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EINSTEIN METRICS ON FLAG MANIFOLDS

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ABSTRACT. In this survey we describe new invariant Einstein metrics on flag manifolds. Following closely San Martin-Negreiros's paper [26] we state results relating Kähler, (1,2)-symplectic and Einstein structures on flags. For the proofs see [11] and [10].

INTRODUCTION

We recall that a Riemannian metric g on a manifold M is called Einstein if $Ric(g) = cg$ for some constant c. As we know Einstein metrics form a special class of metrics on a given manifold (see [4]). In this note we announce properties of these metrics and new examples of Einstein metrics on flag manifolds as described in [11] and [10].

With this purpose in mind, we consider $\mathfrak{g}$ as being a complex semi-simple Lie algebra and Σ a simple root system for $\mathfrak{g}$. If Θ is an arbitrary subset of Σ, $\langle\Theta\rangle$ denotes the roots spanned by Θ. We have

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\langle\Theta\rangle} \mathfrak{g}_\alpha \oplus \sum_{\alpha\in\langle\Theta\rangle} \mathfrak{g}_{-\alpha} \oplus \sum_{\beta\in\Pi^+-\langle\Theta\rangle} \mathfrak{g}_\beta \oplus \sum_{\beta\in\Pi^+-\langle\Theta\rangle} \mathfrak{g}_{-\beta}, \tag{1}$$

where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ is the root space associated to the root α. Let

$$\mathfrak{p}_\Theta := \mathfrak{h} \oplus \sum_{\alpha\in\langle\Theta\rangle} \mathfrak{g}_\alpha \oplus \sum_{\alpha\in\langle\Theta\rangle} \mathfrak{g}_{-\alpha} \oplus \sum_{\beta\in\Pi^+-\langle\Theta\rangle} \mathfrak{g}_\beta, \tag{2}$$

the canonical parabolic subalgebra determined by Θ. Hence

$$\mathfrak{g} = \mathfrak{p}_\Theta \oplus \sum_{\beta\in\Pi^+-\langle\Theta\rangle} \mathfrak{g}_{-\beta}. \tag{3}$$

$\mathbb{F}_\Theta = G/P$ is called a flag manifold, where G has the Lie algebra $\mathfrak{g}$ and P is the normalizer of $\mathfrak{p}$ in G. Each manifold $\mathbb{F}_\Theta$ has a very rich complex geometry, containing families of invariant Hermitian structures denoted by $(\mathbb{F}_\Theta, J, ds^2_\Lambda)$.

The case $\mathbb{F} = \mathbb{F}_\Theta$ for $\Theta = \emptyset$, i.e., the full flag manifold is nowadays well understood. Starting with the work of Borel (cf. [7]), the classification of all invariant Hermitian structures is known and it was derived in [26].

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On the other hand, the case $\mathbb{F}_\Theta$ for $\Theta \neq \emptyset$ is much less known so far. Some partial results are derived in [27] and [28].

We now describe the contents of this survey. In the first two sections we discuss all the invariant Hermitian structures on $\mathbb{F}_\Theta$ and the associated Einstein system of equations. In Section 3 we present new invariant Einstein metrics on generalized flag manifolds of type A_l.We suggest Besse's book [4] as a reference for Einstein manifolds.

In Section 4 we state the classification of all invariant Einstein metrics on $\mathbb{F}(4)$ and state some partial results relating Kähler, (1,2)-symplectic and Einstein structures on $\mathbb{F}(n)$.

For a very stimulating article see [1].

All manifolds and maps between them will be assumed to be C^∞ in this survey.

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1. General results on the invariant Hermitian geometry of flag manifolds

We denote by $\langle\cdot,\cdot\rangle$ the Cartan-Killing form of $\mathfrak{g}$, and we fix a Weyl basis $\{X_\alpha\}_{\alpha\in\Pi}$ for $\mathfrak{g}$. We define the compact real form of $\mathfrak{g}$, as the real subalgebra

$$\mathfrak{u} = \mathrm{span}_{\mathbb{R}}\{i\mathfrak{h}_{\mathbb{R}}, A_\alpha, iS_\alpha : \alpha \in \Pi\},$$

where $A_\alpha = X_\alpha - X_{-\alpha}$ and $S_\alpha = X_\alpha + X_{-\alpha}$.

