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ON THE GEOMETRY OF A CLASS OF CONFORMAL HARMONIC MAPS OF SURFACES INTO $\mathbb{S}_1^n$

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ABSTRACT. This paper deals with certain advances in the understanding of the geometry of superconformal harmonic maps of Riemann surfaces into De Sitter space $\mathbb{S}_1^n$. The character of these notes is mainly expository and we made no attempt to provide complete proofs of the main results, which can be found in reference [12]. Our main analytic tool to study superconformal harmonic maps is a Gram-Schmidt algorithm to produce adapted frames for such maps. This allows us to compute the normal curvatures and obtain identities which are used to study their geometry. Some global properties such as fullness and rigidity are considered and a highest order Gauss transform or polar map is constructed and its main properties are discussed.

1. Introduction

The purpose of these notes is to present some recent advances on the geometry of a class of harmonic maps of surfaces into De Sitter space $\mathbb{S}_1^n$, $(n \geq 3)$. As we shall see these maps have many properties in common with their natural relatives: the so-called *superconformal* harmonic maps of surfaces into Euclidean (round) spheres $\mathbb{S}^n$, introduced and studied by Bolton, Pedit and Woodward in [2] and also by Miyaoka in [14].

The style of the paper is expository so that we have omitted the proofs of the main results. The interested reader can consult reference [12] for details.

Let $\mathbb{R}_1^{n+1}$ denote the flat Lorentz $(n+1)$-space i.e. $\mathbb{R}^{n+1}$ equipped with the Lorentz inner product

$$\langle x, y \rangle = \sum_{j=0}^{n-1} x_j y_j - x_n y_n, \quad x, y \in \mathbb{R}^{n+1}. \tag{1}$$

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The n-dimensional De Sitter space-time of radius $c > 0$ is by definition the pseudosphere

$$\mathbb{S}_1^n(c) = \{x \in \mathbb{R}_1^{n+1} : \langle x, x\rangle = c^2\},$$

on which the ambient metric induces a metric also denoted by $\langle\, , \rangle$ of signature $(n-1,1)$, hence $\mathbb{S}_1^n(c)$ is a Lorentz manifold of constant sectional curvature $\frac{1}{c^2}$. A smooth map $f : M \to \mathbb{S}_1^n(c)$ from a Riemann surface is harmonic if its tension field vanishes on M: $\tau(f) \equiv 0$ [7]. It is easily seen that f is harmonic if and only if on any local complex coordinate in M the following PDE is satisfied by f:

$$\partial\bar{\partial} f = -\langle \partial f, \bar{\partial} f\rangle^c f, \tag{2}$$

where $\langle\, , \rangle^c$ denotes the complex bilinear extension of $\langle\, , \rangle$ to $\mathbb{C}^{n+1}$ and

$$\partial f = \frac{1}{2}(f_x - i f_y), \quad \bar{\partial} f = \frac{1}{2}(f_x + i f_y).$$

Note that equation (2) does not depend on a particular metric on M but only on the conformal structure of M. This is characteristic of harmonic maps of Riemann surfaces.

Let $m \geq 2$ and $n = 2m$ or $n = 2m-1$ and $f : M \to \mathbb{S}_1^n(c)$ a smooth map from a connected Riemann surface and assume that there is an integer $r \geq 1$ such that

$$\begin{aligned} &\langle \partial^\alpha f, \partial^\beta f\rangle^c = 0, \quad 1 \leq \alpha + \beta \leq 2r+1,\ \alpha,\ \beta \text{ positive integers},\\ &\langle \partial^{r+1} f, \partial^{r+1} f\rangle^c \not\equiv 0, \end{aligned}$$

where $\partial^\alpha f = \frac{\partial^\alpha f}{\partial z^\alpha}$. It is not difficult to show that such integer r is independent of complex coordinates on M, so that it depends only on the map f itself. It is called the *isotropy dimension* of f and is denoted by $r(f)$. A smooth map $f : M \to \mathbb{S}_1^n(c)$, $(n \geq 3)$ is called *superconformal* if $r(f) = m-1$, where $m = [\frac{n+1}{2}]$. Here we adapted the notion of isotropy dimension which F. Burstall introduced in [5] to study harmonic maps of surfaces into $\mathbb{S}^n$ and $\mathbb{CP}^n$.

Let $f : M \to \mathbb{S}_1^n(c)$ be a harmonic map with isotropy dimension $r(f) \geq 1$. Then equation (2) implies that the locally defined non-vanishing complex function $\langle \partial^{r+1} f, \partial^{r+1} f\rangle^c$ is holomorphic, hence its zeros are isolated. Moreover the formal expression

$$\varphi_{r+1}(f) := \langle \partial^{r+1} f, \partial^{r+1} f\rangle^C dz^{2r+2},$$

is globally defined on M. It is called the $(r+1)$-th complex Hopf differential of f. We notice here that the topology of the Riemann surface M plays a role. In fact, applying the Riemann-Roch Theorem [13], Ejiri in [8] shows that every space-like harmonic map $f : \mathbb{S}^2 \to \mathbb{S}_1^n(c)$ is isotropic:

$$\langle \partial^\alpha f, \partial^\beta f\rangle^c = 0,$$

for every pair of integers $\alpha, \beta \geq 0$ such that $1 \leq \alpha + \beta$. This says that the isotropy dimension of f is infinite. Then if $f : M \to \mathbb{S}_1^n(c)$ is harmonic with finite isotropy dimension and M is compact, then genus$(M) \geq 1$. Harmonic maps of infinite isotropy dimension in $\mathbb{S}_1^n(c)$ have been considered by Ejiri [8] and also by Erdem [9].

