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# ON THE STABILITY INDEX OF MINIMAL AND CONSTANT MEAN CURVATURE HYPERSURFACES IN SPHERES

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Abstract. The study of minimal and, more generally, constant mean curvature hypersurfaces in Riemannian space forms is a classical topic in differential geometry. As is well known, minimal hypersurfaces are critical points of the variational problem of minimizing area. Similarly, hypersurfaces with constant mean curvature are also solutions to that variational problem, when restricted to volume-preserving variations. In this paper we review about the stability index of both minimal and constant mean curvature hypersurfaces in Euclidean spheres, including some recent progress by the author, jointly with some of his collaborators. One of our main objectives on writing this paper has been to make it comprehensible for a wide audience, trying to be as self-contained as possible.

## 1. Stability and index of minimal hypersurfaces

Let us consider $\psi : \Sigma^n \to \mathbb{S}^{n+1}$ an orientable hypersurface immersed into the unit Euclidean sphere $\mathbb{S}^{n+1}$. We will denote by $A$ the shape operator of $\Sigma$ with respect to a globally defined normal unit vector field $N$. That is, $A : \mathcal{X}(\Sigma) \to \mathcal{X}(\Sigma)$ is the endomorphism determined by

$$AX = -\nabla^o{}_X N = -\overline{\nabla}_X N, \quad X \in \mathcal{X}(\Sigma),$$

where $\nabla^o$ and $\overline{\nabla}$ denote, respectively, the Levi-Civita connections on $\mathbb{R}^{n+2}$ and $\mathbb{S}^{n+1}$. As is well known, $A$ defines a symmetric endomorphism on $\mathcal{X}(\Sigma)$ whose eigenvalues $\kappa_1, \ldots, \kappa_n$ are usually referred to as the principal curvatures of the hypersurface. The mean curvature of $\Sigma$ is then defined as

$$H = \frac{1}{n}\mathrm{tr}(A) = \frac{1}{n}(\kappa_1 + \cdots + \kappa_n).$$

Throughout this paper, we will assume that $\Sigma$ is compact. Every smooth function $f \in \mathcal{C}^\infty(\Sigma)$ induces a normal variation $\psi_t : \Sigma \to \mathbb{S}^{n+1}$ of the original immersion $\psi$, given by

$$\psi_t(p) = \mathrm{Exp}_{\psi(p)}(tf(p)N(p)) = \cos(tf(p))\psi(p) + \sin(tf(p))N(p),$$

where Exp denotes the exponential map in $\mathbb{S}^{n+1}$. Since $\Sigma$ is compact and $\psi_0 = \psi$ is an immersion, there exists $\varepsilon > 0$ such that every $\psi_t$ is also an immersion, for $|t| < \varepsilon$. Then we can consider the area function $\mathcal{A} : (-\varepsilon, \varepsilon) \to \mathbb{R}$ which assigns to

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each $t$ the $n$-dimensional area of $\Sigma$ with respect to the metric induced on $\Sigma$ by the immersion $\psi_t$. That is,

$$\mathcal{A}(t) = \mathrm{Area}(\Sigma_t) = \int_\Sigma d\Sigma_t,$$

where $\Sigma_t$ stands for the manifold $\Sigma$ endowed with the metric induced by $\psi_t$ from the Euclidean metric on $\mathbb{S}^{n+1}$, and $d\Sigma_t$ is the $n$-dimensional area element of that metric on $\Sigma$. The first variation formula for the area [16, Chapter I, Theorem 4] establishes that

$$\delta_f\mathcal{A} = \frac{d\mathcal{A}}{dt}(0) = -n\int_\Sigma fH d\Sigma. \tag{1.1}$$

As a consequence, $\Sigma$ is a minimal hypersurface (that is, $H = 0$ on $\Sigma$) if and only if $\delta_f\mathcal{A} = 0$ for every smooth function $f \in \mathcal{C}^\infty(\Sigma)$. In other words, minimal hypersurfaces are characterized as critical points of the area functional.

The stability operator of this variational problem is given by the second variation formula for the area [16, Chapter I, Theorem 32], which in our case is written as follows

$$\delta_f^2\mathcal{A} = \frac{d^2\mathcal{A}}{dt^2}(0) = -\int_\Sigma (f\Delta f + (|A|^2 + n)f^2)d\Sigma = -\int_\Sigma fJf d\Sigma. \tag{1.2}$$

Here $J = \Delta + |A|^2 + n$, where $\Delta$ stands for the Laplacian operator of $\Sigma$ and $|A|^2 = \mathrm{tr}(A^2)$ is the square of the norm of the shape operator. The operator $J : \mathcal{C}^\infty(\Sigma) \to \mathcal{C}^\infty(\Sigma)$ is called the Jacobi (or stability) operator of the minimal hypersurface $\Sigma$. The Jacobi operator $J$ belongs to a class of operators which are usually referred to as Schrödinger operators, that is, operators of the form $\Delta + q$, where $q$ is any continuous function on $\Sigma$. As is well known, the spectrum of $J$

$$\mathrm{Spec}(J) = \{\lambda_1 < \lambda_2 < \lambda_3 < \cdots\}$$

consists of an increasing sequence of eigenvalues $\lambda_k$ with finite multiplicities $m_k$ and such that $\lim_{k\to\infty} \lambda_k = +\infty$. Moreover, the first eigenvalue is simple ($m_1 = 1$) and it satisfies the following min-max characterization

$$\lambda_1 = \min\left\{ \frac{-\int_\Sigma fJf d\Sigma}{\int_\Sigma f^2 d\Sigma} : \quad f \in \mathcal{C}^\infty(\Sigma), f \neq 0 \right\}. \tag{1.3}$$

Observe that with our criterion, a real number $\lambda$ is an eigenvalue of $J$ if and only if $Jf + \lambda f = 0$ for some smooth function $f \in \mathcal{C}^\infty(\Sigma)$, $f \neq 0$.

The Jacobi operator induces the quadratic form $Q : \mathcal{C}^\infty(\Sigma) \to \mathbb{R}$ acting on the space of smooth functions on $\Sigma$ by

$$Q(f) = -\int_\Sigma fJf d\Sigma,$$

and the index of a minimal hypersurface $\Sigma$, denoted by $\mathrm{Ind}(\Sigma)$, is defined as the maximum dimension of any subspace $V$ of $\mathcal{C}^\infty(\Sigma)$ on which $Q$ is negative definite. That is,

$$\mathrm{Ind}(\Sigma) = \max\{\dim V : V \leqslant \mathcal{C}^\infty(\Sigma), \quad Q(f) < 0 \quad \text{for every } f \in V\}.$$

Equivalently, $\mathrm{Ind}(\Sigma)$ is the number of negative eigenvalues of $J$ (counted with multiplicity), which is necessarily finite and it is given by

$$\mathrm{Ind}(\Sigma) = \sum_{\lambda_k<0} m_k < \infty.$$

A minimal hypersurface would be said to be stable if $Q(f) \geqslant 0$ for every $f \in \mathcal{C}^\infty(\Sigma)$. Equivalently, in terms of the index, stability would mean that $\mathrm{Ind}(\Sigma) = 0$.

Intuitively, $\mathrm{Ind}(\Sigma)$ measures the number of independent directions in which the hypersurface fails to minimize area. To see it, observe that if $Q(f) < 0$ for some $f \in \mathcal{C}^\infty(\Sigma)$, then $\delta_f^2\mathcal{A} < 0$ and therefore $\mathrm{Area}(\Sigma) > \mathrm{Area}(\Sigma_t)$ for small values of $t \neq 0$, in the normal variation of $\Sigma$ induced by $f$. That means that the minimal hypersurface $\Sigma$, while a critical point of the area functional, is not a local minimum. For minimal hypersurfaces in $\mathbb{S}^{n+1}$ this is always the case. In fact, taking the constant function $f = 1$ one has

$$Q(1) = -\int_\Sigma (|A|^2 + n)d\Sigma = -n\ \mathrm{Area}(\Sigma) - \int_\Sigma |A|^2 d\Sigma \leqslant -n\ \mathrm{Area}(\Sigma) < 0. \quad (1.4)$$

In particular, every compact minimal hypersurface in $\mathbb{S}^{n+1}$ is unstable.

## 2. Minimal hypersurfaces with low index

We have just seen that there exists no compact stable minimal hypersurface in $\mathbb{S}^{n+1}$. Equivalently, $\mathrm{Ind}(\Sigma) \geqslant 1$ for every compact minimal hypersurface in the sphere. In [22, Theorem 5.1.1], Simons characterized the totally geodesic equators $\mathbb{S}^n \subset \mathbb{S}^{n+1}$ as the only compact minimal hypersurfaces in $\mathbb{S}^{n+1}$ with $\mathrm{Ind}(\Sigma) = 1$. Later on, Urbano [23] when $n = 2$, and El Soufi [11] for general $n$ (see also [20, Lemma 3.1]), proved that if $\Sigma$ is not a totally geodesic equator, then not only must be $\mathrm{Ind}(\Sigma) > 1$ but in fact it must hold

$$\mathrm{Ind}(\Sigma) \geqslant n + 3.$$

Therefore, we have the following result.

**Theorem 1.** *Let $\Sigma^n$ be a compact orientable minimal hypersurface immersed into the Euclidean sphere $\mathbb{S}^{n+1}$. Then*

(i) *either* $\mathrm{Ind}(\Sigma) = 1$ *(and $\Sigma$ is a totally geodesic equator $\mathbb{S}^n \subset \mathbb{S}^{n+1}$),*
(ii) *or* $\mathrm{Ind}(\Sigma) \geqslant n + 3$.

On the other hand, apart from the totally geodesic equators, which are obtained as intersections of $\mathbb{S}^{n+1}$ with linear hyperplanes of $\mathbb{R}^{n+2}$, the easiest minimal hypersurfaces in $\mathbb{S}^{n+1}$ are the minimal Clifford tori. They are obtained by considering the standard immersions $\mathbb{S}^k(\sqrt{k/n}) \hookrightarrow \mathbb{R}^{k+1}$ and $\mathbb{S}^{n-k}(\sqrt{(n-k)/n}) \hookrightarrow \mathbb{R}^{n-k+1}$, for a given integer $k \in \{1, \dots, n-1\}$, and taking the product immersion $\mathbb{S}^k(\sqrt{k/n}) \times \mathbb{S}^{n-k}(\sqrt{(n-k)/n}) \hookrightarrow \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$. As we will see in the next section, all minimal Clifford tori have $\mathrm{Ind}(\Sigma) = n + 3$. For that reason, it has been conjectured for a long time that minimal Clifford tori are the only compact minimal hypersurfaces in $\mathbb{S}^{n+1}$ with $\mathrm{Ind}(\Sigma) = n + 3$, changing Theorem 1 into the following conjecture.

