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SPECTRAL PROPERTIES OF ELLIPTIC OPERATORS ON BUNDLES OF $\mathbb{Z}_2^k$-MANIFOLDS

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ABSTRACT. We present some results on the spectral geometry of compact Riemannian manifolds having holonomy group isomorphic to $\mathbb{Z}_2^k$, $1 \leq k \leq n-1$, for the Laplacian on mixed forms and for twisted Dirac operators.

INTRODUCTION

This expository article is based on a homonymous talk I gave during the "II Encuentro de Geometría" which took place in La Falda, Sierras de Córdoba, from June 6th to 11th of 2005. It summarizes previous results from [**MP**], [**MP2**], [**MPR**], and [**Po**], answering standard questions in spectral geometry by using a special class of compact Riemannian manifolds.

Spectral Geometry. It is a kind of mixture between Spectral Theory and Riemannian Geometry. The general situation is to consider (pseudo) differential operators acting on sections of bundles of Riemannian manifolds. However, one usually considers a *compact* Riemannian manifold M, a *vector* bundle $E \to M$ and an *elliptic self-adjoint differential* operator D acting on smooth sections of E, i.e. $D : \Gamma^\infty(E) \to \Gamma^\infty(E)$. Since M is compact, D has a discrete *spectrum*, denoted by $Spec_D(M)$, consisting of real eigenvalues of finite multiplicity which accumulate only at infinity. In symbols, we have

- $Spec_D(M) = \{\{\ \lambda \in \mathbb{R} : Df = \lambda f,\ f \in \Gamma^\infty(E)\}\} \subset \mathbb{R}$,
- $0 \leq |\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_i| \nearrow \infty, \qquad \lambda_i \in Spec_D(M),\ i \in \mathbb{N}$.
- $d_\lambda = \dim(H_\lambda) < \infty, \quad H_\lambda = \{f : Df = \lambda f\} = \lambda$-eigenspace.

We can also think of $Spec_D(M)$ as being the set $\{(\lambda, d_\lambda)\} \subset \mathbb{R} \times \mathbb{N}_0$.

Two manifolds M, M' are called *isospectral with respect to D*, or simply D-*isospectral*, if $Spec_D(M) = Spec_D(M')$. That is, if M, M' have the same set of eigenvalues with the same corresponding multiplicities. It is a general fact that the spectrum determines the *dimension* and the *volume* of M. In other words, if M, M' are D-isospectral, then $\dim(M) = \dim(M')$ and $\mathrm{vol}\,(M) = \mathrm{vol}\,(M')$. The spectrum is said to be *asymmetric* if $d_\lambda \neq d_{-\lambda}$ for some $\lambda \in Spec_D(M) \smallsetminus \{0\}$.

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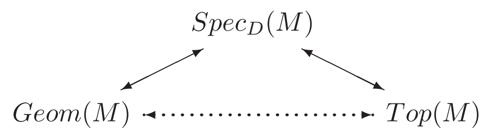
Main Aim. The goal of Spectral Geometry is to study the spectrum of $M$and the interrelations between this object and the geometry or topology of M. That is, knowing the spectrum, what can be said geometrically about M? Conversely, which spectral data can be deduced provided that we know the geometry of M? This can be summarize in the following diagram



Incidentally, by using the diagonal "maps" one could say something about the horizontal "map".

Main Problems. There are several different ways of studying the spectrum. In my opinion, the following are the three most important and interesting ones. In this paper we shall collect results concerning all of them.

- *Computation of* $Spec_D(M)$. The problem is to determine the eigenvalues λ and their multiplicities d_λ. This is in general a difficult task in the sense that this cannot always be done. Indeed, there are few classes of manifolds with explicitly known spectrum for some given operator. The simplest case is the Laplacian Δ acting on smooth functions on the torus $\mathbb{T}^n = \mathbb{Z}^n \backslash \mathbb{R}^n$.

- *Isospectrality*. Physically, it is a problem with more than a century old and inquires about the possibility of changing shape while sounding the same. Mathematically, it begun in 1964 with the famous Kac's question *Can one hear the shape of a drum?* ([**Ka**]) and the negative answer given by Milnor to a related question ([**Mi**]). There are basically two antagonistic approaches to this problem: *criteria* vs. *counterexamples*. In the first case, one seeks sufficient conditions ensuring that two manifolds are isospectral. This is what some people have called *Optimistic Spectral Geometry*. On the contrary, in the second case, one tries to produce examples of pairs of isospectral manifolds which are very similar to each other but differing in some geometrical or topological property $\mathcal{P}$. In this case, we say that this particular property $\mathcal{P}$ *cannot be heard* or that we *cannot hear property* $\mathcal{P}$. This has been fairly called *Pessimistic Spectral Geometry*, but I would faintly call it ***Deaf****ferential Geometry*. One interesting challenge here is to construct big families (the bigger the best) with respect to the dimension n, of isospectral n-manifolds which are topologically very different (the more different the best) to each other. One purpose of this might be to tightly highlight the fact that if some property cannot be heard, it is not merely an isolated casualty but a concrete reality we cannot ignore.

- *Spectral asymmetry*: following the acoustic jargon before, one studies now when our *drum* (the manifold) is *out of tune*. That is, when the positive and the negative spectra differ. The usual devices designed to detect this phenomenon are the eta series and the eta invariant. One wants to compute them explicitly.

Summary of results. Here we give a list of the principal results obtained for $\mathbb{Z}_2^k$-manifolds (that is, compact flat manifolds having holonomy group isomorphic to $\mathbb{Z}_2^k$) relative to the problems mentioned before. The results will be properly stated and explained in the body of the paper.

A. Full Laplacian $\Delta_{\mathcal{F}}$: (1) all $\mathbb{Z}_2^k$-manifolds covered by the same torus (or by isospectral tori) are isospectral on differential forms of mixed degree; *(2)* There are *big* families of $\Delta_{\mathcal{F}}$-isospectral manifolds.

B. Spin structures: (3) we give necessary and sufficient conditions for their existence; *(4)* there are families of $\mathbb{Z}_2^k$-manifolds which *are* spin while there are others which *are not* spin; *(5)* we answer Webb's question: *"Can one hear the property of being spin on a compact Riemannian manifold?"*.

C. Dirac spectrum: (6) we compute the multiplicities of the eigenvalues of twisted Dirac operators D_ρ for an arbitrary spin $\mathbb{Z}_2^k$-manifold.

D. Dirac isospectrality: (7) we obtain several examples of pairs M, M' of D_ρ-isospectral manifolds having different topological, geometrical or spectral properties; *(8)* there are *big* families of D_ρ-isospectral manifolds.

E. Spectral asymmetry. (9) we give a characterization of those manifolds having asymmetric Dirac spectrum; *(10)* explicit expressions for the eta series and the η-invariant are given; *(11)* we answer Schueth's question: *"Can one hear the η-invariants of a compact Riemannian manifold?"*.

