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STABILITY OF HOLOMORPHIC-HORIZONTAL MAPS AND EINSTEIN METRICS ON FLAG MANIFOLDS

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Abstract. In this note we announce several results concerning the stability of certain families of harmonic maps that we call holomorphic–horizontal frames, with respect to families of invariant Hermitian structures on flag manifolds. Special emphasis is given to the Einstein case. See [23] for additional detail and the proofs of the results mentioned in this survey.

1. Introduction

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra and Σ a simple root system for $\mathfrak{g}$. If Θ is an arbitrary subset of Σ, $\langle\Theta\rangle$ denotes the roots spanned by Θ. We have

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\langle\Theta\rangle} \mathfrak{g}_\alpha \oplus \sum_{\alpha\in\langle\Theta\rangle} \mathfrak{g}_{-\alpha} \oplus \sum_{\beta\in\Pi^+-\langle\Theta\rangle} \mathfrak{g}_\beta \oplus \sum_{\beta\in\Pi^+-\langle\Theta\rangle} \mathfrak{g}_{-\beta}, \tag{1}$$

where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and, $\mathfrak{g}_\alpha$ is the root space associated to the root α.

Let

$$\mathfrak{p}_\Theta := \mathfrak{h} \oplus \sum_{\alpha\in\langle\Theta\rangle} \mathfrak{g}_\alpha \oplus \sum_{\alpha\in\langle\Theta\rangle} \mathfrak{g}_{-\alpha} \oplus \sum_{\beta\in\Pi^+-\langle\Theta\rangle} \mathfrak{g}_\beta, \tag{2}$$

the canonical parabolic subalgebra determined by Θ. Thus,

$$\mathfrak{g} = \mathfrak{p}_\Theta \oplus \sum_{\beta\in\Pi^+-\langle\Theta\rangle} \mathfrak{g}_{-\beta}. \tag{3}$$

The flag manifold is defined as $\mathbb{F}_\Theta = G/P$, where G has Lie algebra $\mathfrak{g}$ and P_Θ is the normalizer of $\mathfrak{p}_\Theta$ in G.

$\mathbb{F}_\Theta$ for $\Theta = \emptyset$, is called the full flag manifold and is denoted by $\mathbb{F}$. This case is nowadays well understood. Starting with the work of Borel (cf. [3]), the classification of all invariant Hermitian structures is known and it was described in [25].

The main purpose for this note is to announce some results discussing the stability phenomenon for the energy functional for to a special class of maps $\psi : (M^2, J, g) \to (\mathbb{F}_\Theta, J, ds^2_\Lambda)$, called holomorphic–horizontal frames. The energy functional is taken with respect to several families of invariant Hermitian structures

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on $\mathbb{F}_\Theta$. These maps are deeply connected with the study of harmonic/minimal surfaces in $\mathbb{S}^n$, $\mathbb{CP}^n$, $\mathbb{G}_k(\mathbb{C}^n)$, $\mathbb{HP}^n$, Twistor Theory and so on (cf. [27], [8], [4], [15]).

The layout of the paper is as follows. In the first two sections, we state general results on the invariant Hermitian geometry of $\mathbb{F}_\Theta$, and state the holomorphic and harmonic map equations. We give in this note, examples of families holomorphic–horizontal frames only in the (A_l) case, but in [23] we also discuss the cases (B_l), (C_l) and (D_l).

We generalize and give additional results to the approach initiated by Black in [2] and the author in [21]. As a reference for harmonic maps theory we suggest the Eells-Lemaire [14] article.

In Section 3 we compute the second variation of energy for an arbitrary harmonic map on $\mathbb{F}_\Theta$. We state a basic perturbation lemma and a result for holomorphic–horizontal frames on $\mathbb{F}_\Theta$.

According to the classification results in [25], among all Hermitian invariant structures there are two main classes. Thus, in the last section we state results concerning the stability of holomorphic–horizontal frames regarding metrics in such classes and in particular, the case of metrics that are Einstein and non Kähler on the geometrical flag manifold $\mathbb{F}$.