**Conjecture 2.** *Let $\Sigma^n$ be a compact orientable minimal hypersurface immersed into the Euclidean sphere $\mathbb{S}^{n+1}$. Then*

(i) *either* $\mathrm{Ind}(\Sigma) = 1$ *(and $\Sigma$ is a totally geodesic equator $\mathbb{S}^n \subset \mathbb{S}^{n+1}$),*
(ii) *or* $\mathrm{Ind}(\Sigma) \geqslant n+3$, *with equality if and only if $\Sigma$ is a minimal Clifford torus $\mathbb{S}^k(\sqrt{k/n}) \times \mathbb{S}^{n-k}(\sqrt{(n-k)/n}) \subset \mathbb{S}^{n+1}$.*

In [23], Urbano showed that the conjecture is true when $n = 2$. See also next section for some other partial answers to Conjecture 2 in the general $n$-dimensional case.

On the other hand, using also the constant function $f = 1$ as a test function in (1.3) to estimate $\lambda_1$, from (1.4) one finds that

$$\lambda_1 \leqslant \frac{Q(1)}{\mathrm{Area}(\Sigma)} \leqslant -n - \frac{1}{\mathrm{Area}(\Sigma)} \int_\Sigma |A|^2 d\Sigma \leqslant -n. \tag{2.1}$$

Moreover, equality $\lambda_1 = -n$ holds if and only if $|A| = 0$ on $\Sigma$, that is, if and only if $\Sigma$ is a totally geodesic equator $\mathbb{S}^n \subset \mathbb{S}^{n+1}$. In [22, Lemma 6.1.7] Simons proved that when $\Sigma$ is not a totally geodesic equator, then not only must be $\lambda_1 < -n$ but in fact it must hold

$$\lambda_1 \leqslant -2n.$$

Later on, Wu [24] was able to characterize the case in which equality $\lambda_1 = -2n$ holds, by showing that equality holds if and only if $\Sigma$ is a minimal Clifford torus. More recently, Perdomo [21] has considered again this problem, giving a new proof of that spectral characterization of minimal Clifford tori by the first stability eigenvalue. Summing up, we have the following result.

**Theorem 3.** *Let $\Sigma^n$ be a compact orientable minimal hypersurface immersed into the Euclidean sphere $\mathbb{S}^{n+1}$, and let $\lambda_1$ stand for the first eigenvalue of its Jacobi operator. Then*

(i) *either $\lambda_1 = -n$ (and $\Sigma$ is a totally geodesic equator $\mathbb{S}^n \subset \mathbb{S}^{n+1}$),*
(ii) *or $\lambda_1 \leqslant -2n$, with equality if and only if $\Sigma$ is a minimal Clifford torus $\mathbb{S}^k(\sqrt{k/n}) \times \mathbb{S}^{n-k}(\sqrt{(n-k)/n}) \subset \mathbb{S}^{n+1}$.*

## 3. Proof of Theorem 1 and some partial answers to Conjecture 2

Before giving the proof of Theorem 1, we will fix some notation and establish some basic formulae which will be useful throughout this paper. Let us consider $\psi : \Sigma^n \to \mathbb{S}^{n+1}$ an orientable hypersurface immersed into the unit Euclidean sphere $\mathbb{S}^{n+1}$, with normal unit vector field $N$. If $\nabla$ denotes the Levi-Civita connection on $\Sigma$, then the Gauss and Weingarten formulae for the immersion $\psi$ are given by

$$\nabla^o{}_X Y = \overline{\nabla}_X Y - \langle X, Y\rangle\psi = \nabla_X Y + \langle AX, Y\rangle N - \langle X, Y\rangle\psi, \tag{3.1}$$

and

$$AX = -\nabla^o{}_X N = -\overline{\nabla}_X N, \tag{3.2}$$

for every tangent vector fields $X, Y \in \mathcal{X}(\Sigma)$.

As is well known, the curvature tensor $R$ of the hypersurface $\Sigma$ is described in terms of $A$ by the Gauss equation of $\Sigma$, which can be written as

$$R(X,Y)Z = \langle X, Z\rangle Y - \langle Y, Z\rangle X + \langle AX, Z\rangle AY - \langle AY, Z\rangle AX, \tag{3.3}$$

for $X, Y, Z \in \mathcal{X}(\Sigma)$. Observe that our criterion for the definition of the curvature is

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z.$$

As a consequence of (3.3), the Ricci curvature of $\Sigma$ is given by

$$\mathrm{Ric}(X,Y) = (n-1)\langle X, Y\rangle + nH\langle AX, Y\rangle - \langle AX, AY\rangle. \tag{3.4}$$

It follows from here that the scalar curvature of every minimal hypersurface in $\mathbb{S}^{n+1}$ satisfies

$$S = \mathrm{trace}(\mathrm{Ric}) = n(n-1) - |A|^2 \leqslant n(n-1), \tag{3.5}$$

with equality only at points where $\Sigma$ is totally geodesic. As a consequence, the only minimal hypersurfaces in $\mathbb{S}^{n+1}$ which are isometric to a unit round sphere are the totally geodesic equators. On the other hand, the Codazzi equation of $\Sigma$ is given by

$$\nabla A(X,Y) = \nabla A(Y,X) \tag{3.6}$$

for every tangent vector fields $X, Y \in \mathcal{X}(\Sigma)$, where $\nabla A$ denotes the covariant differential of $A$,

$$\nabla A(X,Y) = (\nabla_Y A)X = \nabla_Y(AX) - A(\nabla_Y X), \quad X, Y \in \mathcal{X}(\Sigma).$$

For a fixed arbitrary vector $v \in \mathbb{R}^{n+2}$, we will consider the functions $\ell_v = \langle \psi, v\rangle$ and $f_v = \langle N, v\rangle$ defined on $\Sigma$. Observe that $\ell_v$ and $f_v$ are, respectively, the coordinates of the immersion $\psi$ and the Gauss map $N$. A standard computation, using Gauss (3.1) and Weingarten (3.2) formulae, shows that the gradient and the hessian of the functions $\ell_v$ and $f_v$ are given by

$$\begin{aligned} \nabla \ell_v &= v^\top, & (3.7)\\ \nabla^2 \ell_v(X,Y) &= -\ell_v\langle X, Y\rangle + f_v\langle AX, Y\rangle, & (3.8) \end{aligned}$$

and

$$\begin{aligned} \nabla f_v &= -A(v^\top), & (3.9)\\ \nabla^2 f_v(X,Y) &= -\langle \nabla A(v^\top, X), Y\rangle + \ell_v\langle AX, Y\rangle - f_v\langle AX, AY\rangle, & (3.10) \end{aligned}$$

for every tangent vector fields $X, Y \in \mathcal{X}(\Sigma)$. Here $v^\top \in \mathcal{X}(\Sigma)$ denotes the tangential component of $v$ along the immersion $\psi$, that is,

$$v = v^\top + f_v N + \ell_v \psi = \nabla \ell_v + f_v N + \ell_v \psi. \tag{3.11}$$

Equation (3.8) directly yields

$$\Delta \ell_v = \mathrm{tr}(\nabla^2 \ell_v) = -n\ell_v + nHf_v. \tag{3.12}$$

On the other hand, using Codazzi equation (3.6) in (3.10) we also get

$$\begin{aligned} \Delta f_v &= \mathrm{tr}(\nabla^2 f_v) = -\mathrm{tr}(\nabla_{v^\top} A) + nH\ell_v - |A|^2 f_v & (3.13)\\ &= -n\langle v^\top, \nabla H\rangle + nH\ell_v - |A|^2 f_v. \end{aligned}$$

Here we are using the fact that trace commutes with the covariant derivative, which yields

$$\mathrm{tr}(\nabla_{v^\top} A) = \nabla_{v^\top}(\mathrm{tr} A) = n\langle v^\top, \nabla H\rangle.$$

Now we are ready to prove Theorem 1.

*Proof of Theorem 1.* We already know from our previous discussions that $\mathrm{Ind}(\Sigma) \geqslant 1$ for every compact minimal hypersurface in $\mathbb{S}^{n+1}$. Moreover, if $\Sigma$ is a totally geodesic equator in $\mathbb{S}^{n+1}$, then the Jacobi operator reduces to $J = \Delta + n$, where $\Delta$ is the Laplacian operator on the unit sphere $\Sigma = \mathbb{S}^n$. In particular, the eigenvalues of $J$ are given by $\lambda_i = \mu_i - n$, where $\mu_i$ denotes the $i$-th eigenvalue of the Laplacian on $\mathbb{S}^n$, with the same multiplicity. Then, $\lambda_1 = -n$ with multiplicity 1 and $\lambda_2 = 0$. In particular, $\mathrm{Ind}(\Sigma) = 1$ for a totally geodesic equator of the sphere.

Therefore, it remains to show that, if $\Sigma$ is not a totally geodesic equator, then $\mathrm{Ind}(\Sigma) \geqslant n + 3$. When $\Sigma$ is not totally geodesic, we already know from the estimate (2.1) that $\lambda_1 < -n$ with multiplicity $m_1 = 1$. Therefore, we will prove that $\mathrm{Ind}(\Sigma) \geqslant n + 3$ by showing that $-n$ is also another negative eigenvalue of $J$ with multiplicity at least $n + 2$. Since $H = 0$, equation (3.13) implies that $\Delta f_v = -|A|^2 f_v$, and then

$$J f_v - n f_v = 0$$

for every $v \in \mathbb{R}^{n+2}$. Thus, whenever $f_v \neq 0$, the functions $f_v$ are eigenfunctions of $J$ with negative eigenvalue $-n$. We claim that if $\Sigma$ is not totally geodesic in $\mathbb{S}^{n+1}$, then the dimension of the linear subspace $V = \{f_v : v \in \mathbb{R}^{n+2}\}$ is $n + 2$. If our claim is true, then the multiplicity of $-n$ as an eigenvalue of $J$ will be at least $n + 2$ and this will finish the proof of the theorem.