It is well known that harmonic maps of surfaces into $\mathbb{S}_1^4(c)$ are related to Willmore surfaces in $\mathbb{R}^3$ and $\mathbb{S}^3$. In fact harmonic maps (superconformal or not) of surfaces in $\mathbb{S}_1^4(c)$ arise as images of the conformal Gauss map of immersed Willmore surfaces in $\mathbb{R}^3$ and in $\mathbb{S}^3$ (see [1, 15]).

Also in [1] Alías and Palmer considered space-like superconformal minimal surfaces into four dimensional Lorentz spaceforms and studied the behaviour of their normal and Gaussian curvatures obtaining interesting results.

On the other hand Sakaki in [16] studied superconformal minimal space-like surfaces in four dimensional Lorentz spaceform satisfying a generalized Ricci-condition. Harmonic maps of surfaces with infinite isotropy dimension (also called isotropic) into $\mathbb{S}_1^n$ were considered by Ejiri [8]. Erdem in [9] obtained a classification of harmonic isotropic maps of Riemann surfaces into $\mathbb{S}_1^n$ with non-degenerate osculating bundle.

The layout of the paper is as follows. Section 1 introduces the Gram-Schmidt algorithm for the construction of harmonic sequences which give rise to the complex line sub bundles L_j. In Section 2 we compute the normal curvatures of a superconformal harmonic map and derive the structural equations. Section 3 deals with global rigidity and Section 4 is devoted to the construction and main properties of the highest order Gauss transform or polar map.

2. Harmonic sequences

In what follows we shall consider only the class of *superconformal* harmonic maps $f : M \to \mathbb{S}_1^n$ with $n = 2m$ or $n = 2m - 1$ $(m \geq 2)$, i.e. those with maximal finite isotropy dimension: $r(f) = m - 1$. Hence in particular f is a (weakly) conformal map

$$\langle \partial f, \partial f \rangle^c \equiv 0, \tag{3}$$

At any point $p \in M$ condition (3) above is equivalent to

$$\|df_p\left(\frac{\partial}{\partial x}\right)\|^2 = \|df_p\left(\frac{\partial}{\partial y}\right)\|^2; \quad \langle df_p\left(\frac{\partial}{\partial x}\right), df_p\left(\frac{\partial}{\partial y}\right)\rangle = 0.$$

Now since the ambient metric has signature $(n-1,1)$, for any $p \in M$ such that df_p is non-singular we have $\|df_p(\frac{\partial}{\partial x})\|^2 = \|df_p(\frac{\partial}{\partial y})\|^2 > 0$, so that f is a *space-like* map i.e. the pull-back metric $f^*\langle\, ,\, \rangle$ is Riemannian at those points $p \in M$ for which df_p is non-singular.

Conversely, if M is an orientable surface and $f : M \to \mathbb{S}^n_1(c)$ is a space-like immersion, then the pullback $f^*\langle\, ,\, \rangle$ is a Riemannian metric on M which determines a conformal or Riemann surface structure on M such that f is a conformal immersion [13]. Hence if one considers on M the induced metric $g = f^*\langle\, ,\, \rangle$, then $f : M \to \mathbb{S}^n_1(c)$ is an isometric space-like immersion.

Let us denote $\mathbb{S}^n_1(1) = \mathbb{S}^n_1$ by simplicity. Now fix a local chart $(U, z) \in M$ and set

$$f_0 := f,\ f_1 := \partial f, \quad f_{j+1} = \partial f_j - \frac{\langle \partial f_j, f_j \rangle}{\|f_j\|^2} f_j. \tag{4}$$

Note that f_{j+1} is just the component of ∂f_j orthogonal to f_j. Moreover f_{j+1} is defined away from the zeros of $\|f_j\|^2$ which are called the *higher order singularities of f*.

The following is our main technical result. It establishes the consistency of the Gram-Schmidt process (4) and assures that the vectors f_j have positive square norms open densely in U. Its proof relies on the fact that harmonic maps of Riemann surfaces into pseudospheres are real analytic maps, an essential observation due to Ejiri [8]. For details, see [12], Section 3.

Lemma 1. [12] *Let M be a connected Riemann surface and $f : M \to \mathbb{S}^n_1$ a superconformal harmonic map, where $n = 2m$ or $n = 2m-1$, $(m \geq 2)$. On any fixed complex chart (U, z) of M, formula (4) generates $\mathbb{C}^{n+1}$-valued maps $f_0, f_1, f_2, \ldots, f_m$, defined on an open and dense subset of U satisfying the following properties:*

i) For each $1 \leq j \leq m-1$ the zeros of f_j are isolated in U and $\|f_j\|^2 > 0$ on an open and dense subset of U.
ii) $\langle f_i, f_j \rangle = 0$ for $0 \leq i \neq j \leq m$.
iii) $\langle f_i, f_j \rangle^c = 0$ for $0 \leq i, j \leq m-1$.