Let x_Θ be the origin of $\mathbb{F}_\Theta$. $T_{x_\Theta}\mathbb{F}_\Theta$ is identified with

$$\begin{aligned} T_{x_\Theta}\mathbb{F}_\Theta &\approx \mathfrak{m}_\Theta = \mathrm{span}_{\mathbb{R}}\{A_\alpha, iS_\alpha : \alpha \notin \langle\Theta\rangle\} = \\ &= \sum_{\alpha\in\Pi\setminus\langle\Theta\rangle=\Pi_\Theta} \mathfrak{u}_\alpha, \end{aligned}$$

where $\mathfrak{u}_\alpha = (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{u} = \mathrm{span}_{\mathbb{R}}\{A_\alpha, iS_\alpha\}$. Complexifying $\mathfrak{m}_\Theta$ we obtain $T^{\mathbb{C}}_{x_\Theta}\mathbb{F}_\Theta$, which can be identified with

$$\mathfrak{m}^{\mathbb{C}}_\Theta = \sum_{\beta\in\Pi\setminus\langle\Theta\rangle} \mathfrak{g}_\beta. \tag{4}$$

A U-invariant almost complex structure J on $\mathbb{F}_\Theta$, is completely determined by a collection of numbers $\varepsilon_\sigma = \pm 1$, $\sigma \in \Pi_\Theta$.

A U-invariant Riemannian metric ds^2_Λ on $\mathbb{F}_\Theta$ is completely characterized by the following inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{m}_\Theta$

$$\langle X, Y\rangle_\Lambda := -\langle \Lambda X, Y\rangle, \tag{5}$$

where $\Lambda : \mathfrak{m}_\Theta \to \mathfrak{m}_\Theta$ is definite-positive with respect to the Cartan-Killing form. On each irreducible component of $\mathfrak{m}_\Theta$, $\Lambda = \lambda_\sigma \mathrm{id}$ with $\lambda_{-\sigma} = \lambda_\sigma > 0$.

Consider $\tau =$ the conjugation of $\mathfrak{g}$ relatively to $\mathfrak{u}$. Hence, $\langle\langle X, Y\rangle\rangle_\Lambda = \langle X, \tau Y\rangle_\Lambda$ is a Hermitian form on $\mathfrak{g}$, that originates a U-invariant Hermitian form on $\mathbb{F}_\Theta$.

If $\Omega = \Omega_{J,\Lambda}$ denotes the corresponding Kähler form then

$$\Omega(X_\alpha, X_\beta) = -\sqrt{-1}\lambda_\alpha \epsilon_\beta \langle X_\alpha, X_\beta\rangle. \tag{6}$$

We recall that a almost-Hermitian manifold is said $(1,2)$-symplectic if $d\Omega(X,Y,Z)=0$ when one of the vectors X, Y, Z is of type $(1,0)$, and the other two are of type $(0,1)$. If J is integrable and $d\Omega\equiv 0$, we say $(\mathbb{F}_\Theta, J, ds^2_\Lambda)$ is a Kähler manifold.