To show our claim, we will follow the ideas of Urbano in [23] for the case $n = 2$. Obviously, $\dim V \leqslant n+2$. If $\dim V < n+2$, then there exists a unit vector $v \in \mathbb{R}^{n+2}$ such that $f_v = 0$ on $\Sigma$. From (3.8), that implies that $\nabla^2 \ell_v = -\ell_v \langle , \rangle$. Moreover, by (3.11) we also have

$$1 = |\nabla \ell_v|^2 + f_v^2 + \ell_v^2 = |\nabla \ell_v|^2 + \ell_v^2,$$

which in particular implies that the function $\ell_v$ cannot be constant on $\Sigma$. Then a classical result by Obata [18, Theorem A] implies that $\Sigma$ is isometric to a unit round sphere. But we have already seen as a consequence of Gauss equation (3.3) that the only minimal hypersurfaces in $\mathbb{S}^{n+1}$ which are isometric to a unit round sphere are the totally geodesic equators. Thus, if $\Sigma$ is not totally geodesic we have $\dim V = n + 2$, as claimed. We also refer the reader to [20, Lemma 3.1] for an alternative proof of our claim, using a more geometric argument. $\square$

Let us consider now a minimal Clifford torus $\mathbb{S}^k(\sqrt{k/n}) \times \mathbb{S}^{n-k}(\sqrt{(n-k)/n})$ in $\mathbb{S}^{n+1}$. At a point $(x, y) \in \mathbb{S}^k(\sqrt{k/n}) \times \mathbb{S}^{n-k}(\sqrt{(n-k)/n})$, the vector field

$$N(x, y) = \left( \sqrt{\frac{n-k}{k}} x, -\sqrt{\frac{k}{n-k}} y \right)$$

defines a normal unit vector at the point $(x, y)$. With respect to this orientation, its principal curvatures are given by

$$\kappa_1 = \cdots = \kappa_k = -\sqrt{\frac{n-k}{k}}, \quad \kappa_{k+1} = \cdots = \kappa_n = \sqrt{\frac{k}{n-k}}.$$

Then, every minimal Clifford torus has $|A|^2 = n$. In particular, its Jacobi operator reduces to $J = \Delta + 2n$, where $\Delta$ is the Laplacian on the product manifold $\Sigma = \mathbb{S}^k(\sqrt{k/n}) \times \mathbb{S}^{n-k}(\sqrt{(n-k)/n})$, and the eigenvalues of $J$ are given by $\lambda_i = \mu_i - 2n$, where $\mu_i$ are the eigenvalues of $\Delta$. Therefore, the index of $\Sigma$ reduces to the number of eigenvalues of $\Delta$ (counted with multiplicity) which are strictly less than $2n$.

To compute it, simply recall that if $\alpha$ is an eigenvalue of the Laplacian on a Riemannian manifold $M$ with multiplicity $m_\alpha$ and $\beta$ is an eigenvalue of the Laplacian on a Riemannian manifold $N$ with multiplicity $m_\beta$, then $\mu = \alpha+\beta$ is an eigenvalue of the Laplacian on the product manifold $M\times N$, and the multiplicity of $\mu$ is the sum of the products $m_\alpha m_\beta$ for all the possible values of $\alpha$ and $\beta$ satisfying $\mu = \alpha+\beta$ [8]. In our case, the eigenvalues of the Laplacian on $\mathbb{S}^k(\sqrt{k/n})$ are given by

$$\alpha_i = \frac{n(i-1)(k+i-2)}{k}, \quad i = 1, 2, 3, \ldots,$$

with multiplicities

$$m_{\alpha_1} = 1, \quad , m_{\alpha_2} = k+1,$$

and

$$m_{\alpha_i} = \binom{k+i-1}{i-1} - \binom{k+i-3}{i-3}, \quad i = 3, 4, \ldots,$$

and the eigenvalues of the Laplacian on $\mathbb{S}^{n-k}(\sqrt{(n-k)/n})$ are given by

$$\beta_j = \frac{n(j-1)(n-k+j-2)}{n-k}, \quad j = 1, 2, 3, \ldots,$$

with multiplicities

$$m_{\beta_1} = 1, \quad , m_{\beta_2} = n-k+1,$$

and

$$m_{\beta_j} = \binom{n-k+j-1}{j-1} - \binom{n-k+j-3}{j-3}, \quad j = 3, 4, \ldots.$$

It easily follows from here that $\mu_1 = 0$ with multiplicity 1, $\mu_2 = \alpha_1+\beta_2 = \alpha_2+\beta_1 = n$ with multiplicity $n+2$ and $\mu_3 = \alpha_2+\beta_2 = 2n$. Therefore, all minimal Clifford tori in $\mathbb{S}^{n+1}$ have $\mathrm{Ind}(\Sigma) = n+3$, which supports Conjecture 2.

In [23] Urbano obtained the following characterization of minimal Clifford tori in $\mathbb{S}^3$, solving Conjecture 2 when $n = 2$.

**Theorem 4.** *Let $\Sigma^2$ be a compact orientable minimal surface immersed into $\mathbb{S}^3$, which is not a totally geodesic equator. Then $\mathrm{Ind}(\Sigma) \geqslant 5$, with equality if and only if $\Sigma^2$ is a minimal Clifford torus $\mathbb{S}^1(\sqrt{1/2}) \times \mathbb{S}^1(\sqrt{1/2}) \subset \mathbb{S}^3$.*

Later on, Guadalupe, Brasil Jr. and Delgado [14] showed that the conjecture is true for every dimension $n$, under the additional hypothesis that $\Sigma$ has constant scalar curvature, obtaining the following result.

**Theorem 5.** *Let $\Sigma^n$ be a compact orientable minimal hypersurface immersed into $\mathbb{S}^{n+1}$, which is not a totally geodesic equator. Assume that $\Sigma$ has constant scalar curvature. Then $\mathrm{Ind}(\Sigma) \geqslant n+3$, with equality if and only if $\Sigma^n$ is a minimal Clifford torus $\mathbb{S}^k(\sqrt{k/n}) \times \mathbb{S}^{n-k}(\sqrt{(n-k)/n}) \subset \mathbb{S}^{n+1}$.*

More recently, Perdomo [20] has showed that the conjecture is also true for every dimension $n$ with an additional assumption about the symmetries of $\Sigma$, and, in particular, the conjecture is true for minimal hypersurfaces with antipodal symmetry.

## 4. Proof of Theorem 3

The proof of Theorem 3 makes use of a celebrated formula for the Laplacian of the function $|A|^2$ on $\Sigma$, which was established by Simons in [22]. Specifically, for the case of minimal hypersurfaces in $\mathbb{S}^{n+1}$, Simons formula reads as follows,

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + (n - |A|^2)|A|^2. \tag{4.1}$$

To give a proof of (4.1), let us introduce the following standard notation. Let $T_1, T_2 : \mathcal{X}(\Sigma) \to \mathcal{X}(\Sigma)$ be two self-adjoint operators. Then

$$\langle T_1, T_2 \rangle = \operatorname{tr}(T_1 \circ T_2) = \sum_{i=1}^{n} \langle T_1 E_i, T_2 E_i \rangle$$

and

$$\langle \nabla T_1, \nabla T_2 \rangle = \sum_{i,j=1}^{n} \langle \nabla T_1(E_i, E_j), \nabla T_2(E_i, E_j) \rangle,$$

where $\{E_1, \ldots, E_n\}$ is a local orthonormal frame on $\Sigma$. Recall that, in our notation,

$$\nabla T_i(X, Y) = (\nabla_Y T_i)X = \nabla_Y(T_i X) - T_i(\nabla_Y X)$$

for $X, Y \in \mathcal{X}(\Sigma)$. On the other hand, the rough Laplacian of an operator $T : \mathcal{X}(\Sigma) \to \mathcal{X}(\Sigma)$ is defined as the operator $\Delta T : \mathcal{X}(\Sigma) \to \mathcal{X}(\Sigma)$ given by

$$\Delta T(X) = \operatorname{tr}(\nabla^2 T(X, \cdot, \cdot)) = \sum_{i=1}^{n} \nabla^2 T(X, E_i, E_i).$$

Recall again that in our notation, $\nabla^2 T(X, Y, Z) = (\nabla_Z \nabla T)(X, Y)$.

Consider now $\Sigma$ an orientable hypersurface immersed in $\mathbb{S}^{n+1}$, which in principle we do not assume to be minimal. A standard tensor computation implies

$$\frac{1}{2}\Delta|A|^2 = \frac{1}{2}\Delta\langle A, A \rangle = |\nabla A|^2 + \langle A, \Delta A \rangle. \tag{4.2}$$

By the Codazzi equation (3.6), we know that $\nabla A$ is symmetric and, hence, $\nabla^2 A$ is also symmetric in its two first variables,

$$\nabla^2 A(X, Y, Z) = \nabla^2 A(Y, X, Z), \quad X, Y, Z \in \mathcal{X}(\Sigma).$$

Regarding to the symmetries of $\nabla^2 A$ in the other variables, it is not difficult to see that

$$\nabla^2 A(X, Y, Z) = \nabla^2 A(X, Z, Y) - R(Z, Y)AX + A(R(Z, Y)X).$$

Thus, using the Gauss equation (3.3) we conclude from here that

$$\begin{aligned} \Delta A(X) &= \sum_{i=1}^{n} \left( \nabla^2 A(E_i, E_i, X) - R(E_i, X)AE_i + A(R(E_i, X)E_i) \right) \quad (4.3)\\ &= \operatorname{tr}(\nabla_X(\nabla A)) - nHX + (n - |A|^2)AX + nHA^2X \\ &= n\nabla_X(\nabla H) - nHX + (n - |A|^2)AX + nHA^2X, \end{aligned}$$

where we have used the facts that trace commutes with $\nabla_X$ and that $\mathrm{tr}(\nabla A) = \nabla H$. In particular, if $\Sigma$ is a minimal hypersurface in $\mathbb{S}^{n+1}$ we conclude that

$$\Delta A = (n - |A|^2)A,$$

which jointly with (4.2) implies (4.1).

As a first application of Simons formula (4.1), we have the following result.

**Theorem 6.** *Let $\Sigma$ be a compact orientable minimal hypersurface immersed into the Euclidean sphere $\mathbb{S}^{n+1}$, and assume that $|A| \leqslant \sqrt{n}$ on $\Sigma$. Then*

(i) *either $|A| = 0$ (and $\Sigma$ is a totally geodesic equator $\mathbb{S}^n \subset \mathbb{S}^{n+1}$),*
(ii) *or $|A| = \sqrt{n}$ and $\Sigma$ is a minimal Clifford torus.*

Part (i) and the sharp bound given in (ii) are due to Simons [22, Corollary 5.3.2]. On the other hand, the characterization of minimal Clifford tori given in (ii), which is local, was obtained independent and simultaneously by Chern, do Carmo and Kobayashi [10] and Lawson [15].