By the above Lemma every superconformal harmonic map $f : M \to \mathbb{S}^n_1$ has isolated higher-order singularities. Moreover it is space-like and df is non-singular on an open and dense subset of M. In particular the induced metric $g = f^*\langle\, ,\, \rangle$ is a Riemannian metric on M with isolated singularities. Considering M with the induced metric g then $f : M \to \mathbb{S}^n_1$ is a branched isometric minimal space-like immersion. The globally defined complex differential $Q(f) = \varphi_m(f) dz^{2m}$, where $\varphi_m(f) = \langle f_m, f_m \rangle^c$, measures the failure of f_m and $\bar{f}_m$ to be orthogonal. It is called the Hopf differential of f and is an important invariant of f. If $Q(f) \equiv 0$

the map f is called isotropic.
Let $f_0, f_1, f_2, \ldots, f_m$ be the finite sequence generated by (4) on U. Defining

$$f_{-j} := (-1)^j \frac{\bar{f}_j}{\|f_j\|^2}, \quad 1 \leq j \leq m, \tag{5}$$

then a straightforward computation shows that the augmented sequence $\{f_{-m}, \ldots, f_{-1}, f_0, f_1, \ldots, f_m\}$ satisfies the following formulae

$$\begin{cases} f_{j+1} = \partial f_j - \partial \log \|f_j\|^2 f_j, & -m \leq j \leq m-1 \\ \bar{\partial} f_j = -\frac{\|f_j\|^2}{\|f_{j-1}\|^2} f_{j-1}, & -m+1 \leq j \leq m \\ \langle f_i, f_j \rangle = 0, \text{ for } 0 < |i-j| \leq 2m-1. \end{cases} \tag{6}$$

Also from (5) it follows that $\langle f_m, f_{-m} \rangle = \frac{(-1)^m}{\|f_m\|^2} \varphi_m$, hence the extremes are orthogonal only at the zeros of the m-th Hopf differential.

Harmonic sequences were thoroughly studied by Bolton and Woodward in [4], who considered harmonic maps of surfaces into complex projective spaces and spheres.

2.1. **The line bundles L_j.** The (pseudo) hermitian inner product on $\mathbb{C}^{n+1}$ is defined by

$$\langle z, w \rangle = z_0 \bar{w}_0 + z_1 \bar{w}_1 + \cdots + z_{n-1} \bar{w}_{n-1} - z_n \bar{w}_n. \tag{7}$$

We denote by $\mathbb{C}_1^{n+1}$ the complex vector space $\mathbb{C}^{n+1}$ equipped with the (pseudo-hermitian) inner product (7). Let $\underline{\mathbb{C}}_1^{2m+1} = \mathbb{C}_1^{2m+1} \times M \to M$ be the trivial bundle equipped with the canonical connection $D_X s = Xs$ where s is any smooth local section of $\underline{\mathbb{C}}_1^{2m+1}$ and $X \in TM$. The map f determines a complex line subbundle

$$L_0 = \{(v, x) \in \underline{\mathbb{C}}_1^{2m+1} : v \in \mathbb{C} f(x)\},$$

equipped with the metric-compatible connection $\nabla_{L_0} = \pi_{L_0} \circ D$, where the projection $\pi_0 : \underline{\mathbb{C}}_1^{2m+1} \to L_0$ along $L_0^\perp$ is well defined since $\langle f, f \rangle = 1$. By the well known Theorem of Koszul-Malgrange [7], ∇_{L_0} determines a unique compatible holomorphic structure on L_0 such that a local smooth section s of L_0 is holomorphic if and only if $\nabla''_{L_0} s = 0$, where $\nabla''_{L_0} = \pi_{L_0} \circ \bar{\partial}$. Hence s is holomorphic if and only if $\bar{\partial} s \in L_0^\perp$. In particular by the harmonic map equation (2) f is a global holomorphic section of L_0. On the other hand the fibers of L_0 determine a map $\varphi_0 : M \to \mathbb{CP}_1^{2m}$ by $\varphi_0(x) = \mathbb{C} f(x)$. Since φ_0 is the composition of f followed by the totally geodesic imbedding $\mathbb{S}_1^{2m} \hookrightarrow \mathbb{CP}_1^{2m}$, it results also harmonic.

In general a complex vector subbundle $E \subset \underline{\mathbb{C}}_1^{n+1}$ can be equipped with the Koszul-Malgrange holomorphic structure provided it is non-degenerate respect to the ambient hermitian indefinite inner product $\langle\,,\,\rangle$. That is, $E \cap E^{\perp} = \{0\}$ fiber-wise, where $\perp$ denotes $\langle\,,\,\rangle$-orthogonal complement.

The bundle operator $A_{L_0} : TM \otimes L_0 \to L_0^{\perp}$ given by $A_{L_0} = \pi_{L_0^{\perp}} \circ D$ splits up into its $(0,1)$ and $(1,0)$ parts $A'_{L_0} = \pi_{L_0^{\perp}} \circ \partial$ and $A''_{L_0} = \pi_{L_0^{\perp}} \circ \bar{\partial}$, according to $D = \partial + \bar{\partial}$. It is shown in [11] that both operators are related by the identity

$$(A'_{L_0})^* = -A''_{L_0^{\perp}}. \tag{8}$$

Now since f is a harmonic map, A'_{L_0} takes holomorphic sections of L_0 to holomorphic sections of $L_0^{\perp}$. This is equivalent to