2. Generalities on invariant Hermitian geometry of flag manifolds

Let Π be a root system and Σ a simple root system for a simple Lie algebra g. If Θ is a subset of Σ, $\langle\Theta\rangle$ denotes the roots generated by Θ. We have the root decomposition :

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\langle\Theta\rangle} \mathfrak{g}_\alpha \oplus \sum_{\alpha\in\langle\Theta\rangle} \mathfrak{g}_{-\alpha} \oplus \tag{4}$$

$$\sum_{\beta\in\Pi^+-\langle\Theta\rangle} \mathfrak{g}_\beta \oplus \sum_{\beta\in\Pi^+-\langle\Theta\rangle} \mathfrak{g}_{-\beta}, \tag{5}$$

where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ is the root space associated to the root α.

Let

$$\mathfrak{p}_\Theta := \mathfrak{h} \oplus \sum_{\alpha\in\langle\Theta\rangle} \mathfrak{g}_\alpha \oplus \sum_{\alpha\in\langle\Theta\rangle} \mathfrak{g}_{-\alpha} \oplus \sum_{\beta\in\Pi^+-\langle\Theta\rangle} \mathfrak{g}_\beta. \tag{6}$$

The space $\mathbb{F}_\Theta = G/P_\Theta$ is called a flag manifold, where $\mathfrak{g}$ and $\mathfrak{p}_\Theta$ are the Lie algebras of G and P_Θ, respectively.

Each manifold $\mathbb{F}_\Theta$ has families of complex geometries, i.e., families of invariant Hermitian structures denoted by $(\mathbb{F}_\Theta, J, ds^2_\Lambda)$.

We denote by $\langle\cdot,\cdot\rangle$ the Cartan-Killing form of $\mathfrak{g}$, and fix once and for all a Weyl basis of $\mathfrak{g}$, which amounts to take $X_\alpha \in \mathfrak{g}_\alpha$ such that $\langle X_\alpha, X_{-\alpha}\rangle = 1$, and $[X_\alpha, X_\beta] = m_{\alpha,\beta}X_{\alpha+\beta}$ with $m_{\alpha,\beta} \in \mathbb{R}$, $m_{-\alpha,-\beta} = -m_{\alpha,\beta}$, and $m_{\alpha,\beta} = 0$ if $\alpha+\beta$ is not a root.

We define the compact real form of $\mathfrak{g}$, as the real subalgebra

$$\mathfrak{u} = \mathrm{span}_{\mathbb{R}}\{i\mathfrak{h}_{\mathbb{R}}, A_\alpha, iS_\alpha : \alpha \in \Pi\}$$

where $A_\alpha = X_\alpha - X_{-\alpha}$ and $S_\alpha = X_\alpha + X_{-\alpha}$.

Let x_Θ be the origin of $\mathbb{F}_\Theta$. $T_{x_\Theta}\mathbb{F}_\Theta$ is identified with

$$\begin{aligned} T_{x_\Theta}\mathbb{F}_\Theta &\approx \eta_\Theta = \mathrm{span}_{\mathbb{R}}\{A_\alpha, iS_\alpha : \alpha \notin \langle\Theta\rangle\} = \\ &= \sum_{\alpha\in\Pi\setminus\langle\Theta\rangle=\Pi_\Theta} \mathfrak{u}_\alpha, \end{aligned}$$

where $\mathfrak{u}_\alpha = (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})\cap\mathfrak{u} = \mathrm{span}_{\mathbb{R}}\{A_\alpha, iS_\alpha\}$. Complexifying η_Θ we obtain $T^{\mathbb{C}}_{x_\Theta}\mathbb{F}_\Theta$, which can be identified with

$$\mathfrak{q}_\Theta = \sum_{\beta\in\Pi\setminus\langle\Theta\rangle} \mathfrak{g}_\beta.$$

We denote the irreducible components of $\mathfrak{q}_\Theta$ as $\mathfrak{g}_\sigma$, where σ is the set of roots α with $\mathfrak{g}_\alpha \subset \mathfrak{g}_\sigma$, thus $\mathfrak{g}_\sigma = \Sigma_{\alpha\in\sigma}\mathfrak{g}_\alpha$.