*Proof of Theorem 6.* Integrating (4.1) on $\Sigma$, and using Stokes' theorem and the hypothesis $|A|^2 \leqslant n$, we obtain that

$$0 \leqslant \int_\Sigma (n - |A|^2)|A|^2 d\Sigma \leqslant -\int_\Sigma |\nabla A|^2 d\Sigma \leqslant 0.$$

Therefore $|\nabla A|^2 = 0$ on $\Sigma$, and either $|A| = 0$ (and $\Sigma$ is totally geodesic) or $|A| = \sqrt{n}$. This proves part (i) of the theorem and the first statement of part (ii). If $|A| = \sqrt{n}$, then a local argument using the fact that $\nabla A = 0$ implies that $\Sigma$ has exactly two constant principal curvatures

$$\mp\sqrt{\frac{n-k}{k}} \quad \text{and} \quad \pm\sqrt{\frac{k}{n-k}}$$

with multiplicities $k \geqslant 1$ and $n-k \geqslant 1$, respectively (for the details, see [10] or the proof of [15, Lemma 1]). In other words, $\Sigma$ is a minimal isoparametric hypersurface of $\mathbb{S}^{n+1}$ with two distinct principal curvatures, and from a classical result by Cartan [9], $\Sigma$ must be a minimal Clifford tori. Actually, in [9] Cartan showed that an isoparametric hypersurface of $\mathbb{S}^{n+1}$ with two distinct principal curvatures must be an open piece of a standard product $\mathbb{S}^k(r)\times\mathbb{S}^{n-k}(\sqrt{1-r^2}) \subset \mathbb{S}^{n+1}$ with $0 < r < 1$, but $\Sigma$ being minimal it must be $r = \sqrt{k/n}$. $\square$

The proof of Theorem 3 below makes use also of the following auxiliary result, which can be found in [7] (see also [24, Lemma 1]).

**Lemma 7.** *Let $\Sigma^n$ be a Riemannian manifold, and consider $T : \mathcal{X}(\Sigma) \to \mathcal{X}(\Sigma)$ a symmetric tensor on $\Sigma$ such that $\mathrm{tr}(T) = 0$ and its covariant differential $\nabla T$ is symmetric. Then*

$$|\nabla|T|^2|^2 \leqslant \frac{4n}{n+2}|T|^2|\nabla T|^2.$$

*Proof of Theorem 3.* We already know from (2.1) that $\lambda_1 \leqslant -n$ with equality if and only if $\Sigma$ is a totally geodesic equator. Then, assume that $\Sigma$ is not totally geodesic and consider, for every $\varepsilon > 0$, the positive smooth function $f_\varepsilon = \sqrt{\varepsilon + |A|^2}$. We will use $f_\varepsilon$ as a test function to estimate $\lambda_1$ in (1.3). Observe that

$$\Delta f_\varepsilon = \frac{1}{2\sqrt{\varepsilon + |A|^2}}\Delta|A|^2 - \frac{1}{4(\varepsilon + |A|^2)^{3/2}}|\nabla|A|^2|^2.$$

Therefore, using Simons formula (4.1) we obtain

$$f_\varepsilon \Delta f_\varepsilon = (n - |A|^2)|A|^2 + |\nabla A|^2 - \frac{1}{4(\varepsilon + |A|^2)}|\nabla|A|^2|^2.$$

On the other hand, Lemma 7 applied to $A$ yields

$$|\nabla A|^2 - \frac{1}{4(\varepsilon + |A|^2)}|\nabla|A|^2|^2 \geqslant \frac{2}{n+2}|\nabla A|^2,$$

so that

$$f_\varepsilon \Delta f_\varepsilon \geqslant (n - |A|^2)|A|^2 + \frac{2}{n+2}|\nabla A|^2.$$

Then,

$$-f_\varepsilon J f_\varepsilon = -f_\varepsilon \Delta f_\varepsilon - (n + |A|^2) f_\varepsilon^2 \leqslant -2n|A|^2 - \frac{2}{n+2}|\nabla A|^2 - \varepsilon(n + |A|^2).$$

Therefore, using $f_\varepsilon$ as a test function in (1.3), we get

$$\begin{aligned} \lambda_1 \int_\Sigma f_\varepsilon^2 d\Sigma &\leqslant -\int_\Sigma f_\varepsilon J f_\varepsilon d\Sigma \\ &\leqslant -2n \int_\Sigma |A|^2 d\Sigma - \frac{2}{n+2}\int_\Sigma |\nabla A|^2 d\Sigma - \varepsilon \int_\Sigma (n + |A|^2) d\Sigma. \end{aligned} \tag{4.4}$$

Since $\Sigma$ is not totally geodesic, then

$$\lim_{\varepsilon \to 0} \int_\Sigma f_\varepsilon^2 d\Sigma = \int_\Sigma |A|^2 d\Sigma > 0.$$

Now, letting $\varepsilon \to 0$ in (4.4) we conclude from here that

$$\lambda_1 \leqslant -2n - \frac{2}{n+2}\frac{\int_\Sigma |\nabla A|^2 d\Sigma}{\int_\Sigma |A|^2 d\Sigma} \leqslant -2n.$$

Moreover, if $\lambda_1 = -2n$ then $|\nabla A|^2 = 0$ on $\Sigma$, and Lemma 7 implies that $|A|^2$ is a positive constant. Thus $J = \Delta + |A|^2 + n$, where $|A|^2 + n$ is a constant, and the first eigenvalue of $J$ is simply the constant $-(|A|^2 + n) = \lambda_1 = -2n$. Therefore, $|A|^2 = n$ and by Theorem 6 we conclude that $\Sigma$ must be a minimal Clifford torus. $\square$

In [21], Perdomo gave another proof of the characterization of minimal Clifford tori by the equality $\lambda_1 = -2n$. His proof is based on a maximum principle, and it works as follows. Let us assume that $\lambda_1 = -2n$. In particular, by (2.1) we know that $\Sigma$ is not totally geodesic. Let $\mathcal{U} = \{p \in \Sigma : |A|(p) > 0\}$ be the (non-empty) open subset of non-geodesic points of $\Sigma$. The function $|A|$ is smooth on $\mathcal{U}$. Writing $|A| = \sqrt{|A|^2}$ and using Simons formula (4.1), we obtain that

$$\Delta|A| = \frac{\Delta|A|^2}{2|A|} - \frac{|\nabla|A|^2|^2}{4|A|^3} = |A|(n - |A|^2) + \frac{1}{|A|}\left(|\nabla A|^2 - \frac{|\nabla|A|^2|^2}{4|A|^2}\right) \tag{4.5}$$

on $\mathcal{U}$. By Lemma 7 we also have

$$|\nabla A|^2 - \frac{|\nabla|A|^2|^2}{4|A|^2} \geqslant \frac{2}{n+2}|\nabla A|^2.$$

Therefore, using this into (4.5) we obtain

$$\Delta|A| \geqslant |A|(n - |A|^2) + \frac{2}{n+2}\frac{|\nabla A|^2}{|A|^2} \geqslant |A|(n - |A|^2) \tag{4.6}$$

on $\mathcal{U}$. Moreover, if equality $\Delta|A| = |A|(n - |A|^2)$ holds at a point $p \in \mathcal{U}$, then $\nabla A(p) = 0$.

As is well known, the first eigenvalue $\lambda_1 = -2n$ is simple, and its eigenspace is generated by a positive smooth function $\varrho \in \mathcal{C}^\infty(\Sigma)$. Then $J\varrho = 2n\varrho$ or, equivalently,

$$\Delta\varrho = (n - |A|^2)\varrho.$$

Observe that

$$\Delta\varrho^{-1} = -\varrho^{-2}\Delta\varrho + 2\varrho^{-3}|\nabla\varrho|^2 = -(n - |A|^2)\varrho^{-1} + 2\varrho^{-3}|\nabla\varrho|^2. \tag{4.7}$$

Consider the smooth function $f$ defined on $\mathcal{U}$ by $f = |A|\varrho^{-1}$. A straightforward computation using (4.6) and (4.7) yields

$$\Delta f = -|A|\varrho^{-1}(n - |A|^2) + \varrho^{-1}\Delta|A| - 2\varrho^{-1}\langle\nabla f, \nabla\varrho\rangle \geqslant -2\varrho^{-1}\langle\nabla f, \nabla\varrho\rangle. \tag{4.8}$$

Summing up,

$$Lf \geqslant 0 \quad \text{on} \quad \mathcal{U}, \tag{4.9}$$

where $L$ is the differential operator on $\mathcal{U}$ given by $Lf = \Delta f + 2\varrho^{-1}\langle\nabla f, \nabla\varrho\rangle$. Let $p_0 \in \mathcal{U}$ be a point where the function $f$ attains its (positive) maximum on $\Sigma$, and let $\Omega \subset \mathcal{U}$ be a region around $p_0$ on which $f$ is greater than some positive constant. Since the maximum of $f$ in $\Omega$ is attained in the interior of $\Omega$, by (4.9) and the maximum principle applied to $L$ we deduce that $f$ is constant on $\Omega$. Since $\Sigma$ is connected, we conclude that $f$ is a positive constant on the whole $\Sigma = \mathcal{U}$. Therefore, equality trivially holds in (4.9). That means that the inequality in (4.8) must be an equality, but this happens if and only if equality happens in (4.6), which implies that $\nabla A = 0$ on $\Sigma$. Then, by Lemma 7 we know that $|A|$ is a positive constant, and since $\Delta|A| = 0 = |A|(n - |A|^2)$, we obtain that $|A|^2 = n$. Then Theorem 6 implies that $\Sigma$ must be a minimal Clifford torus. This finishes Perdomo's proof.

## 5. Stability and index of constant mean curvature hypersurfaces

Let $\psi : \Sigma^n \to \mathbb{S}^{n+1}$ be a compact orientable hypersurface immersed into the unit Euclidean sphere. As another consequence of the first variation formula for the area (1.1), we have that $\Sigma$ has constant mean curvature (not necessarily zero) if and only if $\delta_f \mathcal{A} = 0$ for every smooth function $f \in \mathcal{C}^\infty(\Sigma)$ satisfying the additional condition $\int_\Sigma f d\Sigma = 0$. To see it, assume that $\delta_f \mathcal{A} = 0$ for every $f \in \mathcal{C}^\infty(\Sigma)$ satisfying $\int_\Sigma f d\Sigma = 0$, and write $H = H_0 + (H - H_0)$, where

$$H_0 = \frac{1}{\mathrm{Area}(\Sigma)} \int_\Sigma H d\Sigma.$$

Since $\int_\Sigma (H - H_0) d\Sigma = 0$, then

$$\delta_{H-H_0} \mathcal{A} = -n \int_\Sigma (H - H_0) H d\Sigma = -n \int_\Sigma (H - H_0)^2 d\Sigma = 0,$$

but this implies that $H = H_0$ is constant on $\Sigma$.