1. $\mathbb{Z}_2^k$-MANIFOLDS

What are we talking about? A *Bieberbach group* is a crystallographic group without torsion. That is, a discrete, cocompact, torsion-free subgroup $\Gamma \subset I(\mathbb{R}^n)$ of the isometries of $\mathbb{R}^n$. Such Γ acts properly discontinuously on $\mathbb{R}^n$, thus $M_\Gamma = \Gamma\backslash\mathbb{R}^n$ is a compact flat Riemannian manifold having fundamental group Γ. Any element $\gamma \in I(\mathbb{R}^n) = \mathrm{O}(n) \ltimes \mathbb{R}^n$ decomposes uniquely as $\gamma = BL_b$, where $B \in \mathrm{O}(n)$ and L_b denotes translation by $b \in \mathbb{R}^n$.

By the classical Bieberbach's theorems we have the following two basic results: *(i)* the translations in Γ form a normal maximal abelian subgroup L_Λ of finite index, with Λ a lattice in $\mathbb{R}^n$ which is B-stable for each $BL_b \in \Gamma$ (as usual, one identifies L_Λ with Λ) and *(ii)* the restriction to Γ of the canonical projection $r : I(\mathbb{R}^n) \to \mathrm{O}(n)$ given by $BL_b \mapsto B$ is a group homomorphism with kernel Λ and $F := r(\Gamma)$ is a finite subgroup of $\mathrm{O}(n)$. It turns out that Γ satisfies the exact sequence of groups

$$0 \to \Lambda \to \Gamma \xrightarrow{r} F \to 1.$$

The group $F \simeq \Lambda\backslash\Gamma$ is called the *holonomy group* of Γ and it is isomorphic to the linear holonomy group of the Riemannian manifold M_Γ. The action of F on Λ by conjugation defines an integral representation $F \to \mathrm{GL}_n(\mathbb{Z})$ which is usually called the *integral holonomy representation* of Γ. Note that this representation *does not* determine the group Γ, i.e. there may be many non-isomorphic Bieberbach groups with the same holonomy representation.

A compact Riemannian manifold with holonomy group isomorphic to F will be called an *F-manifold* (see [**Ch**]). We shall only be concerned with $\mathbb{Z}_2^k$-manifolds which, by the Cartan-Ambrose-Singer theorem, are necessarily flat.

Compact flat manifolds. A *flat manifold* is a closed, connected, Riemannian manifold M, whose curvature identically vanishes. Notably, by the Killing-Hopf theorem any *compact* flat manifold M is isometric to a quotient $M_\Gamma = \Gamma\backslash\mathbb{R}^n$, with Γ a Bieberbach group. Putting the Bieberbach theorem's into Riemannian language (see [**Wo**] or [**Ch**]) we get that: *(i)* M_Γ is covered by the associated flat torus $T_\Lambda = \Lambda\backslash\mathbb{R}^n$ and the covering $\pi : T_\Lambda \to M_\Gamma$ is a local isometry, *(ii)* M_Γ is affinely equivalent to $M_{\Gamma'}$ if and only if $\Gamma \simeq \Gamma'$ and *(iii)* there is a finite number of classes of affine equivalence of compact flat manifolds, in each dimension.

Up to equivalence, in dimension 1 there is only one compact flat manifold, the *circle* $\mathbb{S}^1 = \mathbb{Z}\backslash\mathbb{R}$, while in dimension 2 there are two, the *torus* T^2 and the *Klein bottle* K^2:

$$T^2 = \langle L_{e_1}, L_{e_2}\rangle\backslash\mathbb{R}^2, \qquad K^2 = \langle \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right] L_{\frac{e_2}{2}}, L_{e_1}, L_{e_2}\rangle\backslash\mathbb{R}^2.$$

The number of compact flat manifolds grows rapidly with the dimension and a classification is unfortunately known only up to dimension 6.

There are two nice results concerning compact flat manifolds. One says that there are a plethora of them while the other says that all these manifolds bound: *(1) Every finite group F can be realized as the holonomy group of a compact flat manifold* ([**AK**]) and *(2) If M is a compact flat n-manifold then there is a compact $(n+1)$-manifold $\tilde{M}$ such that $\partial\tilde{M} = M$* ([**HR**]).

Into the jungle. A $\mathbb{Z}_2^k$-*manifold* is just a compact flat n-manifold whose holonomy group is isomorphic to $\mathbb{Z}_2^k$, with $1 \le k \le n-1$. Thus, it is of the form $M_\Gamma = \Gamma\backslash\mathbb{R}^n$ where $\Gamma = \langle\gamma_1, \ldots, \gamma_k, \Lambda\rangle$, with $\Lambda = \mathbb{Z}\lambda_1 \oplus \cdots \oplus \mathbb{Z}\lambda_n$ and $\gamma_i = B_i L_{b_i}$ satisfying $B_i \in \mathrm{O}(n)$, $B_i\Lambda = \Lambda$, $b_i \in \mathbb{R}^n$ and $B_i^2 = Id$, $B_iB_j = B_jB_i$ for $1 \le i, j \le k$.

Some friendly tribes. We now introduce some particularly interesting classes of $\mathbb{Z}_2^k$-manifolds that will be used in the rest of the paper.

∘ $\mathbb{Z}_2$-*manifolds.* They generalize the Klein Bottle in the sense that they are quotients of tori divided by a $\mathbb{Z}_2$-action. They are determined by the integral holonomy representation which can be parametrized by the block matrices

$$B_{j,h} = \mathrm{diag}(\underbrace{J, \ldots, J}_{j\ge 0}, \underbrace{-1, \ldots, -1}_{h\ge 0}, \underbrace{1, \ldots, 1}_{l\ge 1}), \qquad J = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right],$$

with $2j + h + l = n$ and $j + h \ne 0$. The corresponding diffeomorphism classes are represented by $M_{j,h} = \langle B_{j,h}L_{\frac{e_n}{2}}, \Lambda\rangle\backslash\mathbb{R}^n$, Λ the canonical lattice. One can compute their first integral homology groups and their Betti numbers. Indeed, $H_1(M_{j,h}, \mathbb{Z}) \simeq \mathbb{Z}^{j+l} \oplus \mathbb{Z}_2^h$ and $\beta_p(M_{j,h}) = \sum_{i=0}^{[p/2]} \binom{j+h}{2i}\binom{j+l}{p-2i}$, for $0 \le p \le n$. (See [**MP**]).

∘ *Primitive $\mathbb{Z}_2^2$-manifolds.* We recall that *primitive* means that $\beta_1(M_\Gamma) = 0$, that is F has trivial center. By a construction due to Calabi (see [**Ca**], [**Wo**]), any compact flat manifold can be obtained from a primitive one. Primitive $\mathbb{Z}_2^2$-manifolds are also determined by the integral holonomy representation, which decomposes as a sum of integral representations of rank ≤ 3 ([**Ti**]).