Let $\Pi(\Theta)$ be the collection of sets σ originating the irreducible components. We write

$$\mathfrak{q}_\Theta = \oplus_{\sigma\in\Pi(\Theta)}\mathfrak{g}_\sigma = \oplus_{\alpha\in\Pi_\Theta}\mathfrak{g}_\alpha.$$

Each $\sigma \in \Pi(\Theta)$ defines a field of complex subspaces $(E_\sigma)_{\sigma\in\Pi(\Theta)}$ such that $T^{\mathbb{C}}_x\mathbb{F}_\Theta = \sum_{\sigma\in\Pi(\Theta)} E_\sigma(x)$ for each $x \in \mathbb{F}_\Theta$.

A U-invariant almost complex structure J on $\mathbb{F}_\Theta$ is completely determined by a linear map $J : \eta_\Theta \to \eta_\Theta$. The map J satisfies $J^2 = -1$ and commutes with the adjoint action of K_Θ on η_Θ. We denote also by J its complexification to $\mathfrak{q}_\Theta$.

The invariance of J entails that $J(\mathfrak{g}_\sigma) = \mathfrak{g}_\sigma$ for all $\sigma \in \Pi(\Theta)$. The eigenvalues of J are $\pm\sqrt{-1}$, and the eigenvectors in $\mathfrak{q}_\Theta$ are X_α, $\alpha \in \Pi_\Theta$. Hence, in each irreducible component $\mathfrak{g}_\sigma$, $J = \sqrt{-1}\epsilon_\sigma \mathrm{id}$ with $\epsilon_\sigma = \pm1$ satisfying $\epsilon_{-\sigma} = -\epsilon_\sigma$. A U-invariant almost complex structure on $\mathbb{F}_\Theta$ is completely determined by the numbers $\varepsilon_\sigma = \pm1$, $\sigma \in \Pi(\Theta)$.

A U-invariant Riemannian metric ds^2_Λ on $\mathbb{F}_\Theta$ is completely determined by the following inner product $\langle\cdot,\cdot\rangle$ on η_Θ

$$\langle X, Y\rangle_\Lambda := -\langle \Lambda X, Y\rangle \tag{7}$$

with $\Lambda : \eta_\Theta \to \eta_\Theta$ definite-positive with respect to the Cartan-Killing form. On each irreducible component of $\mathfrak{q}_\Theta$, $\Lambda = \lambda_\sigma \mathrm{id}$ with $\lambda_{-\sigma} = \lambda_\sigma > 0$.

Consider τ the conjugation of $\mathfrak{g}$ relatively to $\mathfrak{u}$. Hence, $\langle\langle X, Y\rangle\rangle_\Lambda = \langle X, \tau Y\rangle_\Lambda$ is a Hermitian form on $\mathfrak{g}$, that originates a U-invariant Hermitian form on $\mathbb{F}_\Theta$.

If $\Omega = \Omega_{J,\Lambda}$ denotes the corresponding Kähler form then

$$\Omega(X_\alpha, X_\beta) = -\sqrt{-1}\lambda_\alpha\epsilon_\beta\langle X_\alpha, X_\beta\rangle. \tag{8}$$

We recall that an almost-Hermitian manifold is said $(1,2)$-symplectic if $d\Omega(X, Y, Z) = 0$ when one of the vectors X, Y, Z is of type $(1,0)$, and the other two are of type $(0,1)$. If J is integrable and $d\Omega \equiv 0$, we say $(\mathbb{F}_\Theta, J, ds^2_\Lambda)$ is a Kähler manifold.

3. Maps on $\mathbb{F}_\Theta$

From now on, for abuse of notation we will denote a map $\phi : M \to \mathbb{F}_\Theta$ by $(\phi_\sigma)_{\sigma\in\Pi(\Theta)}$ where $\phi_\sigma : M^2 \to E_\sigma$, despite ϕ and $(\phi_\sigma)_{\sigma\in\Pi(\Theta)}$ being completely different objects.