Geometrically, the additional condition $\int_\Sigma f d\Sigma = 0$ means that the variations under consideration preserve a certain volume function. In fact, if $\psi_t$ is the normal variation induced by a smooth function $f \in \mathcal{C}^\infty(\Sigma)$, then the volume function is the function $\mathcal{V} : (-\varepsilon, \varepsilon) \to \mathbb{R}$ defined by

$$\mathcal{V}(t) = \int_{[0,t]\times\Sigma} \Psi^*(dV),$$

where $dV$ denotes the $(n+1)$-dimensional volume element of $\mathbb{S}^{n+1}$ and $\Psi : (-\varepsilon, \varepsilon) \times \Sigma \to \mathbb{S}^{n+1}$ is the variation of $\psi$, $\Psi(t, p) = \psi_t(p)$. Then, the first variation of $\mathcal{V}$ is given by

$$\delta_f \mathcal{V} = \frac{d\mathcal{V}}{dt}(0) = \int_\Sigma f d\Sigma.$$

We refer the reader to [5, 6] for the details. A variation is said to be volume-preserving if $\mathcal{V}(t) = \mathcal{V}(0) = 0$ for all $t$. As shown by Barbosa, do Carmo and Eschenburg in [6, Lemma 2.2], given a smooth function $f \in \mathcal{C}^\infty(\Sigma)$ with $\int_\Sigma f d\Sigma = 0$, there exists a volume-preserving normal variation whose variation vector field is $fN$. As a consequence, $\Sigma$ has constant mean curvature (not necessarily zero) if and only if $\delta\mathcal{A} = 0$ for every volume-preserving variation of $\Sigma$. In other words, constant mean curvature hypersurfaces are characterized as critical points of the area functional when restricted to volume-preserving variations.

As in the case of minimal hypersurfaces, the stability operator of this variational problem is given by the second variation formula of the area (1.2), and similarly the corresponding quadratic form is also given by

$$Q(f) = -\int_\Sigma f J f d\Sigma,$$

with Jacobi operator $J = \Delta + |A|^2 + n$. However, in contrast to the case of minimal hypersurfaces, in the case of hypersurfaces with constant mean curvature one can consider two different eigenvalue problems: the usual Dirichlet problem, associated with the quadratic form $Q$ acting on the whole space of smooth functions on $\Sigma$, and the so called *twisted* Dirichlet problem, associated with the same quadratic form $Q$, but restricted to the subspace of smooth functions $f \in \mathcal{C}^\infty(\Sigma)$ satisfying the additional condition $\int_M \Sigma f d\Sigma = 0$.

Similarly, there are two different notions of stability and index, the strong stability and strong index, denoted by $\mathrm{Ind}(\Sigma)$ and associated to the usual Dirichlet problem, and the weak stability and weak index, denoted by $\mathrm{Ind}_T(\Sigma)$ and associated to the twisted Dirichlet problem. Thus, the strong index is simply

$$\mathrm{Ind}(\Sigma) = \max\{\dim V : V \leqslant \mathcal{C}^\infty(\Sigma), \quad Q(f) < 0 \quad \text{for every } f \in V\},$$

and $\Sigma$ is called strongly stable if and only if $\mathrm{Ind}(\Sigma) = 0$. On the other hand, the weak index is

$$\mathrm{Ind}_T(\Sigma) = \max\{\dim V : V \leqslant \mathcal{C}_T^\infty(\Sigma), \quad Q(f) < 0 \quad \text{for every } f \in V\},$$

where

$$\mathcal{C}_T^\infty(\Sigma) = \{f \in \mathcal{C}^\infty(\Sigma) : \textstyle\int_\Sigma f d\Sigma = 0\},$$

and $\Sigma$ is called weakly stable if and only if $\mathrm{Ind}_T(\Sigma) = 0$. From a geometrical point of view, the weak index is more natural than the strong index. However, from an analytical point of view, the strong index is more natural and easier to use.

In [4], Barbosa and Bérard studied in depth the twisted Dirichlet problem, comparing the eigenvalues of this problem with the eigenvalues of the usual Dirichlet problem. For instance, it easily follows from the min-max principle that both spectra are interwined by

$$\lambda_1 < \lambda_1^T \leqslant \lambda_2 \leqslant \lambda_2^T \leqslant \cdots, \tag{5.1}$$

where

$$\mathrm{Spec}(J) = \{\lambda_1 < \lambda_2 < \lambda_3 < \cdots\}$$

is the usual spectrum of $J$ and

$$\mathrm{Spec}_T(J) = \{\lambda_1^T < \lambda_2^T < \lambda_3^T < \cdots\}$$

is its twisted spectrum.

When dealing with constant mean curvature hypersurfaces, instead of the second fundamental form $A$, it is more convenient to work with the so called traceless second fundamental form, which is given by $\phi = A - HI$, where $I$ denotes the identity operator on $\mathcal{X}(\Sigma)$. Observe that

$$\mathrm{tr}(\phi) = 0 \quad \text{and} \quad |\phi|^2 = |A|^2 - nH^2 \geqslant 0,$$

with equality if and only if $\Sigma$ is totally umbilical. For that reason $\phi$ is also called the total umbilicity tensor of $\Sigma$. In terms of $\phi$, the Jacobi operator is given by

$$J = \Delta + |\phi|^2 + n(1 + H^2).$$

Using again the constant function $f = 1$ as a test function for estimating $\mathrm{Ind}(\Sigma)$ one has

$$\begin{aligned} Q(1) &= -\int_\Sigma (|\phi|^2 + n(1+H^2))d\Sigma = -n(1+H^2)\mathrm{Area}(\Sigma) - \int_\Sigma |\phi|^2 d\Sigma \\ &\leqslant -n(1+H^2)\mathrm{Area}(\Sigma) < 0. \end{aligned}$$

In particular, $\mathrm{Ind}(\Sigma) \geqslant 1$ for every constant mean curvature hypersurface in $\mathbb{S}^{n+1}$, which means that there is no strongly stable constant mean curvature hypersurface in $\mathbb{S}^{n+1}$. It also follows from here that

$$\lambda_1 \leqslant \frac{Q(1)}{\mathrm{Area}(\Sigma)} \leqslant -n(1+H^2) - \frac{1}{\mathrm{Area}(\Sigma)} \int_\Sigma |\phi|^2 d\Sigma \leqslant -n(1+H^2), \tag{5.2}$$

with equality $\lambda_1 = -n(1+H^2)$ if and only if $\Sigma$ is a totally umbilical round sphere $\mathbb{S}^n(r) \subset \mathbb{S}^{n+1}$. Observe that, in general, $\lambda_1 < 0$ contributes to $\mathrm{Ind}(\Sigma)$ but not to $\mathrm{Ind}_T(\Sigma)$ because its eigenspace is generated by a positive smooth function $\varrho \in \mathcal{C}^\infty(\Sigma)$ which does not satisfy the additional condition $\int_\Sigma \varrho d\Sigma = 0$. On the other hand, in [13, Theorem 2.1], El Soufi and Ilias derived a sharp upper bound for the second eigenvalue of a Scrödinger operator of the form $\Delta + q$ of a compact submanifold $\Sigma^n$ of a Riemannian space form, in terms of the total mean curvature of $\Sigma$ and the mean value of the potential $q$. In particular, for the Jacobi operator of a constant mean curvature hypersurface in $\mathbb{S}^{n+1}$, their estimate yields

$$\lambda_2 \leqslant -\frac{1}{\mathrm{Area}(\Sigma)} \int_\Sigma |\phi|^2 d\Sigma \leqslant 0, \tag{5.3}$$

with equality $\lambda_2 = 0$ if and only if $\Sigma$ is totally umbilical. See also [12] for another interesting bound for the second eigenvalue of a Scrödinger operator, and its applications to the study of minimal and constant mean curvature hypersurfaces.

## 6. Weakly stable constant mean curvature hypersurfaces

We have just shown that there is no compact strongly stable constant mean curvature hypersurface in $\mathbb{S}^{n+1}$. In contrast to this, Barbosa, do Carmo and Eschenburg [6, Theorem 1.2] characterized the totally umbilical round spheres $\mathbb{S}^n(r) \subset \mathbb{S}^{n+1}$ as the only compact weakly stable constant mean curvature hypersurfaces in $\mathbb{S}^{n+1}$.

**Theorem 8.** *Let $\Sigma^n$ be a compact orientable constant mean curvature hypersurface immersed into the unit Euclidean sphere $\mathbb{S}^{n+1}$. Then $\Sigma$ is weakly stable if and only if $\Sigma$ is a totally umbilical round sphere $\mathbb{S}^n(r) \subset \mathbb{S}^{n+1}$.*

*Proof.* If $\Sigma$ is a totally umbilical round sphere $\mathbb{S}^n(r)$ in $\mathbb{S}^{n+1}$ with radius $0 < r < 1$, then $1/r^2 = 1 + H^2$ and the Jacobi operator reduces to $J = \Delta + n/r^2$. Therefore, the eigenvalues of $J$ are given by $\lambda_i = \mu_i - n/r^2$, where $\mu_i$ is the $i$-th eigenvalue of the Laplacian operator on $\mathbb{S}^n(r)$, with the same multiplicity. In particular, $\lambda_1 = -n/r^2 < 0$ with multiplicity 1 and its associated eigenfunctions are the constant functions. Therefore, since all the other eigenfunctions of $J$ (for the usual Dirichlet problem) are orthogonal to the constant functions, they do satisfy the additional condition $\int_\Sigma f d\Sigma = 0$. Thus, in this case we have $\lambda_i^T = \lambda_{i+1} = \mu_{i+1} - n/r^2$ for every $i \geqslant 1$. Since $\mu_2 = n/r^2$, it follows from here that $\lambda_1^T = 0$ and $\Sigma$ is weakly stable.