2. Ricci tensor and the Einstein system of equations

We now consider $\{e_\alpha\}$ a $\mathcal{B}$-orthogonal basis adapted to a decomposition of $\mathfrak{m}=\bigoplus_{k=1}^{l}\mathfrak{m}_k$. In other words, $e_\alpha\in\mathfrak{m}_i$ for some $i\in\{1,\cdots,l\}$, and $\alpha<\beta$ if $i<j$ with $e_\alpha\in\mathfrak{m}_i,\ e_\beta\in\mathfrak{m}_j$. Define, as in [29],

$$A^\gamma_{\alpha\beta}=\left(\left[e_\alpha,e_\beta\right],e_\gamma\right), \tag{7}$$

that is,

$$\sum (A^\gamma_{\alpha\beta})^2=\begin{bmatrix} & k & \\ i & & j\end{bmatrix} \tag{8}$$

where in the second equation we take all indices α,β,γ with $e_\alpha\in\mathfrak{m}_i, e_\beta\in\mathfrak{m}_j, e_\gamma\in\mathfrak{m}_k$. Notice that $\begin{bmatrix} & k & \\ i & & j\end{bmatrix}$ is independent of orthonormal frame chosen for $\mathfrak{m}_i,\mathfrak{m}_j,\mathfrak{m}_k$ and $[e_\alpha,e_\beta]=\sum_\gamma A^\gamma_{\alpha\beta}\ e_\gamma$

$$\begin{bmatrix} & k & \\ i & & j\end{bmatrix}=\begin{bmatrix} & k & \\ j & & i\end{bmatrix}=\begin{bmatrix} & j & \\ k & & i\end{bmatrix}$$

Furthermore, if w is an element of Weyl´s group then

$$\begin{bmatrix} w(\gamma) \\ w(\alpha)\ w(\beta)\end{bmatrix}=\begin{bmatrix} \gamma \\ \alpha\ \beta\end{bmatrix} \tag{9}$$

The following result is due to Wang-Ziller [29] (see also [2]):

Lemma 2.1. *The components r_k of the Ricci tensor of an U-invariant metric on $M=U/K$ are given by:*

$$r_k=\frac{1}{2\lambda_k}+\frac{1}{4d_k}\sum_{i,j=1}^{l}\frac{\lambda_k}{\lambda_i\lambda_j}\begin{bmatrix} & k & \\ i & & j\end{bmatrix}-\frac{1}{2d_k}\sum_{i,j=1}^{l}\frac{\lambda_k}{\lambda_i\lambda_j}\begin{bmatrix} & j & \\ k & & i\end{bmatrix}\quad (k=1,\cdots,l), \tag{10}$$

where $\mathfrak{m}=\bigoplus_{k=1}^{l}\mathfrak{m}_k,\quad d_k=\dim\mathfrak{m}_k.$

More generally, Arvanitoyeorgos proved in [3] the following result

Proposition 2.2. *The Ricci tensor of an invariant metric $(\Lambda_\alpha)=\{\lambda_\alpha>0,\ \alpha\in\Pi_M\}$ on a flag manifold $\mathbb{F}_\Theta$ is given by*

$$\begin{aligned} Ric(X_\alpha,X_\beta)&=0, \qquad \text{if}\ \ \alpha,\beta\in\Pi_M,\ \alpha+\beta\notin\Pi_M\\ Ric(X_\alpha,X_{-\alpha})&=\big(\alpha,\alpha\big)+\sum_{\substack{\phi\in\Pi_\Theta\\ \alpha+\phi\in\Pi}} m^2_{\alpha,\phi}+\frac{1}{4}\sum_{\substack{\beta\in\Pi_M\\ \alpha+\beta\in\Pi_M}}\frac{m^2_{\alpha,\beta}(\lambda^2_\alpha-(\lambda_{\alpha+\beta}-\lambda_\beta)^2)}{\lambda_{\alpha+\beta}\lambda_\beta}\end{aligned}$$

We have the following non-homogeneous version of this equation

$$\lambda_{ij} = 2 + \frac{1}{2} \sum_{k \neq i,j} \frac{\lambda_{ij}^2 \; - (\lambda_{ik} - \lambda_{jk})^2}{\lambda_{ij}\lambda_{jk}}$$

With each solution we associate the Einstein constant, which is defined as the value of the Ricci tensor r_{ij} when λ is re-normalized to have unit volume.