Black for the cases B_l, C_l and D_l (see [2]) and the author in the case A_l (see ([21]), obtained the Cauchy-Riemann equations in our situation:

Proposition 3.1. *A map $\phi : M^2 \to (\mathbb{F}_\Theta,\ J)$ is J-holomorphic on $p \in M$ if and only if for every $\sigma \in \Pi(\Theta)$, $\phi_\sigma(p) \neq 0$ implies $\phi_{-\sigma}(p) = 0$.*

We define the energy of ϕ as:

$$\begin{aligned} E(\phi) &= \frac{1}{2}\int_M \begin{pmatrix} \left\langle\left\langle \frac{\partial\phi}{\partial z}, \frac{\partial\phi}{\partial z}\right\rangle\right\rangle_\Lambda + \\ \left\langle\left\langle \frac{\partial\phi}{\partial \bar z}, \frac{\partial\phi}{\partial \bar z}\right\rangle\right\rangle_\Lambda \end{pmatrix} \upsilon_g \\ &= \frac{1}{2}\sum_{\sigma\in\Pi(\Theta)^+}\int_M \begin{pmatrix} \langle\langle \phi_{\epsilon_\sigma\sigma}(p), \phi_{\epsilon_\sigma\sigma}(p)\rangle\rangle_\Lambda + \\ \langle\langle \phi_{\epsilon_{-\sigma}\sigma}(p), \phi_{\epsilon_{-\sigma}\sigma}(p)\rangle\rangle_\Lambda \end{pmatrix} \upsilon_g \end{aligned}$$

To deduce the harmonic map equations, a basic remark is that

$\int_{M^2}(\left\langle\left\langle \frac{\partial q_1}{\partial z}\ , q_2\right\rangle\right\rangle + \left\langle\left\langle q_1\ , \frac{\partial q_2}{\partial \bar z}\right\rangle\right\rangle)\ \upsilon_g = 0$, for any perturbations $q_1, q_2 : M^2 \to g$. In fact, every map $f : M^2 \to \mathbb{C}$ satisfies

$$df(p) = \frac{\partial f}{\partial x}(p)\ dx + \frac{\partial f}{\partial y}(p)dy =$$
$$= \tfrac{1}{2}(\frac{\partial f}{\partial z}(p)dz + \sqrt{-1}\ \frac{\partial f}{\partial z}(p)d(\sqrt{-1}z)\ ) =$$
$$= \tfrac{1}{2}\frac{\partial f}{\partial z}(p)dz + (-\tfrac{\sqrt{-1}}{2})\sqrt{-1}\frac{\partial f}{\partial z}(p) = \frac{\partial f}{\partial z}(p)dz\ .$$

We now consider the map $f = \langle\langle q_1\ , q_2\rangle\rangle : M^2 \to \mathbb{C}$. $\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\langle\langle q_1\ , q_2\rangle\rangle =$ $\frac{\partial}{\partial z}\langle q_1\ , \overline{q_2}\rangle = \left\langle \frac{\partial q_1}{\partial z}\ , \overline{q_2}\right\rangle + \left\langle q_1\ , \frac{\partial}{\partial z}(\overline{q_2})\right\rangle = \left\langle\left\langle \frac{\partial q_1}{\partial z}\ , q_2\right\rangle\right\rangle + \left\langle q_1\ , \overline{(\frac{\partial q_2}{\partial \bar z})}\right\rangle =$ $\left\langle\left\langle \frac{\partial q_1}{\partial z}\ , q_2\right\rangle\right\rangle + \left\langle\left\langle q_1\ , \frac{\partial q_2}{\partial \bar z}\right\rangle\right\rangle$. According to Stokes' Theorem we have $\int_{M^2} df(p)\upsilon_g = \int_{\partial(M^2)} f(p)\upsilon_g = 0$. Thus,

$$\int_{M^2}(\left\langle\left\langle \frac{\partial q_1}{\partial z}\ , q_2\right\rangle\right\rangle + \left\langle\left\langle q_1\ , \frac{\partial q_2}{\partial \bar z}\right\rangle\right\rangle)\ \upsilon_g = 0.$$

We now perturb the map ϕ in the following natural way

$$\phi^t(p) := \mathrm{e}^{tq(p)} \circ \phi(p) \qquad -\epsilon < t < \epsilon$$

We are considering here the natural action of $Gl(n, C)$ on $\mathbb{F}_\Theta$ and taking an arbitrary C^∞ map $q : M^2 \to \mathfrak{gl}(\mathfrak{n}, \mathbb{C})$.