$$A'_{L_0} \circ \nabla''_{L_0} = \nabla''_{L_0^{\perp}} \circ A'_{L_0},$$

which also says that A'_{L_0} is a holomorphic section of $Hom(L_0, L_0^{\perp})$ and by (8) A''_{L_0} is antiholomorphic. Let L_1 be the unique complex line sub bundle of $\underline{\mathbb{C}}_1^{2m+1}$ containing the image of A'_{L_0}. Define L_1 by continuity across the isolated zeros of A'_{L_0}, hence L_1 is a well defined non-degenerate complex line subbundle of $\underline{\mathbb{C}}_1^{2m+1}$ on which the ambient metric $\langle\,,\,\rangle$ is positive definite by Lemma 1, i.e. L_1 is a space-like subbundle of $\underline{\mathbb{C}}_1^{2m+1}$. In particular it has a well defined metric connection $\nabla_{L_1} = \pi_{L_1} \circ D$ and hence a unique compatible holomorphic structure. From (4) it follows that A'_{L_0} sends holomorphic sections of L_0 to holomorphic sections of L_1, in particular $f_1 = A'_{L_0} f_0$ is a local holomorphic section of L_1. In the same way the image of the operator $A_{L_1} = \pi_{L_1^{\perp}} \circ D : L_1 \to L_1^{\perp}$ determines a unique space-like (hence non-degenerate) complex line subbundle $L_2 \subset \underline{\mathbb{C}}_1^{2m+1}$. Thus it has also a well-defined metric connection $\nabla_{L_2} = \pi_{L_2} \circ D$ and hence a unique compatible holomorphic structure. Also from (4) $f_2 = A'_{L_1} f_1$ is a local holomorphic section of L_2. The process continues producing a sequence of mutually orthogonal space-like holomorphic complex line subbundles $L_1, L_2, \ldots, L_{m-1}, \subseteq \underline{\mathbb{C}}_1^{2m+1}$ where each L_j has a well-defined metric connection $\nabla_{L_j} = \pi_{L_j} \circ D$ and a compatible holomorphic structure via the Koszul-Malgrange Theorem. However the last complex subbundle L_m containing the image of $A'_{L_{m-1}}$ may degenerate at some points and its signature may change.

The conjugate bundles $L_{-j} := \bar{L}_j \;\; 1 \leq j \leq m-1$ are also space-like. Including the possibly degenerate subbundles L_{-m}, L_m, then Lemma 1 and formulae (6) imply that the whole sequence $\{L_j : -m \leq j \leq m\}$ satisfies orthogonality relations

$$L_i \perp L_j \quad \text{for } 0 < |i-j| \leq 2m-1. \tag{9}$$

Also from (6) we conclude that $A'_{L_j} : L_j \to L_{j+1}$ defined by

$$A'_{L_j} = \pi_{L_j^\perp} \circ \partial,$$

satisfy $A'_{L_j} f_j = f_{j+1}$. Hence for $-m \leq j \leq m-1$ the bundle operators $A'_{L_j} : L_j \to L_{j+1}$ are holomorphic, i.e. they send holomorphic sections to holomorphic sections. This fact is expressed in an equivalently way by equation

$$A'_{L_j} \circ \nabla''_{L_j} = \nabla''_{L_j^\perp} \circ A'_{L_j},$$

which follows from (4) and the definition of A'_{L_j} and ∇_{L_j}. Also for $-m+1 \leq j \leq m$ the operators $A''_{L_j} := \pi_{L_j^\perp} \circ \bar{\partial} : L_j \to L_{j-1}$ are anti-holomorphic bundle operators as a consequence of the identity $(A'_{L_j})^* = -A''_{L_j^\perp}$ (see [12], page 190, for details). It is not difficult to show that the maps $\varphi_j : M \to \mathbb{CP}_1^{2m}$ defined by $\varphi_j(x) = L_j(x)$ are harmonic for $-m+1 \leq j \leq m-1$. The finite sequence of harmonic maps $\varphi_j : M \to \mathbb{CP}_1^{2m}$, $-(m-1) \leq j \leq m-1$ is called the *harmonic sequence* of the initial superconformal harmonic map $f : M \to \mathbb{S}_1^{2m}$.

The curvature of L_j. The intrinsic curvatures of the complex line bundles L_j are obtained using the Koszul-Malgrange holomorphic structure. In fact, we have

$$[\nabla'_{L_j}, \nabla''_{L_j}] f_j = -\bar{\partial}\partial \log \|f_j\|^2 \, f_j.$$

In the next section we shall see that the quantity $-\bar{\partial}\partial \log \|f_j\|^2$ is just a multiple of the j-th normal curvature K_j by a factor depending on the induced metric g on M.

3. Normal curvatures and Structure equations

We fix the induced metric $g = f^*\langle\, , \,\rangle$ on M, hence the computations that follow hold away from the isolated singularities g. Let ∇ be the pseudo-Riemannian Levi-Civita connection of $\mathbb{S}_1^{2m}$ determined by the Lorentz metric and consider the pull-back bundle

$$T = f^*(T\mathbb{S}_1^{2m}) \subset \underline{\mathbb{R}}_1^{2m+1} := \mathbb{R}_1^{2m+1} \times M,$$

with the pull-back connection denoted also by ∇, and the pull-back Lorentz metric $\langle\, , \,\rangle$. The subspace of T_p generated by the ∇-derivatives of f up to order j at $p \in M$ is called the *j-th osculating space at p* and is denoted by T_p^j. Then $T_p^1 = df_p(TM)$ and T_p^j is a subspace of T_p^{j+1}. The orthogonal complement of T_p^j in T_p^{j+1}, denoted by N_p^j, is called *the j-normal space at p*. Thus

$$T_p^j = T_p^{j-1} \oplus N_p^{j-1}, \quad 2 \leq j \leq m. \tag{10}$$

At generic points one can consider the j-th osculating bundle T^j with $2j$-dimensional fibers T^j_p, and also the j-th normal bundle N^j with 2-dimensional fibers N^j_p. A point p is said to be generic if the fiber of T^j over p coincides with the j-th osculating space at p. It is known that the set of generic points is open and dense in M (see [17]). The set of non-generic points, are nothing but the higher-order singularities of f and consists of isolated points (cf. [6, 17]).