Conversely, assume that $\Sigma$ is a compact orientable hypersurface with constant mean curvature in $\mathbb{S}^{n+1}$ which is weakly stable. This means that

$$Q(f) \geqslant 0$$

for every smooth function $f \in \mathcal{C}^\infty(\Sigma)$ with $\int_\Sigma f d\Sigma = 0$. As in the proof of Theorem 1, we will work with the functions $\ell_v$ and $f_v$, where $v \in \mathbb{R}^{n+2}$ is a fixed arbitrary vector. Since $H$ is constant, writing $|A|^2 = nH^2 + |\phi|^2$ equation (3.13) becomes

$$\Delta f_v = nH\ell_v - |A|^2 f_v = -nH(Hf_v - \ell_v) - |\phi|^2 f_v. \tag{6.1}$$

Let us consider the function $g_v = Hf_v - \ell_v$. From (3.12) we have that $ng_v = \Delta \ell_v$, so that $g_v$ trivially satisfies the condition $\int_\Sigma g_v d\Sigma = 0$. Using (3.12) and (6.1), we easily get

$$\Delta g_v = -n(1 + H^2) g_v - H|\phi|^2 f_v,$$

and then

$$Jg_v = -|\phi|^2 \ell_v.$$

Therefore, we have that

$$Q(g_v) = -\int_\Sigma g_v J g_v d\Sigma = H \int_\Sigma |\phi|^2 f_v \ell_v d\Sigma - \int_\Sigma |\phi|^2 \ell_v^2 d\Sigma \geqslant 0$$

for every fixed arbitrary vector $v \in \mathbb{R}^{n+2}$. Let us choose $v = e_i$ as an element of the standard orthonormal basis of $\mathbb{R}^{n+2}$, $e_i = (0, \ldots, 0, \overset{i}{1}, 0 \ldots, 0)$. Then

$$0 \leqslant \sum_{i=1}^{n+2} Q(g_{e_i}) = H \int_\Sigma |\phi|^2 \sum_{i=1}^{n+2} f_{e_i} \ell_{e_i} d\Sigma - \int_\Sigma |\phi|^2 \sum_{i=1}^{n+2} \ell_{e_i}^2 d\Sigma = -\int_\Sigma |\phi|^2 d\Sigma \leqslant 0,$$

because of

$$\sum_{i=1}^{n+2} f_{e_i} \ell_{e_i} = \langle N, \psi \rangle = 0 \quad \text{and} \quad \sum_{i=1}^{n+2} \ell_{e_i}^2 = \langle \psi, \psi \rangle = 1.$$

But this implies that $|\phi|^2 = 0$ on $\Sigma$, and it must be totally umbilical. □

It is worth pointing out that Theorem 8 can be seen also as a consequence of the estimate (5.3), jointly with the interwining (5.1) of the two spectra of $J$. In fact, if $\Sigma$ is weakly stable, then we have

$$0 \leqslant \lambda_1^T \leqslant \lambda_2 \leqslant -\frac{1}{\text{Area}(\Sigma)} \int_\Sigma |\phi|^2 d\Sigma \leqslant 0,$$

which implies again that $|\phi|^2 = 0$ and $\Sigma$ must be totally umbilical.

## 7. Constant mean curvature hypersurfaces with low index

Apart from the totally umbilical spheres, the easiest constant mean curvature hypersurfaces in $\mathbb{S}^{n+1}$ are the constant mean curvature Clifford tori. They are obtained by considering the standard immersions $\mathbb{S}^k(r) \hookrightarrow \mathbb{R}^{k+1}$ and $\mathbb{S}^{n-k}(\sqrt{1-r^2}) \hookrightarrow \mathbb{R}^{n-k+1}$, for a given radius $0 < r < 1$ and integer $k \in \{1, \ldots, n-1\}$, and taking the product immersion $\mathbb{S}^k(r) \times \mathbb{S}^{n-k}(\sqrt{1-r^2}) \hookrightarrow \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$. Its principal curvatures are given by

$$\kappa_1 = \cdots = \kappa_k = -\frac{\sqrt{1-r^2}}{r}, \quad \kappa_{k+1} = \cdots = \kappa_n = \frac{r}{\sqrt{1-r^2}},$$

and its constant mean curvature $H = H(r)$ is given by

$$nH(r) = \frac{nr^2 - k}{r\sqrt{1-r^2}}.$$

In particular, $H(r) = 0$ precisely when $r = \sqrt{k/n}$, which corresponds to the minimal Clifford torus.

For the constant mean curvature Clifford tori, one has

$$|A|^2 + n = k/r^2 + (n-k)/(1-r^2)$$

and the Jacobi operator reduces to $J = \Delta + k/r^2 + (n-k)/(1-r^2)$, where $\Delta$ is the Laplacian operator on the product manifold $\mathbb{S}^k(r) \times \mathbb{S}^{n-k}(\sqrt{1-r^2})$. In particular, the eigenvalues of $J$ are given by $\lambda_i = \mu_i - (k/r^2 + (n-k)/(1-r^2))$, where $\mu_i$ are the eigenvalues of $\Delta$, and they have the same multiplicity. Thus, $\lambda_1 = -(k/r^2+(n-k)/(1-r^2)) < 0$ with multiplicity 1 and its associated eigenfunctions are the constant functions. Moreover, since all the rest of eigenfunctions of $J$ (for the usual Dirichlet problem) are orthogonal to the constant functions, they do satisfy the additional condition $\int_\Sigma f d\Sigma = 0$. Thus, similarly to the case of totally umbilical round spheres, we have $\lambda_i^T = \lambda_{i+1} = \mu_{i+1} - (k/r^2 + (n-k)/(1-r^2))$ for every $i \geqslant 1$, and $\mathrm{Ind}_T(\Sigma)$ reduces to the number of positive eigenvalues of the Laplacian operator (counted with multiplicity) which are strictly less than $k/r^2 + (n-k)/(1-r^2)$. This yields that $\mathrm{Ind}_T(\Sigma) \geqslant n+2$ for every constant mean curvature Clifford torus, and $\mathrm{Ind}_T(\Sigma) = n+2$ precisely when $k/(n+2) \leqslant r^2 \leqslant (k+2)/(n+2)$ (for the details, see [3, Section 3]). Observe that, in particular, this happens when $r^2 = k/n$, so that, the minimal Clifford tori satisfy $\mathrm{Ind}_T(\Sigma) = n+2$ when regarded as constant mean curvature hypersurfaces.

Motivated by this value of $\mathrm{Ind}_T(\Sigma)$, in [3] Alías, Brasil and Perdomo have recently obtained the following result, which extends Theorem 1 (under the additional hypothesis of constant scalar curvature) and Theorem 5 to the case of constant mean curvature hypersurfaces.

**Theorem 9.** *Let $\Sigma^n$ be a compact orientable hypersurface immersed into the Euclidean sphere $\mathbb{S}^{n+1}$ with constant mean curvature. Assume that $\Sigma$ has constant scalar curvature. Then*

(i) *either* $\mathrm{Ind}_T(\Sigma) = 0$ *(and $\Sigma$ is a totally umbilic sphere in $\mathbb{S}^{n+1}$), or*
(ii) $\mathrm{Ind}_T(\Sigma) \geqslant n+2$, *with equality if and only if $\Sigma$ is a constant mean curvature Clifford torus $\mathbb{S}^k(r) \times \mathbb{S}^{n-k}(\sqrt{1-r^2})$ with radius $\sqrt{k/(n+2)} \leqslant r \leqslant \sqrt{(k+2)/(n+2)}$.*

As observed in [3], the value of the index of the constant mean curvature Clifford tori $\mathbb{S}^k(r) \times \mathbb{S}^{n-k}(\sqrt{1-r^2})$ converges to infinity as $r$ converges either to 0 or 1. For that reason, in contrast to the case of minimal Clifford tori, it is not possible, in general, to find a characterization of all constant mean curvature Clifford tori in terms of their index.

*Proof.* We already know from Theorem 8 that $\mathrm{Ind}_T(\Sigma) = 0$ for the totally umbilical round spheres, whereas $\mathrm{Ind}_T(\Sigma) \geqslant 1$ for the rest of compact constant mean curvature hypersurfaces in $\mathbb{S}^{n+1}$, without any additional hypothesis. Then, first we need to show that, if the scalar curvature of $\Sigma$ is constant (or equivalently, $|A|^2$ is constant; see equation (3.5)), then $\mathrm{Ind}_T(\Sigma) \geqslant n+2$ for every compact constant mean curvature hypersurface in $\mathbb{S}^{n+1}$ which is not totally umbilical. That is, we need to find a subspace $V$ of $\mathcal{C}_T^\infty(\Sigma)$ with $\dim V \geqslant n+2$ on which $Q$ is negative definite.

As in the proof of Theorem 8, we will consider the functions $\ell_v$ and $f_v$, where $v \in \mathbb{R}^{n+2}$ is a fixed arbitrary vector. When $H=0$, we can take $V=\{f_v : v \in \mathbb{R}^{n+2}\}$. In fact, since $H=0$ and $|A|^2=|\phi|^2$ is a positive constant, then (6.1) implies that the functions $f_v$ are eigenfunctions of $J$ with negative eigenvalue $-n$ and that they also satisfy the condition $\int_\Sigma f_v d\Sigma = 0$. Moreover, we have also seen in the proof of Theorem 1 that, when $\Sigma$ is minimal and not totally geodesic, then $\dim V = n+2$.

Therefore, in what follows we assume that $H$ is a non-zero constant. In that case, we take $V = U_- \oplus U_+$ where

$$U_- = \{\ell_v - \alpha_- f_v : v \in \mathbb{R}^{n+2}\} \quad \text{and} \quad U_+ = \{\ell_v - \alpha_+ f_v : v \in \mathbb{R}^{n+2}\},$$

and $\alpha_\pm$ are the two different real roots of the quadratic equation

$$nH\alpha^2 + (n-|A|^2)\alpha - nH = 0.$$

That is,

$$\alpha_\pm = \frac{|A|^2 - n \pm \sqrt{D}}{2nH}, \quad \text{where} \quad D = (|A|^2-n)^2 + 4n^2H^2 > 0.$$

Using (3.12) and (6.1), it is not difficult to see that $Jf + \lambda_\pm f = 0$ for every $f \in U_\pm$, where

$$\lambda_- = \frac{-(n+|A|^2) - \sqrt{D}}{2} < \lambda_+ = \frac{-(n+|A|^2) + \sqrt{D}}{2} < 0,$$

and that they also satisfy the condition $\int_\Sigma f d\Sigma = 0$. Therefore

$$\mathrm{Ind}_T(\Sigma) \geqslant \dim V = \dim U_- + \dim U_+.$$

It remains to estimate $\dim U_- + \dim U_+$. Taking into account that $U_\pm = \mathrm{im}\varphi_\pm$, where $\varphi_\pm : \mathbb{R}^{n+2} \to \mathcal{C}^\infty(\Sigma)$ is the linear map given by $\varphi_\pm(v) = \ell_v - \alpha_\pm f_v$, we deduce that

$$\dim U_\pm = n+2 - \dim\ker\varphi_\pm.$$

On the other hand, using that $\ker\varphi_- \cap \ker\varphi_+ = \{0\}$ we also have

$$\dim\ker\varphi_- + \dim\ker\varphi_+ = \dim(\ker\varphi_- \oplus \ker\varphi_+) \leqslant n+2,$$

which implies

$$\dim U_- + \dim U_+ = 2(n+2) - (\dim\ker\varphi_- + \dim\ker\varphi_+) \geqslant n+2,$$

and therefore $\mathrm{Ind}_T(\Sigma) \geqslant n+2$. We refer the reader to [3, Section 4] for further details about this proof.