∘ *Diagonal type.* A compact flat manifold M_Γ is *of diagonal type* if there is an orthonormal $\mathbb{Z}$-basis $\{e_i, \ldots, e_n\}$ of Λ satisfying $Be_i = \pm e_i$, $1 \leq i \leq n$, for every $BL_b \in \Gamma$. In this case we say that Γ have *diagonal holonomy representation*. One can assume that Λ is the *canonical lattice* and that $b \in \frac{1}{2}\Lambda$. They necessarily have holonomy group $F \simeq \mathbb{Z}_2^k$. (See [**MR3**]).

∘ *Hantzsche-Wendt manifolds.* (Or, HW-manifolds, for short). They are the orientable $\mathbb{Z}_2^{n-1}$-manifolds in odd dimension n. Any such manifold M_Γ is given by $\Gamma = \langle B_1 L_{b_1}, \ldots, B_n L_{b_n}, \Lambda\rangle$ where B_i fixes e_i, $B_i e_j = -e_j$ for $1 \leq i \leq n$, if $j \neq i$, and $\Lambda = \mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_n$ (note that $B_n = B_1 B_2 \cdots B_{n-1}$). They generalize the only orientable $\mathbb{Z}_2^2$-manifold existing in dimension 3, historically called the Hantzsche-Wendt manifold, and were studied in [**MR**]. They are primitive, of diagonal type and, furthermore, they are rational homology spheres, i.e. $H_*(M, \mathbb{Q}) = H_*(\mathbb{S}^n, \mathbb{Q})$ for every HW-manifold M. Also, one can associate certain directed graphs to them.

∘ *Generalized Hantzsche-Wendt manifolds.* (Or GHW-manifolds). They are simply the $\mathbb{Z}_2^{n-1}$-manifolds in dimension n. They share many properties with HW-manifolds but they are not primitive in general. There are $[\frac{n+1}{2}]$ different integral holonomy representations, all of diagonal type. (See [**RS**]).

A little bit of numerology. As we have seen, we have the following natural inclusions *HW* $\subset$ *GHW* $\subset$ *Diagonal type* $\subset$ $\mathbb{Z}_2^k$-*manifolds*. In the table below we compare the cardinality of these families. We see that, at least in low dimensions, the class of $\mathbb{Z}_2^k$-manifolds represents more than half of the compact flat manifolds.

# manifolds	dim 1	dim 2	dim 3	dim 4	dim 5	dim 6
compact flat	1	2	10	74	1.060	38.746
$\mathbb{Z}_2^k$	-	1	6	43	650	27.515
GHW	-	1	3	12	123	2.536
HW	-	-	1	-	2	-

2. The full Laplacian

Consider the Laplacian on p-forms Δ_p. It is a first order elliptic differential operator acting on smooth sections of the p-exterior bundle $\Lambda^p(TM)$ of M. The spectrum of this operator on compact flat manifolds was studied in [**MR2**] (see also [**MR3**], [**MR4**]). The multiplicity of the eigenvalue $4\pi^2\mu$ of Δ_p has the expression

$$d_{p,\mu}(\Gamma) = \tfrac{1}{|F|} \sum_{\gamma = BL_b \in \Lambda\backslash\Gamma} \chi_p(B)\, e_{\mu,\gamma}$$

where $e_{\mu,\gamma} = \sum_{v \in \Lambda_\mu^* : Bv = v} e^{-2\pi i v \cdot b}$, with $\Lambda_\mu^* = \{v \in \Lambda^* : \|v\| = \mu\}$, and χ_p is the character of the p-exterior representation. For Γ of diagonal type, this character is given by integer values of certain polynomials. In fact, for $BL_b \in \Gamma$, we have

$$\chi_p(B) = K_p^n(n - n_B) \qquad \text{with} \qquad K_p^n(x) := \sum_{t=0}^{p} (-1)^t \tbinom{x}{t}\tbinom{n-x}{p-t}$$

where $n_B = \dim(\mathbb{R}^n)^B$ and $K_p^n(x)$ is the (binary) *Krawtchouk polynomial* of order n and degree p. They are discrete orthogonal polynomials (see [**KL**]). The first ones have the expressions $K_0^n(x) = 1$, $K_1^n(x) = -2x+n$, $K_2^n(x) = 2x^2 - 2nx + \binom{n}{2}$, $K_3^n(x) = -\frac{4}{3}x^3 + 2nx^2 - (n^2 - n + \frac{2}{3})x + \binom{n}{3}$, etc.

From now on in this section we refer to [**MPR**]. One can simply define a Laplacian on arbitrary forms by considering the p-Laplacians altogether, that is, we can take

$$\Delta_{\mathcal{F}} := \oplus \sum_{p=0}^{n} \Delta_p.$$

This *full Laplacian* is again a first order elliptic differential operator which acts on sections of the *full* exterior bundle $\Lambda(TM) = \bigoplus_{p=0}^n \Lambda^p(TM)$ of M. The eigenvalues are still of the form $4\pi^2\mu$, but their multiplicities are now given by the sum $d_{\mathcal{F},\mu}(\Gamma) = \sum_{p=0}^n d_{p,\mu}(\Gamma)$. Clearly, p-isospectrality (i.e. isospectrality with respect to Δ_p) for all p implies $\Delta_{\mathcal{F}}$-isospectrality, but the converse is far from being true, as will be shown in Example 2.2 below.

We have the following curious *"optimistic"* result from [**MPR**]:

Theorem 2.1. *Let Γ be a Bieberbach group of dimension n with translation lattice Λ and holonomy group $\mathbb{Z}_2^k$. Then, the eigenvalue $4\pi^2\mu$ of $\Delta_{\mathcal{F}}$ has multiplicity $d_{\mathcal{F},\mu}(\Gamma) = 2^{n-k}|\Lambda_\mu^*|$ where $\Lambda_\mu^* = \{v \in \Lambda^* : ||v|| = \mu\}$.*

Thus, two $\mathbb{Z}_2^k$-manifolds $M_\Gamma, M_{\Gamma'}$ are $\Delta_{\mathcal{F}}$-isospectral if and only if the translation lattices Λ, Λ' are isospectral. In particular, for fixed Λ and k, all *$\mathbb{Z}_2^k$-manifolds with covering torus T_Λ are $\Delta_{\mathcal{F}}$-isospectral.*