For $1 \leq j \leq m-1$ the fibres of each complex line bundle L_j determined by f are isotropic space-like complex lines in $\mathbb{C}^{2m+1}_1$. Hence each $L_j,\ 1 \leq j \leq m-1$, may be identified with an oriented real space-like 2-plane subbundle of $\underline{\mathbb{R}}^{2m+1}_1$ in the following way: on a complex chart $(U, z) \in M$ f_j is a local holomorphic section of L_j generated by (4). We define real vector fields F_{2j-1}, F_{2j} on U by setting

$$f_j = \frac{\|f_j\|}{\sqrt{2}}(F_{2j-1} - iF_{2j}), \quad 1 \leq j \leq m-1. \tag{11}$$

Then since $\langle f_j, f_j\rangle^c = 0$, the fields are orthogonal $\langle F_{2j-1}, F_{2j}\rangle = 0$ and of unit norm $\|F_{2j-1}\|^2 = \|F_{2j}\|^2 = 1$. Thus, for $j = 1$, F_1, F_2 are local generating sections of the first osculating bundle (or tangent bundle) $T^1 = df(TM)$ of f, and for $2 \leq j \leq m-1$ F_{2j-1}, F_{2j} are local generating sections of the $(j-1)$-th normal bundle N^{j-1} of f. This exhibits the identification of $L_1 \equiv df(TM)$ and of $L_j \equiv N^{j-1}$ for $2 \leq j \leq m-1$. Consequently the (complex) maximal isotropic space-like subbundle

$$L_1 \oplus L_2 \oplus \cdots \oplus L_{m-1} \subset T^{\mathbb{C}},$$

identifies with the $(m-1)$-th osculating bundle $T^{m-1} \subset T$ of f. Also from (11) we have

$$df(TM)^{\mathbb{C}} = \bar{L}_1 \oplus L_1, \quad (N^{j-1})^{\mathbb{C}} = \bar{L}_j \oplus L_j,\ 2 \leq j \leq m-1.$$

It follows from Lemma 1 and our discussion above that T^{m-1} is a real space-like $2(m-1)$-dimensional vector subbundle of T. Now if f is linearly full, we have $T^m = T$ and by (10) the last normal bundle $N^{m-1} = (T^{m-1})^{\perp}$ of f is a real non-degenerate oriented Lorentz 2-plane subbundle of T: i.e. the restriction of the Lorentz metric to the fibers of N^{m-1} has signature $(1, 1)$. Then it is easily seen that there are local generating sections F_{2m-1}, F_{2m} of N^{m-1} satisfying

$$\langle F_{2m-1}, F_{2m}\rangle = 0, \quad \|F_{2m-1}\|^2 = -\|F_{2m}\|^2 = 1. \tag{12}$$

In particular $(N^{m-1})^{\mathbb{C}} = \bar{L}_m \oplus L_m$ and hence there are (local) complex functions α, β such that

$$f_m = \alpha F_{2m-1} - \beta F_{2m}, \tag{13}$$

so that L_m identifies with N^{m-1}. Note that the direct sum subbundle $L_2 \oplus L_3 \oplus \cdots \oplus L_m$ identifies with the normal bundle of f

$$\nu(f) = N^1 \oplus N^2 \oplus \cdots \oplus N^{m-1},$$

and ∇ restricted to $\nu(f)$ coincides with the normal connection $\nabla^\perp$ on $\nu(f)$. Also ∇ restricted to T^1 coincides with the Levi-Civita Riemannian connection on M determined by the induced metric g. The projection of $\nabla^\perp$ onto each normal 2-plane subbundle N^{j-1} defines a metric-compatible connection $\nabla^\perp_{j-1}$, $2 \leq j \leq m$ which is Riemannian for $2 \leq j \leq m-1$, whereas $\nabla^\perp_{m-1}$ is pseudo-Riemannian.

Let $\omega_j = \langle \nabla^\perp_{j-1} F_{2j}, F_{2j-1} \rangle$ be the connection forms of T^1, $N^1, N^2, \ldots, N^{m-1}$. Then the equation $d\omega_j = K_j dA$ defines the curvature function K_j, where

$$dA = 2\|f_1\|^2 dx \wedge dy,$$

is the area element of the induced metric g respect to a local complex coordinate $z = x + iy$ (cf. [17]). It is shown in [12] that

$$K_j = -\frac{1}{2}\Delta_g \log \|f_j\|^2, \quad \text{for } j = 1, \ldots, m-1, \tag{14}$$

where

$$\Delta_g = 2\|f_1\|^{-2}\partial\bar{\partial}, \tag{15}$$

is the Laplacian operator of the induced metric $g = 2\|f_1\|^2 dz d\bar{z}$ on M. Note that K_1 is just the Gauss curvature of the induced metric g. The expression of the last normal curvature K_m looks a little bit different

$$K_m = 2\|f_1\|^{-2}.\|f_{m-1}\|^{-2} Im(\alpha.\bar{\beta}), \tag{16}$$

where α, β are given by (13). See [12], page 194 for details.

3.1. **Curvature identities and consequences.** The compatibility or integrability equations satisfied by the harmonic sequence of a superconformal harmonic map $f : M \to \mathbb{S}^{2m}_1$ are $\bar{\partial}\partial f_j = \partial\bar{\partial} f_j$, $1 \leq j \leq m$ which as consequence of (4) and (6) are given by

$$\bar{\partial}\partial \log \|f_j\|^2 = \frac{\|f_{j+1}\|^2}{\|f_j\|^2} - \frac{\|f_j\|^2}{\|f_{j-1}\|^2}.$$

