\mathbb{R}^2$ having (3) as its Levi-Civita connection. Now assume that there exists a semi-Riemannian metric on $\mathbb{R}^2$ having (3) as its Levi-Civita connection and denote by $\mathfrak{g}$ the Lie algebra spanned by $\mathcal{S}$. By Corollary 4.3, either $\mathfrak{g}' = 0$ or $\mathfrak{g}'$ is one-dimensional and it is spanned by an invertible 2×2 matrix. Let us show that the second possibility cannot occur. If $\mathfrak{g}' \neq 0$ then $2 \leq \dim(\mathfrak{g}) \leq 4$. If $\dim(\mathfrak{g}) = 4$ then $\mathfrak{g} = \mathrm{gl}(2, \mathbb{R})$ and $\mathfrak{g}'$ is three-dimensional, which is not possible. If either $\dim(\mathfrak{g}) = 2$ or $\dim(\mathfrak{g}) = 3$ then by Lemmas 4.6 and 4.7 there exist $X, Z \in \mathfrak{g}$ with $[Z, X] = X$ and such that X spans $\mathfrak{g}'$. By Lemma 4.8, X is not invertible and we obtain a contradiction. □

5. Left-Invariant Connections on Lie Groups

Let G be a Lie group and ∇ be a left-invariant connection on G. The connection ∇ is determined by a bilinear map $\Gamma : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, i.e.:

$$\nabla_X Y = \Gamma(X, Y),$$

for any left-invariant vector fields X, Y on G.

The torsion of ∇ is given by:

$$T(X, Y) = \Gamma(X, Y) - \Gamma(Y, X) - [X, Y].$$

Observe that ∇ is torsion-free if and only if there exists a symmetric bilinear map $B : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ with $\Gamma(X, Y) = B(X, Y) + \frac{1}{2}[X, Y]$, for all $X, Y \in \mathfrak{g}$. If we identify Γ with the linear map $\mathfrak{g} \ni X \mapsto \Gamma(X, \cdot) \in \mathrm{Lin}(\mathfrak{g})$ then ∇ is torsion-free if and only if Γ is a Lie algebra homomorphism. The curvature tensor of ∇ is given by:

$$R(X, Y) = [\Gamma(X), \Gamma(Y)] - \Gamma\big([X, Y]\big);$$

observe that the first bracket is the commutator in $\mathrm{Lin}(\mathfrak{g})$ and the second is the Lie algebra product of $\mathfrak{g}$.

Given a curve γ on G, we identify vector fields along γ with curves on $\mathfrak{g}$ by left translation. Using this identification, the parallel transport of $Y \in \mathfrak{g}$ along a one-parameter subgroup $t \mapsto \exp(tX) \in G$ is given by $t \mapsto e^{-t\Gamma(X)}Y \in \mathfrak{g}$.

Proposition 5.1. *Assume that ∇ is torsion-free and let $h : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be a nondegenerate symmetric bilinear form. The following condition is necessary and sufficient for the existence of an extension of h to a semi-Riemannian metric on a neighborhood of the identity of G whose Levi-Civita connection is ∇:*

$$e^{\Gamma(Z)}\Big([\Gamma(X), \Gamma(Y)] - \Gamma\big([X, Y]\big)\Big)e^{-\Gamma(Z)} \in \mathfrak{so}(h), \quad \text{for all } X, Y, Z \in \mathfrak{g}. \tag{5}$$

In (5) *we have denoted by* $\mathfrak{so}(h)$ *the Lie subalgebra of* $\mathrm{Lin}(\mathfrak{g})$ *consisting of* h*-anti-symmetric linear operators.*

Proof. Set $\Lambda = \mathfrak{g}$ and consider the one-parameter family of curves $\psi : \mathbb{R} \times \Lambda \to G$ defined by $\psi(t, \lambda) = \exp(t\lambda)$. If U is an open neighborhood of the origin of $\mathfrak{g}$ that is mapped diffeomorphically by exp onto an open neighborhood $\exp(U)$ of the identity of G then a local right inverse for ψ can be defined by setting $\alpha(g) = \big(1, (\exp|_U)^{-1}(g)\big)$, for all $g \in \exp(U)$. The conclusion follows from Proposition 1.1. □

Corollary 5.2. *Assume that* ∇ *is torsion-free and let* $h : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ *be a nondegenerate symmetric bilinear form. If* G *is (connected and) simply-connected then condition* (5) *is necessary and sufficient for the existence of a globally defined semi-Riemannian metric on* G *whose Levi-Civita connection is* ∇*.*

Proof. It follows from Proposition 5.1 and from Proposition 1.2 observing that left-invariant objects on a Lie group are always real-analytic. □

Lemma 5.3. *Condition* (5) *is equivalent to:*

$$\mathrm{ad}^n_{\Gamma(Z)}\Big([\Gamma(X), \Gamma(Y)] - \Gamma\big([X, Y]\big)\Big) \in \mathfrak{so}(h), \quad \textit{for all } X, Y, Z \in \mathfrak{g},\ n \geq 0,$$

where $\mathrm{ad}_A(B) = [A, B]$*, for all* $A, B \in \mathrm{Lin}(\mathfrak{g})$*.*

Proof. Replace Z by tZ in (5) and compute the Taylor expansion in powers of t of the corresponding expression. □

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