Sketch of proof. Let $F = \langle B_1, \dots, B_k\rangle \simeq \mathbb{Z}_2^k$. Then, the B_i's diagonalize simultaneously with eigenvalues ± 1. Thus, every B_i is conjugate in $\mathrm{GL}_n(\mathbb{R})$ to the diagonal matrix $D_B := \mathrm{diag}(-I_{n-n_B}, I_{n_B})$ where I_m is the identity matrix in $\mathbb{R}^m$. Thus $\chi_p(B) = \chi_p(D_B) = K_p^n(n - n_B)$. Hence, we have $d_{p,\mu}(\Gamma) = 2^{-k}\left(\binom{n}{p}|\Lambda_\mu^*| + \sum_{\gamma\in\Lambda\backslash\Gamma\smallsetminus\{Id\}} K_p^n(n-n_B)\, e_{\mu,\gamma}\right)$ and, adding over p

$$d_{\mathcal{F},\mu}(\Gamma) = 2^{n-k}|\Lambda_\mu^*| + 2^{-k} \sum_{\gamma\in\Lambda\backslash\Gamma\smallsetminus\{Id\}} \Big(\sum_{p=0}^{n} K_p^n(n - n_B)\Big)\, e_{\mu,\gamma}.$$

Now, for $j \neq 0$, one can show that $\sum_{p=0}^n K_p^n(j) = 0$ ($\bigstar$). Since $n - n_B = 0$ if and only if $B = Id$, we finally get that $d_{\mathcal{F},\mu}(\Gamma) = 2^{n-k}|\Lambda_\mu^*|$, as asserted. □

Note 1. The proof seems to be of an entirely combinatorial nature since it only depends on ($\bigstar$), and the Krawtchouk polynomials at integer values $K_p^n(j)$ have some combinatorial interpretations in the literature.

Example 2.2 (*$\mathbb{Z}_2$-manifolds of dim 3*)**.** We illustrate the theorem in the simplest non-trivial case, i.e. when $n = 3$ and $k = 1$. Up to diffeomorphism, there are only three $\mathbb{Z}_2$-manifolds in dimension 3 (see [**Wo**]). They are $M_{1,0}, M_{0,2}$ and $M_{0,1}$, in the notation of page 138.

In the tables below we give the multiplicities for Δ_p, with $0 \le p \le 3$, and also for $\Delta_{\mathcal{F}}$, of the 2 lowest non trivial eigenvalues.

$\mu = 1$	d_0	d_1	d_2	d_3	$d_{\mathcal{F}}$
$M_{1,0}$	2	8	10	4	24
$M_{0,2}$	2	10	10	2	24
$M_{0,1}$	3	9	9	3	24

$\mu = \sqrt{2}$	d_0	d_1	d_2	d_3	$d_{\mathcal{F}}$
$M_{1,0}$	7	19	17	5	48
$M_{0,2}$	6	18	18	6	48
$M_{0,1}$	4	16	20	8	48

The values in the tables show that the manifolds are not p-isospectral to each other for any $0 \le p \le 3$. However, we can see how all these multiplicities balance, that is how they manage to distribute themselves in order to have equal sums for each eigenvalue, in each case.

We really need the hypothesis $F \simeq \mathbb{Z}_2^k$ in the theorem. The “magical” averaging phenomenon, present when considering all the p-Laplacians Δ_p simultaneously, only seems to work in the case considered. The result does *not* hold, in general, for holonomy groups different from $\mathbb{Z}_2^k$. There is a pair of 6-dimensional orientable $\mathbb{Z}_4$-manifolds, $M_\Gamma, M_{\Gamma'}$, which are not $\Delta_{\mathcal{F}}$-isospectral ([**MPR**, Ex. 3.5]), even though they are isospectral on functions. In fact, let $\Lambda = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_6$ and take $\Gamma = \langle B_1 L_{b_1}, \Lambda\rangle$ and $\Gamma' = \langle B_1' L_{b_1'}, \Lambda\rangle$ where $B_1 = \operatorname{diag}(\tilde{J}, \tilde{J}, 1, 1)$, $b_1 = \frac{e_5}{4}$, $B_1' = \operatorname{diag}(\tilde{J}, 1, -1, -1, 1)$, $b_1' = \frac{e_6}{4}$ with $\tilde{J} = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. Following [**MR2**] one can prove that $M_\Gamma, M_{\Gamma'}$ are 0-isospectral (and hence 6-isospectral, by orientability) but they are not p-isospectral for $1 \le p \le 5$. Since $d_{p,0}(\Gamma) = \beta_p(M)$ we have that $d_{\mathcal{F},0}(M) = \sum_{p=0}^{6} \beta_p(M)$. By [**Hi**], the Betti numbers β_p, $0 \le p \le 6$, for M and M' are respectively given by 1,2,5,8,5,2,1 and 1,2,3,4,3,2,1. In this way we get that $d_{\mathcal{F},0}(M) = 24$ while $d_{\mathcal{F},0}(M') = 16$, hence M, M' are not isospectral on forms.

Note 2. One can also consider the *Laplacian on even/odd forms* given by $\Delta_e = \sum_{p \text{ even}} \Delta_p$ and $\Delta_o = \sum_{p \text{ odd}} \Delta_p$. The results in Theorem 2.1 hold *mutatis mutandis* but now with $d_{o,\mu}(\Gamma) = d_{e,\mu}(\Gamma) = \frac{1}{2} d_{\mathcal{F},\mu}(\Gamma)$. In particular, $\chi(M_\Gamma) = \sum_{p=0}^{n} (-1)^p \beta_p(M_\Gamma) = d_{o,0}(\Gamma) - d_{e,0}(\Gamma) = 0$.

Big families of $\Delta_{\mathcal{F}}$-isospectral manifolds. As a straight consequence of the result in the previous theorem, we can exhibit several *big* families of $\Delta_{\mathcal{F}}$-isospectral $\mathbb{Z}_2^k$-manifolds in arbitrary dimension n. Here, big alludes to the fact that the cardinality of the family grows polynomially —or even *exponentially*— with respect to n and, also, that all the manifolds in each family are not homeomorphic to each other. It is worth noting that for the Laplacian on functions, or on p-forms, there are not known examples of such *exponential families*. In the famous *isospectral deformations* of Gordon and Wilson (see [**GW**]), the manifolds have different metrics but they are all homeomorphic to each other.

Consider the following families: $\mathcal{F}_1 = \{\mathbb{Z}_2\text{-}manifolds\}$, $\mathcal{F}_2 = \{primitive\ \mathbb{Z}_2^2\text{-}manifolds\}$ and $\mathcal{F}_3 = \{HW\text{-}manifolds\}$. For simplicity, we assume that all the groups have the canonical lattice of translations $\Lambda = \mathbb{Z}^n$. Thus, by Theorem 2.1, all manifolds belonging to each family are mutually isospectral on forms. We now indicate the order of the cardinality of each family. We have that $\#\mathcal{F}_1 = o(n^2)$

and $\#\mathcal{F}_2 = o(n^5)$ (see [**MP**], [**Ti**]). By using a small subfamily of HW-manifolds it is proved in [**MR**] that $\#\mathcal{F}_3 > \frac{2^{n-3}}{n-1}$. Furthermore, based on an example given in [**LS**], Rossetti constructed a very big family $\mathcal{F}_4$ of GHW-manifolds with $\#\mathcal{F}_4 = o((\sqrt{2})^{n^2})$ (see [**MPR**]).