In terms of the functions $u_j := \log \|f_j\|$, $1 \leq j \leq m-1$, $\sigma_m := \langle \partial F_{2m}, F_{2m-1} \rangle$, and α, β such that $f_m = \alpha F_{2m-1} - \beta F_{2m}$, the compatibility equations above are expressed by the following system of partial differential equations

$$\begin{cases} 2\partial\bar{\partial}u_j = e^{2(u_{j+1}-u_j)} - e^{2(u_j-u_{j-1})}, & j = 1,\dots,m-2, \\ 2\partial\bar{\partial}u_{m-1} = (|\alpha|^2 - |\beta|^2)e^{-2u_{m-1}} - e^{2(u_{m-1}-u_{m-2})}, & \\ Im(\bar{\partial}\sigma_m) = -e^{-2u_{m-1}}.Im(\alpha\bar{\beta}), & \\ \bar{\partial}\alpha = \bar{\sigma}_m\beta, & \\ \bar{\partial}\beta = \bar{\sigma}_m\alpha. & \end{cases} \tag{17}$$

Using the expressions for $K_1,\dots,K_m$ obtained before, and (15), we obtain the normal curvatures in terms of u_j and α,β:

$$\begin{cases} K_j = e^{-2u_1}[e^{2(u_j-u_{j-1})} - e^{2(u_{j+1}-u_j)}] & j = 1,\dots,m-2, \\ K_{m-1} = e^{-2u_1}[e^{2(u_{m-1}-u_{m-2})} - (|\alpha|^2 - |\beta|^2)e^{-2u_{m-1}}], & \\ K_m = 2e^{-2u_1}e^{-2u_{m-1}}Im(\alpha\bar{\beta}). & \end{cases} \tag{18}$$

Hence the sum of the first $m-1$ curvatures gives

$$\sum_{j=1}^{m-1} K_j - 1 = -\|f_1\|^{-2}\|f_{m-1}\|^{-2}(|\alpha|^2 - |\beta|^2). \tag{19}$$

Note that from (19) the sign of $\sum_{j=1}^{m-1} K_j - 1$ depends on the sign of $\|f_m\|^2 = |\alpha|^2 - |\beta|^2$.

On the other hand squaring (16) and adding (19) we obtain

$$\left(1 - \sum_{j=1}^{m-1} K_j\right)^2 + K_m^2 = \|f_1\|^{-4}\|f_{m-1}\|^{-4}|\varphi_m|^2. \tag{20}$$

Away from the zeros of the m-th Hopf differential $Q = \varphi_m(f)dz^{2m}$ we can take log at both sides of (20) and since $\log|\varphi_m|^2$ is a local harmonic function, we obtain the following identity

$$\Delta_g \log[(1 - \sum_{j=1}^{m-1} K_j)^2 + K_m^2] = 4(K_1 + K_{m-1}). \tag{21}$$

This identity generalizes a formula obtained by Alías and Palmer in [1] and is the key point to prove the following characterization of superconformal harmonic maps of tori given in [12], which is a generalization of a Theorem by Sakaki in [16]

Theorem 1. *([12], Theorem 8.1) Let M be a compact connected Riemann surface and $f : M \to \mathbb{S}_1^{2m}$ a linearly full superconformal harmonic map having no higher-order singularities. If the Gaussian and normal curvatures of f satisfy $(1 - \sum_{j=1}^{m-1} K_j)^2 + K_m^2 > 0$ on M, then M is topologically a 2-torus.*

Conversely, if M is a 2-torus then passing to the universal covering space $\mathbb{C}$ of M it is possible to normalize $\varphi_m \equiv 1$ globally on M. Hence if the full superconformal harmonic map $f : M \to \mathbb{S}_1^{2m}$ has no higher-order singularities, the inequality

$(1-\sum_{j=1}^{m-1} K_j)^2 + K_m^2 > 0$ holds on M as a consequence of (20).

Also as an application of identities (19) and (20) we obtain the following result which gives information on the global behaviour of a non-full superconformal harmonic map f in terms of the normal curvatures

Theorem 2. *(*[12]*, Theorem 5.4) Let $f : M \to \mathbb{S}_1^{2m}$ be a superconformal harmonic map of a Riemann surface. Then $K_m \equiv 0$ if and only if $f(M)$ lies fully in a unique non-degenerate hyperplane V. In this case*
a) $\sum_{j=1}^{m-1} K_j - 1 \geq 0$ if and only if the induced metric on V has signature $(2m-1, 1)$ *and f is superconformal harmonic full into $\mathbb{S}_1^{2m-1}(V)$.*
b) $\sum_{j=1}^{m-1} K_j - 1 \leq 0$ if and only if V is space-like and f is a superconformal harmonic full into the Euclidean unit sphere $\mathbb{S}^{2m-1}(V) \subset V$.
In both cases identity (20) *implies that $\sum_{j=1}^{m-1} K_j = 1$ can occur only at the zeros of Q which are isolated.*

3.2. **Toda affine equations and Toda frames.** Away from the isolated zeros of the m-th Hopf differential of f it is possible to find a local complex coordinate (U, z) which normalizes φ_m, i.e. $\varphi_m \equiv 1$ on U (a proof of this fact is given in [2]). In terms of α and β condition, $\varphi_m = \langle f_m, f_m \rangle^c \equiv 1$ is just $\alpha^2 - \beta^2 = 1$ on U, so let ξ be a complex function defined on U such that $\alpha = \cosh \xi$ and $\beta = \sinh \xi$. Introduce new local sections F'_{2m-1}, F'_{2m} of N^{m-1} by

$$\begin{aligned} F'_{2m-1} &= \cosh(r) F_{2m-1} + \sinh(r) F_{2m}, \\ F'_{2m} &= \sinh(r) F_{2m-1} + \cosh(r) F_{2m}, \end{aligned} \tag{22}$$

where $r = Re(\xi)$. It is easily seen that $\|F'_{2m-1}\|^2 = -\|F'_{2m}\|^2 = 1$ and $\langle F'_{2m-1}, F'_{2m} \rangle = 0$. In this new frame we have