Moreover, if $\mathrm{Ind}_T(\Sigma) = n+2$ then $\dim(\ker\varphi_- \oplus \ker\varphi_+) = n+2$ which means that $\mathbb{R}^{n+2}$ splits as direct sum of the two subspaces $\ker\varphi_-$ and $\ker\varphi_+$. Then, at any point $p \in \Sigma$ the tangent space $T_p\Sigma$ splits also as a direct sum of two subspaces

$$T_p\Sigma = T_p\Sigma \cap \mathbb{R}^{n+2} = T_p\Sigma^- \oplus T_p\Sigma^+,$$

where $T_p\Sigma^- = T_p\Sigma \cap \ker\varphi_-$ and $T_p\Sigma^+ = T_p\Sigma \cap \ker\varphi_+$. Using that $\Sigma$ is not totally umbilical, and equations (3.7) and (3.9), we can see that, at each point $p \in \Sigma$, $T_p\Sigma^-$ and $T_p\Sigma^+$ are subspaces of principal directions of $T_p\Sigma$ with constant principal curvatures $-1/\alpha_-$ and $-1/\alpha_+$, respectively (see [3, Section 4] for the details). As a consequence, $\Sigma$ is a compact isoparametric hypersurface of $\mathbb{S}^{n+1}$ with two distinct principal curvatures, and from the well known rigidity result by Cartan [9] we conclude that $\Sigma$ is a standard product of the form $\mathbb{S}^k(r) \times \mathbb{S}^{n-k}(\sqrt{1-r^2})$ with radius $0 < r < 1$. Finally, from our previous discussion about the values of $\mathrm{Ind}_T(\Sigma)$ for those hypersurfaces, we conclude that it must be $\sqrt{k/(n+2)} \leqslant r \leqslant \sqrt{(k+2)/(n+2)}$. □

## 8. A sharp estimate for the first eigenvalue of the Jacobi operator

As we already know from (5.2), the first eigenvalue of the Jacobi operator of a compact hypersurface $\Sigma^n$ with constant mean curvature $H$ in $\mathbb{S}^{n+1}$ satisfies

$$\lambda_1 \leqslant -n(1+H^2),$$

with equality if and only if $\Sigma$ is a totally umbilical round sphere. As an extension of Theorem 3 to the case of constant mean curvature, Alías, Barros and Brasil in [2] have recently proved that when $\Sigma$ is not totally umbilical, then not only must be $\lambda_1 < -n(1+H^2)$ but it fact it must hold

$$\lambda_1 \leqslant -2n(1+H^2) + \frac{n(n-2)}{\sqrt{n(n-1)}}|H| \max_{\Sigma} |\phi|,$$

where $|\phi| = \sqrt{|A|^2 - nH^2}$ is the norm of the total umbilicity tensor of $\Sigma$. Moreover, they were also able to characterize the case where equality holds, obtaining the following extension of Theorem 1.

**Theorem 10.** *Let $\Sigma^n$ be a compact orientable hypersurface immersed into the Euclidean sphere $\mathbb{S}^{n+1}$ with constant mean curvature $H$, and let $\lambda_1$ stand for the first eigenvalue of its Jacobi operator. Then*

- (i) *either $\lambda_1 = -n(1+H^2)$ (and $\Sigma$ is a totally umbilic sphere in $\mathbb{S}^{n+1}$), or*
- (ii) *$\lambda_1 \leqslant -2n(1+H^2) + \frac{n(n-2)}{\sqrt{n(n-1)}}|H| \max_\Sigma |\phi|$, with equality if and only if*
  - (a) *$H = 0$ and $\Sigma$ is a minimal Clifford torus $\mathbb{S}^k(\sqrt{k/n}) \times \mathbb{S}^{n-k}(\sqrt{(n-k)/n})$, with $k = 1, \ldots, n-1$;*
  - (b) *$H \neq 0$, $n = 2$, and $\Sigma^2$ is a constant mean curvature Clifford torus $\mathbb{S}^1(r) \times \mathbb{S}^1(\sqrt{1-r^2})$ with $0 < r < 1$, $r \neq \sqrt{1/2}$;*
  - (c) *$H \neq 0$, $n \geqslant 3$, and $\Sigma^n$ is a constant mean curvature Clifford torus $\mathbb{S}^{n-1}(r) \times \mathbb{S}^1(\sqrt{1-r^2})$ with $0 < r < \sqrt{(n-1)/n}$.*

For the proof of Theorem 10, we will need the following result due to Alencar and do Carmo, which extends Theorem 6 to the case of constant mean curvature hypersurfaces.

**Theorem 11.** *Let $\Sigma^n$ be a compact orientable hypersurface immersed into the Euclidean sphere $\mathbb{S}^{n+1}$ with constant mean curvature $H$, and assume that $|\phi| \leqslant \alpha_H$,* *where*

$$\alpha_H = \frac{\sqrt{n}}{2\sqrt{n-1}}\left(\sqrt{n^2H^2+4(n-1)}-(n-2)|H|\right)$$

*is the positive root of the polynomial*

$$P_H(x) = x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}|H|x - n(1+H^2). \tag{8.1}$$

*Then*

(i) *either* $|\phi| = 0$ *(and* $\Sigma$ *is a totally umbilic sphere in* $\mathbb{S}^{n+1}$*), or*

(ii) $|\phi| = \alpha_H$ *and*

  (a) $H = 0$ *and* $\Sigma$ *is a minimal Clifford torus* $\mathbb{S}^k(\sqrt{k/n})\times\mathbb{S}^{n-k}(\sqrt{(n-k)/n})$, *with* $k = 1, \ldots, n-1$*;*

  (b) $H \neq 0$, $n = 2$, *and* $\Sigma^2$ *is a constant mean curvature Clifford torus* $\mathbb{S}^1(r)\times\mathbb{S}^1(\sqrt{1-r^2})$ *with* $0 < r < 1$, $r \neq \sqrt{1/2}$*;*

  (c) $H \neq 0$, $n \geqslant 3$, *and* $\Sigma^n$ *is a constant mean curvature Clifford torus* $\mathbb{S}^{n-1}(r)\times\mathbb{S}^1(\sqrt{1-r^2})$ *with* $0 < r < \sqrt{(n-1)/n}$.

The proofs of Theorem 10 and Theorem 11 make use of an extension of Simons formula (4.1) for the case of hypersurfaces with constant mean curvature, which is due to Nomizu and Smyth [17]. To see it, consider $\phi = A - HI$ the total umbilicity tensor and recall that $|\phi|^2 = |A|^2 - nH^2$. Since we are assuming now that $H$ is constant, we have that $\nabla\phi = \nabla A$ and $\Delta\phi = \Delta A$, and (4.2) can be written in terms of $\phi$ as follows

$$\frac{1}{2}\Delta|\phi|^2 = \frac{1}{2}\Delta|A|^2 = |\nabla\phi|^2 + \langle\phi,\Delta\phi\rangle + H\langle I,\Delta\phi\rangle. \tag{8.2}$$

Moreover, by (4.3) we also have

$$\begin{aligned}\Delta\phi &= \Delta A = nHA^2 + (n-|A|^2)A - nHI\\ &= nH\phi^2 + (n(1+H^2)-|\phi|^2)\phi - H|\phi|^2 I.\end{aligned}$$

Then, taking into account that $\mathrm{tr}(\phi) = 0$, (8.2) becomes

$$\frac{1}{2}\Delta|\phi|^2 = |\nabla\phi|^2 + (n(1+H^2)-|\phi|^2)|\phi|^2 + nH\mathrm{tr}(\phi^3). \tag{8.3}$$

As a first application of equation (8.3), we may give the proof of Theorem 11. For the proof, we will also need the following auxiliary result, known as Okumura lemma, which can be found in [19] and [1, Lemma 2.6].

**Lemma 12.** *Let* $a_1, \ldots, a_n$ *be real numbers such that* $\sum_{i=1}^n a_i = 0$. *Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}(\sum_{i=1}^n a_i^2)^{3/2} \leqslant \sum_{i=1}^n a_i^3 \leqslant \frac{n-2}{\sqrt{n(n-1)}}(\sum_{i=1}^n a_i^2)^{3/2}.$$

*Moreover, equality holds in the right-hand (respectively, left-hand) side if and only if* $(n-1)$ *of the* $a_i$*'s are nonpositive (respectively, nonnegative) and equal.*

*Proof of Theorem 11.* Since $\operatorname{tr}(\phi)=0$, we may use Lemma 12 to estimate $\operatorname{tr}(\phi^3)$ as follows

$$|\operatorname{tr}(\phi^3)| \leqslant \frac{n-2}{\sqrt{n(n-1)}}|\phi|^3,$$

and then

$$nH\operatorname{tr}(\phi^3) \geqslant -n|H||\operatorname{tr}(\phi^3)| \geqslant -\frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi|^3.$$

Using this in (8.3), we find

$$\begin{aligned} \frac{1}{2}\Delta|\phi|^2 &\geqslant |\nabla\phi|^2 + (n(1+H^2) - |\phi|^2)|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi|^3 \qquad (8.4) \\ &= |\nabla\phi|^2 - |\phi|^2 P_H(|\phi|), \end{aligned}$$

where $P_H$ is given by (8.1). That is,

$$0 \leqslant |\nabla\phi|^2 \leqslant \frac{1}{2}\Delta|\phi|^2 + |\phi|^2 P_H(|\phi|).$$