3. New Einstein metrics

Using the Einstein system of equations described above, we describe now the known and new Einstein metrics on $\mathbb{F}(n)$ as in [11] and [10].

a) The normal metric. We notice this metric is not Kähler.

b) Kähler-Einstein metrics

On the flag manifold $\mathbb{F}(n)$ $(n \geq 3)$, up to permutation there is a unique integrable structure J, and associated with it a unique (up to scaling) Kähler-Einstein metric (which corresponds to the choice $c = \delta = \frac{1}{2}\sum_\beta \beta$ according to Matsushima [19] or [4]):

$$\Lambda = \begin{pmatrix} 0 & \frac{1}{2n} & \frac{1}{n} & \cdots & \frac{n-1}{2n} \\ \frac{1}{2n} & 0 & \frac{1}{2n} & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{2n} & 0 & \ddots & \frac{1}{n} \\ \vdots & \ddots & \ddots & \ddots & \frac{1}{2n} \\ \frac{n-1}{2n} & \cdots & \frac{1}{n} & \frac{1}{2n} & 0 \end{pmatrix}.$$

Thus, counting in the symmetry of this metric, we have $\frac{n!}{2}$ Kähler-Einstein metrics on $\mathbb{F}(n)$.

c) The Arvanitoyeorgos metrics

Arvanitoyeorgos ([3]) considers for all $s \in [1, n]$ metrics in $\mathbb{F}(n)$ $(n \geq 4)$ satisfying

$$\lambda_{ij} = A \;\; (\, s \in \{i, j\}), \qquad\qquad \lambda_{ij} = B \quad \text{otherwise}$$

The Einstein system is reduced to the equations in A, B whose solution is $A = n - 1$ and $B = n + 1$. Counting permutations, we get n Arvanitoyeorgos metrics whose Einstein constant is seen to be

$$c_{Arv.} = \frac{(n^2 - n + 2)\sqrt[n]{(n-1)^2(n+1)^{n-2}}}{4n(n-1)^2}.$$

d) The Sakane-Senda metrics

Sakane and Senda in [25] consider metrics in $\mathbb{F}(2m)$ $(m \geq 3)$ satisfying

$$\lambda_{ij} = A \;\; (\, i, j \leq m \text{ or } i, j > m), \qquad\qquad \lambda_{ij} = B \;\; \text{otherwise}$$

Again, the the Einstein system is reduced to two equations in A, B whose solution is $A = m + 2$ and $B = 3m - 2$.

e) A new family

If $m \geq 6$ we find another solution in $\mathbb{F}(2m)$, for $A = m + 5$ and $B = 3m - 5$.

f) Two new families

On $\mathbb{F}(2m+1)$ $(m \geq 6)$ we consider

$\lambda_{ij} = A \;\; (i,j \leq m+1 \text{ or } i,j > m+1), \qquad \lambda_{ij} = B \;\; \text{otherwise}$

There are two families as solution of the Einstein system. The Einstein constants for these two families are, respectively,

$$c_{\pm} = (\tfrac{1}{2} + \tfrac{1}{4n}(\frac{2n-2}{(\frac{n+3}{2}) \pm \sqrt[2]{(n-1)^2-4n+16}})^2)\sqrt[4n]{(\frac{n+3 \pm \sqrt[2]{(n-5)(n-13)}}{4})^{n-1}}.$$

g) A new metric

Still assuming the same pattern, with $m = 2$, we find on $\mathbb{F}(5)$ the invariant Einstein metric with $A = 1$ and $B = 2$. The Einstein constant of this metric is $\frac{11\sqrt[5]{4}}{40}$.

We define the class $ENK = \{ds^2_\Lambda;\; ds^2_\Lambda$ is a Einstein non-Kähler metric in **a)** or **c)** or **d)** or **e)** or **f)** or **g**). $\}$.

A complete classification of the Einstein metrics for $\mathbb{F}(n)$ $(n \neq 3, 4)$ is completely unknown. It is not even know if the number of such metrics is finite (the Bohn-Wang-Ziller conjecture).