3. Spin structures

Spin structures play a role in geometry and physics. One relevant fact is that they allow to define Dirac operators. On an arbitrary Riemannian manifold M, the Laplacian on functions and the Laplacian on p-forms are always defined. On the other hand, for the Dirac operator to be defined, M needs to have an extra geometric structure. More precisely, to each spin structure on M one can construct the so called spinor bundle and a Dirac operator acting on sections of it. However, one needs to have some care here, since not every Riemannian manifold admits a spin structure (see [**LS**]).

Let M be an oriented Riemannian n-manifold and let $\mathsf{B}(M) \xrightarrow{\pi} M$ be the $\mathrm{SO}(n)$-principal bundle of oriented frames. Inside the group of units of the Clifford algebra $Cl(n)$ of $\mathbb{R}^n$ lies the compact connected Lie group

$$\mathrm{Spin}(n) = \{v_1 \cdots v_{2k} \,:\, v_j \in \mathbb{S}^1(\mathbb{R}^n), j = 1, \ldots, 2k\}$$

where $\mathbb{S}^1(\mathbb{R}^n) = \{x \in \mathbb{R}^n : ||x|| = 1\}$. This group satisfies the exact sequence

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \xrightarrow{\mu} \mathrm{SO}(n) \to 1$$

where $\mu(g)(x) = gxg^{-1}$. Thus, μ is a double covering and since $\mathrm{Spin}(n)$ is simply connencted, for $n \geq 3$, it is the universal covering of $\mathrm{SO}(n)$.

A *spin structure* on an orientable manifold M is an equivariant 2-fold covering $\tilde{\mathsf{B}}(M) \xrightarrow{p} \mathsf{B}(M)$ such that $\pi \circ p = \tilde{\pi}$, where $\tilde{\mathsf{B}}(M) \xrightarrow{\tilde{\pi}} M$ is a $\mathrm{Spin}(n)$-principal bundle. A manifold endowed with a spin structure is called a *spin manifold*.

Fortunately, for compact flat manifolds we can get rid of this complicated geometrical–topological definition by using the following result. The spin structures of M_Γ are in a 1–1 correspondence (see [**LM**] or [**Fr2**]) with the group homomorphisms ε commuting the diagram

$$\begin{array}{ccc} & & \mathrm{Spin}(n) \\ & \overset{\varepsilon}{\nearrow} & \downarrow \mu \\ \Gamma & \xrightarrow[r]{} & \mathrm{SO}(n) \end{array} \tag{3.1}$$

This gives a purely algebraic alternative definition, simpler than the original one, which is in fact a criterion to decide the existence of spin structures. It can be used not only to construct such structures, but also to count them.

Spin structures on $\mathbb{Z}_2^k$-manifolds. In this subsection we refer to [**MP**]. Let $M_\Gamma = \Gamma\backslash\mathbb{R}^n$ be a $\mathbb{Z}_2^k$-manifold and ε a spin structure as in (3.1). Since $r(\Lambda) = Id$, then $\varepsilon(L_\lambda) \in \{\pm 1\}$ for any $\lambda \in \Lambda$. Let $\lambda_1, \ldots, \lambda_n$ be a $\mathbb{Z}$-basis of Λ and put $\delta_i = \varepsilon(L_{\lambda_i})$. For $\lambda = \sum_i m_i \lambda_i \in \Lambda$, with $m_i \in \mathbb{Z}$, we have $\varepsilon(L_\lambda) = \delta_1^{m_1} \delta_2^{m_2} \cdots \delta_n^{m_n}$.

For any $\gamma = BL_b \in \Gamma$ we fix a distinguish (though arbitrary) element $u_B \in \mu^{-1}(B)$. Then,

$$\varepsilon(\gamma) = \sigma_B \, u_B \tag{3.2}$$

where $\sigma_B \in \{\pm 1\}$ depends on γ and on the choice of u_B.

The map ε is determined by its action on the generators of Γ, and so we can identify it with the $(n+k)$-tuple $\varepsilon \equiv (\delta_1, \ldots, \delta_n, \sigma_1, \ldots, \sigma_k) \in \{\pm 1\}^{n+k}$. Since ε is a group homomorphism it must satisfy, for every $\gamma = BL_b$, the following conditions

$$(\bigstar_1) \quad \varepsilon(\gamma^2) = {u_B}^2, \qquad\qquad (\bigstar_2) \quad \varepsilon(L_{B\lambda}) = \varepsilon(L_\lambda) \qquad (\lambda \in \Lambda).$$

Note that, since $B^2 = Id$, these are conditions for ε over Λ, i.e. for the character $\delta_\varepsilon = \varepsilon_{|\Lambda} \in \mathrm{Hom}(\Lambda, \{\pm 1\})$. We define the set

$$\hat{\Lambda}(\Gamma) := \big\{ \chi \in \mathrm{Hom}(\Lambda, \{\pm 1\}) \, : \, \chi \text{ satisfies } (\bigstar_1) \text{ and } (\bigstar_2) \big\}.$$

The following theorem says that the above necessary conditions $(\bigstar)$'s for the existence of spin structures on $\mathbb{Z}_2^k$-manifolds, are also sufficient .

Theorem 3.1. *Let $\Gamma = \langle \gamma_1, \ldots, \gamma_k, \Lambda \rangle$ be an n-dimensional Bieberbach group with holonomy group $F = \langle B_1, \ldots, B_k \rangle \simeq \mathbb{Z}_2^k$, $F \subset SO(n)$, and let $\sigma_1, \ldots, \sigma_k$ be as in (3.2). Then, the map*

$$\varepsilon \mapsto (\varepsilon_{|\Lambda}, \sigma_1, \ldots, \sigma_k)$$

defines a bijective correspondence between the spin structures of M_Γ and the set $\hat{\Lambda}(\Gamma) \times \{\pm 1\}^k$. Hence, the number of spin structures of M_Γ is either 0 *or* 2^r *for some $r \geq k$.*

Applications. By applying Theorem 3.1 we can: *(1)* study the existence of spin structures in particular families of $\mathbb{Z}_2^k$-manifolds, *(2)* give a simple method to obtain spin manifolds and *(3)* determine the *audibility* of the spin structures, that is, whether spin structures can be heard or not.

Spin structures in families. *(i)* Every orientable $\mathbb{Z}_2$-manifold *is* spin and orientable $\mathbb{Z}_2$-manifolds of diagonal type have 2^n spin structures (see [**MP**]), the same as for any n-torus (see [**Fr**]). *(ii)* Orientable primitive $\mathbb{Z}_2^2$-manifolds *are* spin. *(iii)* The 3-dimensional HW-manifold *is* spin. HW-manifolds of dimension n, $n \geq 5$, *are not* spin. See [**Po**] for the case $n = 4k+1$. The general case, i.e. n odd, was proved independently by J. P. Rossetti ([**Ro**], by using a criterion in [**MP**]) and by S. Console ([**Co**], by computing the second Stiefel-Whitney classes ω_2).

