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HYPERSURFACES WITH CONSTANT MEAN CURVATURE

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Abstract. The hypersurfaces with constant mean curvature (cmc) are studied under different aspects:

1- As critical Points of a Variational Problem.
2- As solutions of a Dirichlet Problem.
3- Under the point of view of harmonicity of the Gauss map.

We explain, in a short wave, the principal technical and some results obtained in each aspects.

1. Hypersurfaces of cmc as critical points of a Variational Problem.

An $M^n \to N^{n+1}$ isometric immersion with constant mean curvature is a critical point of the area functional for variations preserving volume.

When the critical point is a *local minimum,* M^n is called *stable.* In particular, the stable hypersurfaces, when compact, bound *isoparametric domains* (frontier with area minima for a given fixed volume).

In $N^{n+1} = R^{n+1}$, S^{n+1} or H^{n+1} the unique compact and stable hypersurfaces with cmc. are the geodesic spheres [B-dC-E, 1984], [Heintze, 1988].

In P^3, the real projective space of dim 3, the compact stable surfaces are embedded, have genus 0 or 1, geodesic sphere or flat torus, respectively. Ritoré-Ros, 1992, [R-R]. In this paper, the authors give the isoperimetrical profile of these surfaces.

In the torus T^3, the stable compact surfaces M^2 immersed with cmc are not classified. Partial results are known (see [F-R1] and references):
- the genus g of M^2 must be $g \leq 5$
- if $g = 4$ or 5, then $H = 0$
- if $H \geq$ some constant $\neq 0$ (depending on the radius of the Torus) then the stable surface with cmc H is embedded and $g = 0$ or 1
- It is known a stable surface with genus 3 and $H = 0$.

Here H denotes the mean curvature of M.

The stable non compact hypersurface with cmc. are also studied. In R^{n+1} they are hyperplanes. In H^{n+1} and CH^{n+1}, the horospheres and tubes with sufficiently large radius around totally geodesic cod 2 -submanifolds are cmc stable hypersurfaces.

2. Constant Mean Curvature hypersurfaces as Solutions of a Dirichlet Problem.

It is known that if $M^n \subset R^{n+1}$ is the graph of a differentiable function $u : \Omega \subset R^n \to R$, then

$$nH = -div \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right)$$

where ∇ denotes the *gradient* in R^n.

Therefore, the existence of a graph with constant mean curvature is equivalent to assure the existence of the solution of a Dirichlet problem:

$$\begin{aligned} Q_H(u) &= div \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) + nH = 0 \\ u_{|\partial\Omega} &= \Phi, \quad \Phi \in C^0(\partial\Omega) \end{aligned}$$

In this case, Q_H is a quasi linear elliptic operator of second order and it satisfies the Maximum Principle.

For limited domains, with appropriate conditions in the mean curvature of the boundary, the Dirichlet problem above has a unique solution (Serrin, 1969) [S].

For not limited domains $\Omega \subset R^{n+1}$ the existence of solutions depends on the construction of appropriate barriers (Perron's method).

In S^{n+1} and H^{n+1}, it is possible to define the "graph" of a real function whose domain is a region Ω contained in a totally geodesic submanifold of dim n. The Dirichlet problem is similar to the above problem, but the operator Q_H can be quite complicated.

These graphs have been recently studied, some references are: Guio- Sá Earp, 2005 [G-S], Fornari, Lira, Ripoll, 2002 [F,L,R], Dajczer -Ripoll, preprint [D-R], Alias-Dajczer, preprint [A-D].

3. Constant Mean Curvature hypersurface and harmonicity of the Gauss map.

Given a complete surface M in R^3, it is well known that:

$$\Delta\eta = -\|B\|^2 \eta$$

where η is the unit normal to a surface, $\eta : M \to S^2$, Δ is the Laplacian of M and $\|B\|$ is the norm of the second fundamental of M.

The above equation has been used to obtain many important results, one of them, due to Hoffman, Osserman and Shoen, 1982, [H-O-S] said:
"*If $\eta(M) \subset$ closed hemisphere of S^2 then M is a plane or a right circular cylinder*"
or equivalently

"*If the function* $f := \langle \eta, V \rangle$ *does not change sign on* M, *for same fixed vector* $V \in R^3$, *then* M *is invariant by a one parameter subgroup of translations of* R^3"

In a joint work with Espirito-Santo, Frensel and Ripoll, [ES-F-F-R], 2003, we obtain similar results for hypersurfaces M immersed with cmc H on a Lie group with bi–invariant metric. Later on, in 2005, with J. Ripoll, [F-R,2] we extend this result for hypersurfaces M immersed with cmc H on a $(n+1)$-dimensional Riemannian manifold N:

Consider the function $f := \langle \eta, V \rangle$ on M, where V is a Killing field of N. We prove that:

$$\Delta f = -n \langle \nabla H, V \rangle - \left(Ric(\eta) + \|B\|^2 \right) f$$

Using this result, similarly to those in R^3, we prove

"*if* f *does not change sign on* M *and* $\mathrm{Ric}_N + 2H^2 \geq 0$ *then* M *is invariant by the one parameter subgroup of isometries of* N *determined by* V *or* M *is umbilic*".

As consequence of this result we obtain a stability criterion for cmc surfaces in a manifold of dim 3 :

"*Let* M *be a surface of constant mean curvature* H *(not necessarily complete) in* N^3 *with* $\mathrm{Ric}_N + 2H^2 \geq 0$ *and let* D *be a domain in* M *such that* $\overline{D} \subset int(M)$. *Let* V *be a Killing vector field on* N *and assume that* $f = \langle V, \eta \rangle$ *has a sign on* $\overline{D}$. *Then* $\overline{D}$ *is stable".*

It follows that any radial or horizontal cmc graph in the half space model for the hyperbolic 3−space is stable.

If the manifold N admits $n+1$ linearly independ Killing vector fields at each point, it is possible to define a *normal Killing translation map* $\gamma : M \to R^{n+1}$

$$\gamma(p) = \Gamma_p(\eta) = \sum_{i=1}^{n+1} \langle \eta, V_i \rangle \, e_i$$

This map is a natural extension of the Gauss map of a hypersurface in R^{n+1}.

We obtain:

$$\Delta\gamma(p) = -n\Gamma_p(\nabla H) - (\mathrm{Ric}(\eta) + ||B||^2)\gamma(p)$$

Using this formula we prove the following results:

Let M^n *be a compact riemannian manifold immersed with cmc in a riemannian manifold* N^{n+1} *with* $\mathrm{Ric}_N + nH^2 \geq 0$. *If* $\gamma(M)$ *is contained in a half space of* R^{n+1} *then* M *is invariant by a one parameter subgroup of isometries of* N *or* M *is umbilic.*

In particular, it arises in this context a natural and interesting extension of a conjecture of M. P. do Carmo which asserts that the Gauss image of a complete cmc surface in R^3 which is not a plane nor a cylinder contains a neighborhood of some equator of the sphere:

Conjecture (An extension of a conjecture of M. P. do Carmo):
Let N be a $n+1$ dimensional Killing parallelizable riemannian manifold and let $\mathcal{B}$ be a Killing basis of TN. Let M be a complete constant mean curvature hypersurface immersed in N and let $\gamma : M \to \mathbb{R}^{n+1}$ be the normal Killing translation map associated to $\mathcal{B}$. If M is not invariant by a Killing field generated (over the real numbers) by $\mathcal{B}$, then the radial projection of $\gamma(M)$ on the unit sphere covers a neighborhood of some equator of the sphere.

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