$$f_m = \cos(\theta) F'_{2m-1} + i \sin(\theta) F'_{2m},$$

so that $\alpha' = \cos(\theta)$, $\beta' = -i \sin(\theta)$, where $\theta = Im(\xi)$. It follows that $\|f_m\|^2 = \cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$. The fourth and fifth compatibility equations in (17) yield $\bar{\partial}\theta = i\sigma_m$ and hence $\bar{\partial}\partial\theta = -i\bar{\partial}\sigma_m$. Also from the third equation of (17) we get $Im(\bar{\partial}\sigma) = -\cos(\theta)\sin(\theta) e^{-2u_{m-1}}$, from which θ satisfies

$$2\partial\bar{\partial}\theta = -\sin(2\theta) e^{-2u_{m-1}}. \tag{23}$$

We have shown that in a local coordinate chart (U, z) where $\varphi_m \equiv 1$ it is possible to find a local frame

$$\begin{array}{ll} F = (f, F_1, F_2, \dots, F_{2m-2}, F'_{2m-1}, F'_{2m}), & \\ f_j = \frac{e^{u_j}}{\sqrt{2}}(F_{2j-1} - iF_{2j}), & 1 \leq j \leq m-1 \\ f_m = \cos(\theta)F'_{2m-1} + i\sin(\theta)F'_{2m}, & \end{array}$$

such that its compatibility equations $\bar{\partial}\partial F = \partial\bar{\partial}F$ become the following system of elliptic non-linear partial differential called Toda affine equations associated to the pair $(\mathfrak{so}(2m+1, \mathbb{C}), \mathfrak{so}(2m, 1))$

$$\begin{cases} 2\partial\bar{\partial}u_j = e^{2(u_{j+1}-u_j)} - e^{2(u_j-u_{j-1})}, \ j = 1, \dots, m-2, \\ 2\partial\bar{\partial}u_{m-1} = \cos(2\theta)e^{-2u_{m-1}} - e^{2(u_{m-1}-u_{m-2})}, \\ 2\partial\bar{\partial}\theta = -\sin(2\theta)e^{-2u_{m-1}}. \end{cases} \tag{24}$$

Thus locally and away from the zeros of the m-th Hopf differential $Q(f) = \varphi_m dz^{2m}$ the geometry of a superconformal harmonic map f is completely determined from a solution of the above system. The frame $F = (F_1, F_2, \dots, F_{2m-2}, F'_{2m-1}, F'_{2m})$ considered above is called a Toda frame by Bolton Pedit and Woodward (cf. [2]). Toda equations are well known examples of completely integrable systems. For a survey on Toda equations and other soliton equations arising in geometry, see [10] and the bibliography cited there.

In terms of θ the last normal curvature K_m is given by

$$K_m = \sin(2\theta)e^{-2(u_1+u_{m-1})}. \tag{25}$$

Also from (19) we see that

$$\sum_{j=1}^{m-1} K_j - 1 = -e^{-2(u_1+u_{m-1})}\cos(2\theta). \tag{26}$$

Thus at points where $\sum_{j=1}^{m-1} K_j \neq 1$ (hence on an open and dense subset) we have

$$\arctan\left(\frac{K_m}{\sum_{j=1}^{m-1} K_j - 1}\right) = -2\theta. \tag{27}$$

Applying Δ_g to both sides of (27) we get the following identity which generalizes formula (3.1) of Alías and Palmer in [1]

$$\Delta_g \arctan\left(\frac{K_m}{\sum_{j=1}^{m-1} K_j - 1}\right) = 2K_m. \tag{28}$$

Integrating this identity respect to dA and using the Divergence Theorem we obtain the following result

Theorem 3. *([12], Lemma 5.5) Let M be a compact connected Riemann surface and $f : M \to \mathbb{S}_1^{2m}$ a full superconformal harmonic immersion for which $\sum_{j=1}^{m-1} K_j \neq 1$ at each point of M. Then*

$$\int_M K_m dA = 0.$$

Note that under the hypothesis of Theorem 3, $K_m \not\equiv 0$ so that K_m must be a signed function on M.

4. Rigidity

Here we consider the problem of determining invariants which determine a superconformal harmonic map $f : M \to \mathbb{S}_1^n$ up to ambient isometries. We obtained the following result which is analogous to that obtained in [4, 11] when the target is $\mathbb{S}^n$ and $\mathbb{H}^n$ respectively

Theorem 4. *([12], Theorem 6.1) Let $f, h : M \to \mathbb{S}_1^{2m}$ be superconformal harmonic maps from a connected Riemann surface. If they induce the same metric on M and have the same m-th Hopf differentials, then there is an isometry Φ of $\mathbb{S}_1^{2m}$ such that $\Phi \circ f = h$.*

The construction of the isometry Φ uses the harmonic sequence of f which by hypothesis coincides with that of g, and the Toda equations (24).

A manifestation of the complete integrability of the Toda system (24) describing the geometry of a superconformal harmonic map $f : M \to \mathbb{S}_1^n$ is the fact that for a simply connected M there is an associated S^1-family $f_\lambda : M \to \mathbb{S}_1^n$, $\lambda \in S^1$ of isometric deformations of the given f. The proof of the following Theorem is consequence of a result by Bolton and Woodward in [4] when the target is the Euclidean sphere $\mathbb{S}^n$. For superconformal harmonic maps into $\mathbb{S}_1^n$, the proof is analogous and uses the machinery of harmonic sequences which we developed before.