In [23] we deduce the following Euler-Lagrange equations for our variational problem

Proposition 3.2. *A map $\phi : (M^2, g) \to (\mathbb{F}_\Theta, ds^2_\Lambda)$ is harmonic if and only if $\frac{d}{dt}|_{t=0}\, E(\phi_t) = 0$ if and only if*

$$\mathrm{Re}\left(\sum_{\alpha\in\Pi} \lambda_\alpha \nabla_{\overline{z}}\phi_\alpha(p)\right) = 0, \text{ for every } p \in M \tag{9}$$

We will use a generalization of J to an f-structure following Yano ([29]). An f -structure $\mathcal{F}$ on $\mathbb{F}_\Theta$ is a section of $End(T(\mathbb{F}_\Theta))$ such that $\mathcal{F}^3 + \mathcal{F} = 0$.

An invariant f-structure is given by the matrix $\epsilon(\mathcal{F}) = (f_\alpha)_{\alpha\in\Pi_\Theta}$ with $f_\alpha = 1$, -1 or 0, according to the eigenvalues of $\mathcal{F}$.

We now state the Cauchy-Riemann equations in the case of f-structures

Proposition 3.3. *A map $\phi : (M^2, J) \to (F_\Theta, \mathcal{F})$ is $\mathcal{F}$-holomorphic if and only if it is subordinate to $\epsilon(\mathcal{F})$.*

Definition 3.1. *Consider an invariant f-structure $\mathcal{F}$ on $\mathbb{F}_\Theta$. Let*

$$\mathcal{F}_+ := \sum_{\substack{\alpha\in\Pi_\Theta \\ f_\alpha=1}} f_\alpha$$

and

$$\mathcal{F}_- := \sum_{\substack{\alpha\in\Pi_\Theta \\ f_\alpha=-1}} f_\alpha.$$

$\mathcal{F}$ is said horizontal if $[\mathcal{F}_+, \mathcal{F}_-] \subset \mathfrak{p}_\Theta$.

The following theorem due to Black ([2]) is essential in our study

Theorem 3.4. *Let $\phi = (\phi_\alpha) : (M^2, J, g) \to (\mathbb{F}_\Theta, \mathcal{F}, ds^2_\Lambda)$ be subordinate to a horizontal f-structure $\mathcal{F}$. Then ϕ is equi-harmonic.*

Consider now $\dfrac{\partial\varphi}{\partial z}(p) = \sum_{\alpha\in\Pi_\Theta} x_\alpha(p)X_\alpha$. We can prove that $\mathcal{H}^\varphi := \{\alpha \in \Pi_\Theta,\, x_\alpha(p) = 0\}$ is a horizontal f-structure, and we will call it by f-structure associated to φ.

We will now exhibit families of equi-harmonic and holomorphic maps $\psi : (M^2, J, g) \to (\mathbb{F}_\Theta, J, ds^2_\Lambda)$ subordinate to an horizontal f-structure, thus all of them are equi-harmonic according to Theorem 3.4. Any map in these families, is called a holomorphic and horizontal frame.

Let $h : M^2 \to \mathbb{C}P^{n-1}$ be a holomorphic and non-degenerate map. We consider its associate curve $\theta_k : M^2 \to G_k(\mathbb{C}^n)$, where $\theta_k(p) := h(z)\wedge\frac{\partial h}{\partial z}(p)\wedge\cdots\wedge\frac{\partial^{(k-1)}h}{\partial z}(p)$ and $\Pi_k := \theta_k \cap \theta_{k-1}^\perp$.

We define the map $\Psi = (\Pi_1, \Pi_2, ..., \Pi_n) : M^2 \to F(n)$. We can prove that $\frac{\partial\Psi}{\partial z} = \sum_{\alpha\in\Pi} x_\alpha(p)X_\alpha$, with $x_\alpha \equiv 0$ if $\alpha \in \Pi - \sum$, where $\sum = \{\alpha_{12}, \alpha_{23}, ..., \alpha_{(n-1)n}\}$ denotes a simple root system for $sl\,(n+1, \mathbb{C})$.