In [11] and [10] we use the procedure described above in order to obtain new Einstein metrics on non-maximal A_l-type manifolds. Our notation will be $\mathbb{F}(n; n_1, ..., n_k)$ where $(n_1, ..., n_k)$ represents block-matrices of size $n = \sum_{i=1}^{k} n_i$. All the entries in each block are equal, so that the metric is completely expressed by a reduced $k \times k$ matrix, which we denote by $\widetilde{\Lambda}$.

Theorem 3.1. ***a)*** *On* $\mathbb{F}(5; 2, 1, 1, 1)$. *The set of restrictions* $\lambda_{12} = \lambda_{13} = \lambda_{14}$ *and* $\lambda_{23} = \lambda_{24} = \lambda_{34}$, *produce two invariant non-Kähler Einstein. On the other hand the restrictions* $\lambda_{12} = \lambda_{13} = \lambda_{23} = \lambda_{24}$ *and* $\lambda_{14} = \lambda_{34}$ *do not produce any solution.*

b) On $\mathbb{F}(n; k, q, q, \cdots, q) = \frac{U(n)}{U(k)\times U(q)^s}$, i.e. $n = k + sq$ ($q(\sqrt{s^2-4} + 2 - s) < 2k,$) we look for a $(s+1) \times (s+1)$ reduced matrix $\widetilde{\Lambda}$ with

$\widetilde{\lambda_{ij}} = A$ $(1 \in \{i, j\})$, $\qquad B$ otherwise

In this way we can produce two non-Kähler Einstein metrics.

c) On $\mathbb{F}(n; k, k, \cdots, k)$ with $n = sk$ the invariant metric represented by the $s \times s$ matrix $\widetilde{\Lambda}$ is Einstein if, and only if, the same matrix represents an Einstein metric on $\mathbb{F}(s)$.

4. Results on the classification of Einstein metrics on $\mathbb{F}(n)$

Gray and Hervella in [13] gave a complete classification of triples (M, g, J) into sixteen classes for arbitrary almost Hermitian manifolds. San Martin-Negreiros discussed in [26] the case where M is a maximal flag manifold. They have proved that the invariant almost Hermitian structures on maximal flag manifolds can be divided only in three classes, namely

(a) $W_1 \oplus W_2$
(b) $W_1 \oplus W_3$
(c) $W_1 \oplus W_2 \oplus W_3$,where the class $W_1 \oplus W_2 \oplus W_3$ contains any invariant almost Hermitian structures.

In [26] it is proved that an invariant pair $(J, \Lambda) \in W_1 \oplus W_2$ if and only if for all $\{1,2\}$-triple of roots $\{\alpha, \beta, \gamma\}$

$$\epsilon_\alpha \lambda_\alpha + \epsilon_\beta \lambda_\beta + \epsilon_\gamma \lambda_\gamma = 0. \tag{11}$$

The next lemma characterizes the Hermitian structures belonging to $W_1 \oplus W_3$ (see [26]) for more details.

LEMMA 4.1. *A necessary and sufficient condition for an invariant pair* (J, Λ) *to be in* $W_1 \oplus W_3 \approx W_1 \oplus W_3 \oplus W_4$ *is* $\lambda_\alpha = \lambda_\beta = \lambda_\gamma$ $\forall$ $\{0,3\}$*-triple* $\{\alpha, \beta, \gamma\}$.

In [11] or [10] the following result is proved:

THEOREM 4.2. *If* $ds^2_\Lambda \in ENK$ *for* $n \geq 4$*, then this metric belongs to* $W_1 \oplus W_3$.

This result leads us to conjecture that any invariant Einstein non-Kähler metric on $\mathbb{F}(n)$ is in $W_1 \oplus W_3$. One result supporting this conjecture is

THEOREM 4.3. *The space* $\mathbb{F}(4)$ *admits (up to scaling) precisely 3 classes of invariant Einstein metrics: The Kähler-Einstein* [7]*, the 4 Arvanitoyeorgos's class* [3]*, and the class of the normal metric* [30].

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