How to get spin manifolds easily? By using the *doubling procedure* in [**JR**] or [**BDM**]. Let Γ be an n-dimensional Bieberbach group with translation lattice Λ and holonomy group F. The *double of* Γ is the Bieberbach group defined by $\mathrm{d}\Gamma = \langle \mathrm{d}B \, L_{(b,b)}, L_{(\lambda_1,\lambda_2)} : BL_b \in \Gamma, \lambda_1, \lambda_2 \in \Lambda \rangle$, with $\mathrm{d}B = \left[\begin{smallmatrix} B & 0 \\ 0 & B \end{smallmatrix}\right]$. It follows that $\mathrm{d}\Gamma$ has translation lattice $\Lambda \oplus \Lambda$ and holonomy group F. The associated manifold $M_{\mathrm{d}\Gamma}$ has dimension $2n$ and is Kähler (see [**DM**]). Now, doubling an orientable manifold of diagonal type gives a spin manifold (see [**MP2**]). If the manifold is not orientable, then one has to double twice.

Spin structures are not audible. Take $\Lambda = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_6$ and consider the manifolds $M = \Gamma\backslash\mathbb{R}^6$ and $M' = \Gamma'\backslash\mathbb{R}^6$ where $\Gamma = \langle B_1L_{b_1}, B_2L_{b_2}, \Lambda\rangle$ and $\Gamma' = \langle B_1L_{b_1'}, B_2L_{b_2'}, \Lambda\rangle$ are Bieberbach groups of diagonal type given, in diagonal notation (see [**MR2**], [**MP**]), in the following table. For example, the 1/2 in the first column means that $L_{b_1} = \frac{e_3}{2}$. Also, $B_3 = B_1B_2$ and $b_3 = B_2b_1 + b_2$, $b_3' = B_2b_1' + b_2'$.

B_1	L_{b_1}	$L_{b_1'}$	B_2	L_{b_2}	$L_{b_2'}$	B_3	L_{b_3}	$L_{b_3'}$
1			1	1/2	1/2	1	1/2	1/2
1		1/2	1	1/2		1	1/2	1/2
1	1/2		-1			-1	1/2	
-1			1		1/2	-1		1/2
-1			1			-1		
1			-1			-1		

It is easy to see that M, M' are orientable $\mathbb{Z}_2^2$-manifolds of dimension 6 and that they are p-isospectral for every p, $0 \le p \le 6$. Note that they are not primitive since $\beta_1(M) = \beta_1(M') = 2$. Using Theorem 3.1 we can check that M has no spin structures while M' has 2^5 spin structures of the form $\varepsilon = (-1, -1, \delta_3, 1, \delta_5, \delta_6; \sigma_1, \sigma_2)$, with $\delta_i, \sigma_j \in \{\pm 1\}$. Thus, we cannot hear the spin structures of Riemannian manifolds!

4. Dirac spectrum

We begin with the ingredients necessary to define twisted Dirac operators on Riemannian manifolds. Let M_Γ be an orientable compact flat manifold endowed with a spin structure ε as in (3.1), denoted by (M_Γ, ε) from now on. Let L_n : $\mathrm{Spin}(n) \to \mathrm{GL}(\mathrm{S}_n)$ be *the spin representation*, that is the restriction to $\mathrm{Spin}(n)$ of any irreducible complex representation of the complexified Clifford algebra $Cl(n) \otimes \mathbb{C}$. It is wellknown that $\dim_{\mathbb{C}}(\mathrm{S}_n) = 2^{[n/2]}$ and that L_n is irreducible if n is odd while, if n is even, L_n splits into two inequivalent irreducible representations $(L_n^\pm, \mathrm{S}_n^\pm)$ of the same dimension, called the *half-spin representations*. Let ρ : $\Gamma \to U(V)$ be a unitary representation such that $\rho_{|\Lambda} = 1$. As usual, we take $\chi_\rho(\gamma) = \mathrm{Tr}\,\rho(\gamma)$ and $d_\rho = \dim(V)$.

Now, the morphism ε allows to construct the *spinor bundle twisted by* ρ

$$S_\rho(M_\Gamma, \varepsilon) := \Gamma\backslash(\mathbb{R}^n \times (\mathrm{S}_n \otimes V)) \to \Gamma\backslash\mathbb{R}^n$$

with action given by $\gamma \cdot (x, w \otimes v) = (\gamma x, L_n(\varepsilon(\gamma))(w) \otimes \rho(\gamma)(v))$. One can identify the space of smooth sections of this bundle, $\Gamma^\infty(S_\rho(M_\Gamma, \varepsilon))$, with the set $\{f : \mathbb{R}^n \to \mathrm{S}_n \otimes V \text{ smooth } : f(\gamma x) = (L_n \circ \varepsilon \otimes \rho)(\gamma) f(x)\}$.

With the above identification, the *Dirac operator twisted by* ρ on compact flat manifolds is $D_\rho : \Gamma^\infty(S_\rho(M_\Gamma, \varepsilon)) \to \Gamma^\infty(S_\rho(M_\Gamma, \varepsilon))$ given by

$$D_\rho\, f(x) = \sum_{i=1}^{n} e_i \cdot \tfrac{\partial f}{\partial x_i}(x)$$

where e_i acts by $L_n(e_i) \otimes Id$ in $\mathrm{S}_n \otimes V$ and $e_1, \ldots, e_n$ is an orthonormal basis of $\mathbb{R}^n$. If $(\rho, V) = (\mathbf{1}, \mathbb{C})$ we have *the classical* Dirac operator D. D_ρ is a first order elliptic differential operator, symmetric and essentially self-adjoint. It does not depend on the choice of the orthonormal basis of $\mathbb{R}^n$. Also, it is a *formal* square root of the Laplacian, that is $D_\rho^2 = -\Delta_{s,\rho}$, called the *twisted spinor Laplacian.* If $f \in \ker D_\rho$, f is called a *harmonic spinor.*

D_ρ has a discrete spectrum consisting of real eigenvalues $\pm 2\pi\mu$, $\mu \geq 0$, of finite multiplicity $d^{\pm}_{\rho,\mu}$. Explicit expressions for $d^{\pm}_{\rho,\mu}$ for an arbitrary pair (M_Γ, ε) were obtained in [**MP2**]. We now recall this result.