We can prove that any such map Ψ is holomorphic and subordinate to $\mathcal{H}^\Psi$, thus, again according to Theorem 3.4, it is an equi-harmonic map.

More generally, we will now construct families of holomorphic and equi-harmonic maps $\Psi : M^2 \to \mathbb{F}(n; n_1, n_2, ..., n_k)$.

Let $\mathbb{F}$ be the geometric flag manifold $\mathbb{F} = \frac{U(n)}{U(n_1)\times U(n_2)\times \ldots \times U(n_k)} = \mathbb{F}(n; n_1, n_2, ..., n_k)$, where $n_i > 0$, $k \geq 3$ and $n_1 + n_2 + ... + n_k = n$.

A root system of height one with respect to Θ, is given by:

$$\sum(\Theta) = \left\{ \begin{array}{c} \alpha_{1(n_1+1)}, ..., \alpha_{1(n_1+n_2)}, ..., \alpha_{n_1(n_1+1)}, ..., \\ \alpha_{n_1(n_1+1)}, ..., \alpha_{(n_1+1)(n_1+n_2+1)}, ..., \alpha_{(n_1+1)(n_1+n_2+n_3)}, \\ , ..., \alpha_{(n_1+n_2)(n_1+n_2+1)}, ..., \alpha_{(n_1+n_2)(n_1+n_2+n_3)}, ..., \\ ..., \alpha_{(n_1+n_2+...+n_{k-1})(n_1+...+n_{k-1}+1)}, ..., \\ ..., \alpha_{(n_1+n_2+...+n_{k-1})n} \end{array} \right\}$$

Let $h : M^2 \to CP^{n-1}$ any holomorphic and nondegenerate map, and $\Psi_0 = (\Pi_1, ..., \Pi_n) : M^2 \to F(n)$ as we have defined above.

We define the map $\Psi = (\Psi_1, ..., \Psi_k) : M^2 \to F(n; n_1, ..., n_k)$ by: $\Psi_1 = \Pi_1 + ... + \Pi_{n_1}, ..., \Psi_k = \Pi_{n_{k-1}+1} + ... + \Pi_{n_k}$.

We prove in [23] that any such Ψ is a holomorphic–horizontal frame.

4. The second variation of energy and stability on $\mathbb{F}_\Theta$

We compute now the second variation of the energy in our situation.

Theorem 4.1. *Consider a harmonic map $\phi : (M^2, g) \to (\mathbb{F}_\Theta, ds^2_\Lambda)$.*

Thus,

$$\frac{d^2}{dt^2}\bigg|_{t=0} E(\phi_t) = I^\phi_\Lambda(q) = \operatorname{Re}\left(\begin{array}{c} \int_M (\left\langle \left[q(p), \frac{\partial q}{\partial z}(p)\right], \frac{\partial \phi}{\partial z}(p)\right\rangle_\Lambda + \\ + \left\langle \left[q(p), \frac{\partial q}{\partial \bar z}(p)\right], \frac{\partial \phi}{\partial \bar z}(p)\right\rangle_\Lambda V_g \end{array} \right) \tag{10}$$

$$+\frac{1}{2}\operatorname{Re}\left(\int_M \left\langle \frac{\partial q}{\partial z}(p), \frac{\partial q}{\partial z}(p)\right\rangle_\Lambda \nu_g\right) \tag{11}$$

$$+\frac{1}{2}\operatorname{Re}\left(\int_M \left\langle \frac{\partial q}{\partial \bar z}(p), \frac{\partial q}{\partial \bar z}(p)\right\rangle_\Lambda \nu_g\right) \tag{12}$$

$$= \operatorname{Re}\left(\begin{array}{c} \int_M (\left\langle \left[q(p), \frac{\partial q}{\partial z}(p)\right], \frac{\partial \phi}{\partial z}(p)\right\rangle_\Lambda + \\ + \left\langle \left[q(p), \frac{\partial q}{\partial \bar z}(p)\right], \frac{\partial \phi}{\partial \bar z}(p)\right\rangle_\Lambda) \upsilon_g \end{array} \right), \tag{13}$$

$$+\frac{1}{2}\operatorname{Re}(\sum_{\alpha\in\Pi}\int_M \lambda_\alpha(\langle q_\alpha(p), q_\alpha(p)\rangle + \langle q_{-\alpha}(p), q_{-\alpha}(p)\rangle)\upsilon_g) \tag{14}$$

where the map $q : M \to \mathfrak{gl}(\mathfrak{n}, \mathfrak{C})$ is defined by $\frac{\partial q}{\partial z}(p) = \sum_{\alpha\in\Pi} q_\alpha(p) X_\alpha$.