Integrating this inequality on $\Sigma$, and using Stokes' theorem and the hypothesis $|\phi| \leqslant \alpha_H$, we find that

$$0 \leqslant \int_\Sigma |\nabla\phi|^2 d\Sigma \leqslant \int_\Sigma |\phi|^2 P_H(|\phi|) d\Sigma \leqslant 0,$$

because of $P_H(x) \leqslant 0$ when $x \in [0, \alpha_H]$. Therefore, $|\nabla\phi| = |\nabla A| = 0$ on $\Sigma$, and either $|\phi| = 0$ (and $\Sigma$ is totally umbilical) or $|\phi| = \alpha_H$. This proves part (i) of Theorem 11 and the first statement of part (ii). If $H=0$, then $|\phi| = |A|$, $\alpha_0 = \sqrt{n}$ and part (ii)(a) just follows from (ii) in Theorem 6. If $H \neq 0$ and $|\phi| = \alpha_H$, then a local argument using the facts that $\nabla A = 0$ and that equality holds in the right-hand side of Lemma 12 implies that $\Sigma$ has exactly two constant principal curvatures, with multiplicities $(n-1)$ and 1. Then, by Cartan's result on isoparametric hypersurfaces of the sphere [9] we conclude that $\Sigma$ must be a constant mean curvature Clifford torus of the form $\mathbb{S}^{n-1}(r) \times \mathbb{S}^1(\sqrt{1-r^2})$ with $0<r<1$ and $r \neq \sqrt{(n-1)/n}$, since we are assuming $H \neq 0$. Finally,to identify which constant mean curvature Clifford tori do appear, a direct computation shows that when $n=2$ we have $|\phi|^2 = \alpha_H^2$ for all of them, but when $n \geqslant 3$ we have

$$|\phi|^2 = \frac{n}{4(n-1)}\left(\sqrt{n^2H^2 + 4(n-1)} - (n-2)|H|\right)^2 = \alpha_H^2$$

when $r < \sqrt{(n-1)/n}$, and

$$|\phi|^2 = \frac{n}{4(n-1)}\left(\sqrt{n^2H^2 + 4(n-1)} + (n-2)|H|\right)^2 > \alpha_H^2$$

when $r > \sqrt{(n-1)/n}$ (for the details see [1, p. 1227] or [2, p. 878]) □

Now we are ready to prove Theorem 10.

*Proof of Theorem 10.* We already know that $\lambda_1 \leqslant -n(1+H^2)$ with equality if and only if $\Sigma$ is totally umbilical, which proves part (i). Then we may assume that $\Sigma$ is not totally umbilical and consider, for every $\varepsilon > 0$, the positive smooth function $f_\varepsilon = \sqrt{\varepsilon + |\phi|^2}$. As in the proof of Theorem 3, we will use $f_\varepsilon$ as a test function to estimate $\lambda_1$ in (1.3). We observe that

$$f_\varepsilon \Delta f_\varepsilon = \frac{1}{2}\Delta|\phi|^2 - \frac{1}{4(\varepsilon+|\phi|^2)}|\nabla|\phi|^2|^2,$$

which using (8.4) yields

$$f_\varepsilon \Delta f_\varepsilon \geqslant |\nabla\phi|^2 - \frac{1}{4(\varepsilon+|\phi|^2)}|\nabla|\phi|^2|^2 - |\phi|^2 P_H(|\phi|). \tag{8.5}$$

From Lemma 7 applied to $\phi$, we also have

$$|\nabla\phi|^2 - \frac{1}{4(\varepsilon+|\phi|^2)}|\nabla|\phi|^2|^2 \geqslant \frac{2}{n+2}|\nabla\phi|^2,$$

which jointly with (8.5) gives

$$f_\varepsilon \Delta f_\varepsilon \geqslant \frac{2}{n+2}|\nabla\phi|^2 - |\phi|^2 P_H(|\phi|).$$

Then,

$$-f_\varepsilon J f_\varepsilon \leqslant |\phi|^2 P_H(|\phi|) - \frac{2}{n+2}|\nabla\phi|^2 - (\varepsilon+|\phi|^2)(|\phi|^2 + n(1+H^2)).$$

Therefore, using $f_\varepsilon$ as a test function in (1.3), we get

$$\begin{aligned} \lambda_1 \int_\Sigma f_\varepsilon^2 d\Sigma &\leqslant \int_\Sigma |\phi|^2 P_H(|\phi|) d\Sigma - \frac{2}{n+2}\int_\Sigma |\nabla\phi|^2 d\Sigma \\ &\quad - \int_\Sigma (\varepsilon+|\phi|^2)(|\phi|^2 + n(1+H^2)) d\Sigma. \end{aligned} \tag{8.6}$$

Since $\Sigma$ is not totally umbilical, then

$$\lim_{\varepsilon\to 0}\int_\Sigma f_\varepsilon^2 d\Sigma = \int_\Sigma |\phi|^2 d\Sigma > 0,$$

and letting $\varepsilon \to 0$ in (8.6) we conclude that

$$\begin{aligned} \lambda_1 &\leqslant -2n(1+H^2) + \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\frac{\int_\Sigma |\phi|^3 d\Sigma}{\int_\Sigma |\phi|^2 d\Sigma} - \frac{2}{n+2}\frac{\int_\Sigma |\nabla\phi|^2 d\Sigma}{\int_\Sigma |\phi|^2 d\Sigma} \\ &\leqslant -2n(1+H^2) + \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\max_\Sigma|\phi|. \end{aligned}$$

This proves the first statement of part (ii). Moreover, if

$$\lambda_1 = -2n(1+H^2) + \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\max_\Sigma|\phi|,$$

then $\nabla\phi = 0$ on $\Sigma$ and by Lemma 7 we know that $|\phi|^2$ is a positive constant. Thus $J = \Delta + (|\phi|^2 + n(1+H^2))$, where $|\phi|^2 + n(1+H^2)$ is a constant, and the first eigenvalue of $J$ is simply the constant

$$-(|\phi|^2 + n(1+H^2)) = \lambda_1 = -2n(1+H^2) + \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi|.$$

That is, $P_H(|\phi|) = 0$ and $|\phi| = \alpha_H$, and by Theorem 11 we know that it must hold either (a), (b) or (c). Conversely, we already know from Theorem 3 that

$\lambda_1 = -2n$ for all minimal Clifford tori in $\mathbb{S}^{n+1}$, and it is not difficult to see that $\lambda_1 = -2n(1+H^2) + \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi|$ for the constant mean curvature Clifford tori in (b) and (c). See [2] for the details. □

Finally, it is worth pointing out that Perdomo's technique in [21] also works here to characterize the equality case. For the details about this claim, see [2, Section 4].

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## References

1. H. Alencar and M. do Carmo, Hypersurfaces with constant mean curvature in spheres, *Proc. Amer. Math. Soc.* **120** (1994), 1223–1229.

2. L.J. Alías, A. Barros and A. Brasil Jr., A spectral characterization of the $H(r)$-torus by the first stability eigenvalue, *Proc. Amer. Math. Soc.* **133** (2005), no. 3, 875–884.

3. L.J. Alías, A. Brasil Jr. and O. Perdomo, On the stability index of hypersurfaces with constant mean curvature in spheres, to appear in *Proc. Amer. Math. Soc.*.

4. J.L. Barbosa and P. Bérard, Eigenvalue and "twisted" eigenvalue problems. Applications to CMC surfaces, *J. Math. Pures Appl. (9)* **79** (2000), 427–450.

5. J.L. Barbosa and M. do Carmo, Stability of hypersurfaces with constant mean curvature, *Math. Z.* **185** (1984), 339–353.

6. J.L. Barbosa, M. do Carmo and J. Eschenburg, Stability of hypersurfaces with constant mean curvature in Riemannian manifolds, *Math. Z.* **197** (1988), 123–138.

7. A. Barros, A. Brasil Jr. and L.A.M. Sousa Jr., A new characterization of submanifolds with parallel mean curvature vector in $\mathbb{S}^{n+p}$, *Kodai. Math. J.* **27** (2004), 45–56.

8. M. Berger, P. Gauduchon, and E. Mazet, Le spectre d'une variété riemannienne. Lecture Notes in Mathematics, Vol. 194 Springer-Verlag, Berlin-New York 1971.

9. E. Cartan, Familles de surfaces isoparamétriques dans les espaces à courbure constante, *Annali di Mat.* **17** (1938), 177–191.

10. S.S. Chern,M. do Carmo, and S. Kobayashi, Minimal submanifolds of a sphere with second fundamental form of constant length. 1970 *Functional Analysis and Related Fields* (Proc. Conf. for M. Stone, Univ. Chicago, Chicago, Ill., 1968) pp. 59–75 Springer, New York.

11. A. El Soufi, Applications harmoniques, immersions minimales et transformations conformes de la sphére, *Compositio Math.* **85** (1993), 281–298.

12. A. El Soufi and S. Ilias, Majoration de la seconde valeur propre d'un opérateur de Schrödinger sur une variété compacte et applications, *J. Funct. Anal.* **103** (1992), 294–316.

13. A. El Soufi and S. Ilias, Second eigenvalue of Schrödinger operators and mean curvature, *Comm. Math. Phys.* **208** (2000), 761–770.

14. I. Guadalupe, A. Brasil Jr., and J.A. Delgado, A characterization of the Clifford torus. *Rend. Circ. Mat. Palermo (2)* **48** (1999), 537–540.

15. H.B. Lawson Jr., Local rigidity theorems for minimal hypersurfaces, *Ann. of Math. (2)* **89** (1969), 187–197.

16. H.B. Lawson Jr., Lectures on minimal submanifolds. Vol. I. Second edition. Mathematics Lecture Series, 9. Publish or Perish, Inc., Wilmington, Del., 1980.

17. K. Nomizu and B. Smyth, A formula of Simons' type and hypersurfaces with constant mean curvature, *J. Differential Geometry* **3** (1969), 367–377.

18. M. Obata, Certain conditions for a Riemannian manifold to be iosometric with a sphere, *J. Math. Soc. Japan* **14** (1962), 333–340.

19. M. Okumura, Hypersurfaces and a pinching problem on the second fundamental tensor, *Amer. J. Math.* **96** (1974), 207–213.

20. O. Perdomo, Low index minimal hypersurfaces of spheres, *Asian J. Math.* **5** (2001), 741–749.

21. O. Perdomo, First stability eigenvalue characterization of Clifford hypersurfaces, *Proc. Amer. Math. Soc.* **130** (2002), 3379–3384

22. J. Simons, Minimal varieties in Riemannian manifolds, *Ann. of Math. (2)*, **88** (1968) 62–105.

23. F. Urbano, Minimal surfaces with low index in the three-dimensional sphere, *Proc. Amer. Math. Soc.* **108** (1990), 989–992.

24. C. Wu, New characterizations of the Clifford tori and the Veronese surface, *Arch. Math. (Basel)*, **61** (1993), 277–284.

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