Let $F_1 = \{B \in F = r(\Gamma) : n_B = 1\}$ where $n_B = \dim \ker(B - Id)$. Put $\Lambda^*_\varepsilon = \{u \in \Lambda^* : \varepsilon(\lambda) = e^{2\pi i \lambda\cdot u}, \lambda \in \Lambda\}$, with Λ^* the dual lattice of Λ, and

$$\Lambda^*_{\varepsilon,\mu} = \{u \in \Lambda^*_\varepsilon : \|u\| = \mu\}.$$

Now, for each $\gamma = BL_b \in \Gamma$, let $(\Lambda^*_{\varepsilon,\mu})^B$ denotes the set of elements fixed by B in $\Lambda^*_{\varepsilon,\mu}$, that is $(\Lambda^*_{\varepsilon,\mu})^B = \{u \in \Lambda^*_{\varepsilon,\mu} : Bv = v\}$. Furthermore, for $\gamma \in \Gamma$, let x_γ be a fixed (though arbitrary) element in the maximal torus of $\mathrm{Spin}(n-1)$, conjugate in $\mathrm{Spin}(n)$ to $\varepsilon(\gamma)$. Finally, define a sign $\sigma(u, x_\gamma)$, depending on u and on the conjugacy class of x_γ in $\mathrm{Spin}(n-1)$, in the following way. If $\gamma = BL_b \in \Lambda\backslash\Gamma$ and $u \in (\Lambda^*_\varepsilon)^B \smallsetminus \{0\}$, let $h_u \in \mathrm{Spin}(n)$ such that $h_u\, u\, h_u^{-1} = \|u\| e_n$. Hence, $h_u \varepsilon(\gamma) h_u^{-1} \in \mathrm{Spin}(n-1)$. Take $\sigma_\varepsilon(u, x_\gamma) = 1$ if $h_u\varepsilon(\gamma)h_u^{-1}$ is conjugate to x_γ in $\mathrm{Spin}(n-1)$ and $\sigma_\varepsilon(u, x_\gamma) = -1$ otherwise. As a consequence, $\sigma(-u, x_\gamma) = -\sigma(u, x_\gamma)$ and $\sigma(\alpha u, x_\gamma) = \sigma(u, x_\gamma)$ for every $\alpha > 0$ (see Definition 2.3, Remark 2.4 and Lemma 6.2 in [**MP2**] for details).

For n odd, the multiplicity of the eigenvalue $\pm 2\pi\mu$, for $\mu > 0$, is given by

$$d^{\pm}_{\rho,\mu}(\Gamma,\varepsilon) = \tfrac{1}{|F|}\Bigg(\sum_{\substack{\gamma \in \Lambda\backslash\Gamma \\ B \notin F_1}} \chi_\rho(\gamma) \sum_{u \in (\Lambda^*_{\varepsilon,\mu})^B} e^{-2\pi i u\cdot b} \cdot \chi_{L^{\pm}_{n-1}}(x_\gamma) \; + \sum_{\substack{\gamma \in \Lambda\backslash\Gamma \\ B \in F_1}} \chi_\rho(\gamma) \sum_{u \in (\Lambda^*_{\varepsilon,\mu})^B} e^{-2\pi i u\cdot b} \cdot \chi_{L_{n-1}^{\pm\sigma(u,x_\gamma)}}(x_\gamma)\Bigg), \tag{4.1}$$

while for n even, it is given by the first term in (4.1), where the sum is taken over all $\gamma \in \Lambda\backslash\Gamma$, with $L^{\pm}_{n-1}$ replaced by L_{n-1}. For $\mu = 0$, with n even or odd, we have that $d_0(\Gamma,\varepsilon) = \frac{1}{|F|}\sum_{\gamma\in\Lambda\backslash\Gamma} \chi_\rho(\gamma)\,\chi_{L_n}(\varepsilon(\gamma)) = \dim \mathrm{S}^F$, if $\varepsilon_{|\Lambda} = 1$, and $d_0(\Gamma,\varepsilon) = 0$, otherwise.

Dirac spectrum of $\mathbb{Z}_2^k$-manifolds. In the particular case when $F \simeq \mathbb{Z}_2^k$, the formula (4.1) becomes more tractable and allows one to give shorter expressions for the multiplicities. Also, one can characterize all the spin $\mathbb{Z}_2^k$-manifolds (M_Γ, ε) having asymmetric twisted Dirac spectrum. To wit

Theorem 4.1. *Let (M_Γ,ε) be an n-dimensional spin $\mathbb{Z}_2^k$-manifold.*

(i) If $F_1 = \emptyset$, then the spectrum $Spec_{D_\rho}(M_\Gamma,\varepsilon)$ is symmetric and the non-zero eigenvalue $\pm 2\pi\mu$ of D_ρ has multiplicity

$$d^{\pm}_{\rho,\mu}(\Gamma,\varepsilon) = 2^{m-k-1}\, d_\rho\, |\Lambda^*_{\varepsilon,\mu}|. \tag{4.2}$$

(ii) If $F_1 \neq \emptyset$, then $Spec_{D_\rho}(M_\Gamma,\varepsilon)$ is asymmetric if and only if: $n = 4r+3$ and there exists $\gamma = BL_b \in \Gamma$ with $n_B = 1$ and $\chi_\rho(\gamma) \neq 0$ such that $B_{|\Lambda} = -\delta_\varepsilon Id$. In

this case, the asymmetric spectrum is the set

$$\mathcal{A} = \{\pm 2\pi\mu_j : \mu_j = (j + \tfrac{1}{2})\|f\|^{-1},\, j \in \mathbb{N}_0\}$$

where $\Lambda^B = \mathbb{Z}f$ *and, if we put* $\sigma_\gamma := \sigma(\langle f, 2b\rangle f, g_m)$*, we have:*

$$d^{\pm}_{\rho,\mu}(\Gamma,\varepsilon) = \begin{cases} 2^{m-k-1}\left(d_\rho\,|\Lambda^*_{\varepsilon,\mu}| \pm 2\sigma_\gamma(-1)^{r+j}\chi_\rho(\gamma)\right) & \mu = \mu_j, \\ 2^{m-k-1}\,d_\rho\,|\Lambda^*_{\varepsilon,\mu}| & \mu \neq \mu_j. \end{cases}$$

Also, by (iii), M_Γ *has no non-trivial harmonic spinors.*

If $Spec_{D_\rho}(M_\Gamma,\varepsilon)$ *is symmetric then* $d^{\pm}_{\rho,\mu}(\Gamma,\varepsilon)$ *is given by* (4.2).

(iii) The number of independent harmonic spinors is given by

$$d_{\rho,0}(\Gamma,\varepsilon) = 2^{m-k}d_\rho$$

if $\varepsilon_{|\Lambda} = 1$*, and by* $d_{\rho,0}(\Gamma,\varepsilon) = 0$*, otherwise. If* $k > m$ *then* M_Γ *has no spin structures of trivial type, hence,* M_Γ *has no harmonic spinors. Furthermore, if* M_Γ *has exactly* $2^m d_\rho$ *harmonic spinors then* $M_\Gamma = T_\Lambda$ *and* $\varepsilon = 1$.