Theorem 5. *Let M be a simply connected Riemann surface and let $f, \tilde{f} : M \to \mathbb{S}_1^{2m}$ be full superconformal harmonic maps inducing the same metric on M. If $Q(f) = \lambda Q(\tilde{f})$ for some function $\lambda : M \to S^1$, then λ is constant and $\tilde{f}$ is congruent with some f_λ of the family.*

5. A higher order Gauss transform

According to Theorem 2 the image of a non-linearly full superconformal harmonic map $f : M \to \mathbb{S}_1^{2m}$ lies fully in a non-degenerate hyperplane $V \subset \mathbb{R}_1^{2m+1}$

which may be either space-like or have signature $(2m-1,1)$. In this case the sequence L_j generated by f is periodic:

$$L_{2m+j} = L_j, \quad \forall j \in \mathbb{Z}, \tag{29}$$

and the last line bundle of f is non-degenerate and satisfies $L_m = \bar{L}_m = L_{-m}$. Our discussion below needs the following result which also has an independent interest

Proposition 1. *([12], Proposition 4.2) Let $f : M \to \mathbb{S}_1^{2m}$ be a superconformal harmonic map. Then for every local complex chart on M the following inequality holds*

$$| \, \|f_m\|^2 | \leq |\varphi_m|. \tag{30}$$

If f is not full then its image $f(M)$ lies fully in a non-degenerate hyperplane $V \subset \mathbb{R}_1^{2m+1}$ and equality holds in (30).

We are ready now to define the higher order Gauss transform or polar map of f as follows. If the equality holds in (30) we have $\|f_m\|^2 = \pm|\varphi_m|$. Then according to the signature of the metric induced on V we have:

(i) The hyperplane V is space-like and consequently $\|f_m\|^2 = |\varphi_m|$. In particular f_m and $\sqrt{\varphi_m}$ have the same order zeros so that one can extend the vector $\frac{f_m}{\sqrt{\varphi_m}}$ across its singularities by continuity (cf. [14]). It can be easily checked that it is a real vector and has square norm one. Moreover $\frac{f_m}{\sqrt{\varphi_m}}$ is independent of coordinates of M. The Gauss transform of f is well defined by

$$f^* = \frac{f_m}{\sqrt{\varphi_m}} : M \to \mathbb{S}^{2m-1}(V) \subset V, \tag{31}$$

where $\mathbb{S}^{2m-1}(V) = \{x \in V : \langle x, x \rangle = 1\}$ is the unit sphere of V.

(ii) The induced metric on the hyperplane V has signature $(2m-1,1)$ and so it is isometric to $\mathbb{R}_1^{2m}$. Here note that the square norm of f_m is non-positive since $\|f_m\|^2 = -|\varphi_m|$. Like in the previous case the vector $\frac{f_m}{\sqrt{\varphi_m}}$ can be extended by continuity across its singularities and does not depend on local coordinates in M. However it is not a real vector since as consequence of $\bar{f}_m = -\frac{\bar{\varphi}_m}{|\varphi_m|} f_m$ we have,

$$\overline{\left(\frac{f_m}{\sqrt{\varphi_m}} \right)} = -\frac{\overline{\varphi_m} f_m}{|\varphi_m| \sqrt{\overline{\varphi_m}}} = -\frac{\sqrt{\overline{\varphi_m}} f_m}{\sqrt{\varphi_m \overline{\varphi_m}}} = -\frac{f_m}{\sqrt{\varphi_m}}.$$

In this case defining $f^* = \frac{\pm i f_m}{\sqrt{\varphi_m}}$ $(i = \sqrt{-1})$, it follows that f^* is a real vector with square norm -1 lying in V which is independent of local coordinates of M. We define the Gauss map of f in this case by

$$f^* = \frac{\pm i f_m}{\sqrt{\varphi_m}} : M \to \mathbb{H}^{2m-1}(V) \subset V, \tag{32}$$

where the sign in (32) depends on a choice of the sheets of the hyperboloid $\{x \in V : \langle x, x\rangle = -1\}$ defining the hyperbolic space $\mathbb{H}^{2m-1}(V)$.

The main result of this section is the following

Theorem 6. *([12], Theorem 7.1) Let $f : M \to \mathbb{S}^{2m}_1$ be a non-full superconformal harmonic map. If the image $f(M)$ lies in a space-like hyperplane $V \subset \mathbb{R}^{2m-1}_1$ then the Gauss transform $f^* = \frac{f_m}{\sqrt{\varphi_m}} : M \to \mathbb{S}^{2m-1}(V)$ is a full superconformal harmonic map into the Euclidean unit sphere of V which has the same m-th Hopf differential as f.*

If $f(M)$ lies in a $(2m-1, 1)$-hyperplane $V \subset \mathbb{R}^{2m+1}_1$ then the Gauss transform $f^* = \frac{\pm i f_m}{\sqrt{\varphi_m}} : M \to \mathbb{H}^{2m-1}(V)$ is a full superconformal harmonic map in the sense of [11]. In this case the m-th Hopf differentials of f and f^* have opposite signs.

Since there are no non-constant harmonic maps of compact surfaces into $\mathbb{H}^n$ we obtain

Corollary 1. *There exist no non-constant superconformal harmonic map of a compact surface into odd-dimensional De Sitter space $\mathbb{S}^{2m-1}_1$ $(m \geq 2)$.*

Note that the simplest case $m = 2$ in the above Corollary is interesting since a conformal minimal immersion $f : M \to \mathbb{S}^3_1$ is superconformal if and only if its umbilic points are isolated.

Corollary 2. *There is no non-constant conformal minimal immersion of a compact surface M into $\mathbb{S}^3_1$ with isolated umbilic points.*

Other applications of higher order Gauss transforms of maps into $\mathbb{S}^{2m-1}_1$ will be discussed elsewhere.

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