5. Dirac isospectrality

We now deal with the isospectral problem for twisted Dirac operators D_ρ. We claim the existence of twisted Dirac isospectral manifolds having different spectral, geometrical or topological properties. We will compare D_ρ-isospectrality with other notions of isospectrality such as isospectrality with respect to the spinor Laplacian Δ_s and the p-Laplacian Δ_p. We will also look at the *spectrum of closed geodesic* with and without multiplicities, that is the so called $[L]$-spectrum and L-spectrum, respectively.

By using $\mathbb{Z}_2^k$-manifolds one can obtain the following results from [**MP2**].

Theorem 5.1. *There are families* $\mathcal{F}$ *of Riemannian n-manifolds, pairwise non homeomorphic, which are mutually* D_ρ*-isospectral for each* ρ*, but they are neither isospectral on functions nor L-isospectral. Furthermore,* $\mathcal{F}$ *can be chosen satisfying any of the following extra properties:*

(i) Every $M \in \mathcal{F}$ *has (or has no) harmonic spinors.*

(ii) All M's in $\mathcal{F}$ *have the same p-Betti numbers for* $1 \leq p \leq n$ *and they are p-isospectral to each other for any p odd.*

Theorem 5.2. *There are pairs of non-homeomorphic spin manifolds which, for each* ρ*, are:*

(i) $\Delta_{s,\rho}$*-isospectral but not* D_ρ*-isospectral; or*

(ii) p-isospectral for $0 \leq p \leq n$ *and* $[L]$*-isospectral such that they are* D_ρ*-isospectral, or not, depending on the choice of the spin structure; or*

(iii) D_ρ*-isospectral and p-isospectral for* $0 \leq p \leq n$*, which are L-isospectral but not* $[L]$*-isospectral.*

We can summarize the previous results in the table below

D-isospectrality vs. other types of isospectrality

D_ρ	$\Delta_{s,\rho}$	Δ_p	$[L]$	L	dim.	F	[**MP2**]
Yes	Yes	No (generically)	No	No	$n \geq 3$	$\mathbb{Z}_2$	Ex. 4.3 (i)
Yes	Yes	Yes (if p odd)	No	No	$n = 4t$	$\mathbb{Z}_2$	Ex. 4.3 (iii)
No	Yes	No	No	No	$n \geq 7$	$\mathbb{Z}_2$	Ex. 4.4 (i)
Yes/No	Yes/No	Yes ($0 \leq p \leq n$)	Yes	Yes	$n \geq 4$	$\mathbb{Z}_2^2$	Ex. 4.5 (i)
Yes/No	Yes/No	Yes ($0 \leq p \leq n$)	No	Yes	$n \geq 4$	$\mathbb{Z}_2^2$	Ex. 4.5 (ii)

Theorem 5.3. *There are big families of D_ρ-isospectral manifolds. More precisely, there exists a family $\mathcal{F}$ of pairwise non-homeomorphic Riemannian n-manifolds that are all mutually D_ρ-isospectral, for many different choices of spin structures, with the cardinality of $\mathcal{F}$ depending exponentially on n or, even better, on n^2.*

6. ETA SERIES AND ETA INVARIANTS

Let A be a self-adjoint elliptic differential operator of order d on a compact n-manifold M. To study the *spectral asymmetry* of A, Atiyah, Patodi and Singer introduced in [**APS**] the so called *eta series* defined by

$$\eta_A(s) = \sum_{0 \neq \lambda \in Spec_A} \operatorname{sign}(\lambda)\, |\lambda|^{-s}.$$

This series converges for $Re(s) > \frac{n}{d}$ and defines a holomorphic function $\eta_A(s)$ which has a meromorphic continuation to $\mathbb{C}$ with simple poles (possibly) at $s = n - k$, $k \in \mathbb{N}_0$. It is a non trivial fact that this function is really finite at $s = 0$ (See [**APS2**] for n odd, [**Gi**] for n even, and [**Wod**] using different methods). The number $\eta_A(0)$ is a spectral invariant, called the *eta invariant*, which does not depend on the metric, although $\eta_A(s)$ does. It gives a measure of the spectral asymmetry of A and it is important because it appears in the "correction term" of some Index Theorems for manifolds with boundary. For example, if $A = D$, the classical Dirac operator, and M is a compact spin manifold with $N = \partial M$, under certain global boundary conditions, the index of D is given by

$$Ind(D) = \int_M \hat{\mathcal{A}}(p) - \frac{d_0 + \eta_N}{2}$$

where $\hat{\mathcal{A}}(p)$ is the Hirzebruch $\hat{\mathcal{A}}$-polynomial in the Pontrjagin forms p_i, η_N is the eta-invariant associated to $D_{|N}$, and $d_0 = \dim \ker D_{|N}$ (see [**APS2**]). Note the beauty of the above expression relating topological, geometrical and spectral data!

For $A = D_\rho$, the twisted Dirac operator, we have the following result:

Theorem 6.1 ([**MP2**]). *Let (M_Γ, ε) be a spin $\mathbb{Z}_2^k$-manifold of dimension $n = 2m + 1 = 4r + 3$. If $Spec_{D_\rho}(M_\Gamma, \varepsilon)$ is asymmetric then:*

$$\eta_{(\Gamma,\rho,\varepsilon)}(s) = (-2)^r\ \sigma_\gamma\ \chi_\rho(\gamma)\ 2^{m-k}\ \tfrac{\|f\|^s}{(4\pi)^s}\ \Big(\zeta(s,\tfrac{1}{4}) - \zeta(s,\tfrac{3}{4})\Big)$$

where $\zeta(s,\alpha) = \sum_{j=0}^{\infty} \frac{1}{(j+\alpha)^s}$, *with* $Re(s) > 1$ *and* $\alpha \in (0,1]$, *denotes the Hurwitz zeta function and* f *and* σ_γ *are as defined in Theorem 4.1. In particular,* $\eta_{(\Gamma,\rho,\varepsilon)}(s)$ *has an analytic continuation to* $\mathbb{C}$ *that is everywhere holomorphic.*

Furthermore, the eta invariant is given by $\eta_{(\Gamma,\rho,\varepsilon)} = (-1)^r\ \sigma_\gamma\ \chi_\rho(\gamma)\ \ 2^{m-k}$.

Eta invariants are not audible. There are 7-dimensional $\mathbb{Z}_2^2$-manifolds M, M' which are p-isospectral for $0 \le p \le 7$ such that $\eta(M) = 0$ but $\eta(M') = 2$ (see [**Po**]). The trick is to pick one manifold having symmetric spectrum while the other not. It turns out that 7 is the minimum dimension in which this can be done. The moral is that we cannot hear the η-invariant of compact Riemannian manifolds!

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