Definition 4.1. *A harmonic map $\phi : (M^2, g) \to (\mathbb{F}_\Theta, ds^2_\Lambda)$ is said stable if $I^\phi_\Lambda(q) \geq 0$, for any variation $q : M^2 \to g$. Otherwise, ϕ is said unstable.*

The following Theorem due to Lichnerowicz ([19]) is fundamental in our study of stability on flags.

Theorem 4.2. *Let $\phi : (M^2, J, g) \to (\mathbb{F}_\Theta, J, ds^2_\Lambda)$ be a J-holomorphic map and $(\mathbb{F}_\Theta, J, ds^2_\Lambda)$ a Kähler structure. Then ϕ is stable.*

Definition 4.2. *We say that $\Lambda'_{\mathcal{P}} = (\lambda'_\sigma)_{\sigma\in\Pi(\Theta)}$ is a $\mathcal{P}$-perturbation of $\Lambda = (\lambda_\sigma)_\sigma$ subordinate to $\psi = (\psi_\sigma)_{\sigma\in\Pi(\Theta)} : M^2 \to \mathbb{F}_\Theta$ if*

1. $\mathcal{P} \subset \Pi(\Theta)$;
2. $\lambda'_\sigma = \lambda_\sigma$ if $\sigma \in \mathcal{P}$;
3. $\lambda'_\sigma = \xi_\sigma + \lambda_\sigma > 0$, $\xi_\sigma \in \mathbb{R}$ if $\sigma \in \Pi(\Theta) - \mathcal{P}$;
4. $\psi_\sigma = 0$ if $\sigma \in \Pi(\Theta) - \mathcal{P}$.

Regarding the families of the holomorphic and horizontal frames we have just defined, we can simply consider $\mathcal{P} = \Sigma(\Theta)$.

Using the above definition of perturbation we derive the following basic lemma.

Lemma 4.3. *Let $\psi \approx (\psi_\sigma)_\sigma : (M^2, g) \to (\mathbb{F}_\Theta, ds^2_{\Lambda_\Theta})$ a holomorphic and horizontal frame. Then,*

$$I^{\psi}_{\Lambda'_{\mathcal{P}}}(q) = I^{\psi}_{\Lambda}(q) + \tag{15}$$

$$+ \sum_{\sigma \in \Pi(\Theta) - \mathcal{P}} \xi_\sigma \left(\begin{array}{c} \int_M \lambda_\sigma(\langle q_\sigma(p), q_\sigma(p)\rangle + \\ \langle q_{-\sigma}(p), q_{-\sigma}(p)\rangle) \upsilon_g \end{array} \right). \tag{16}$$

According to Gray-Hervella ([16]) the almost Hermitian structures can be decomposed into four irreducible components. For instance, $\{0\}$ corresponds to Kähler metrics, $W_1 \oplus W_2$ to the $(1,2)$-symplectic ones and, so on. See [25] and [26].

Lemma 4.4. *A necessary and sufficient condition for $\left(\mathbb{F}, J, ds^2_\Lambda\right)$ to be in $W_1 \oplus W_3$ is: $\lambda_\alpha = \lambda_\beta = \lambda_\gamma$ if $\{\alpha, \beta, \gamma\}$ is a $(0,3)$-triple.*

As an immediate consequence of this lemma we notice that the Cartan-Killing structure is in $W_1 \oplus W_3$. We will now consider perturbations of the Cartan-Killing structure.

We consider a $J = (\epsilon_\alpha)$ and denote by $C(J)$ the subset of roots α such that there exists a $(0,3)$-triple $\{\alpha, \beta, \gamma\}$.

Let $ds^2_{\Lambda^0 = (\lambda^0_\alpha)}$ given by $\lambda^0_\alpha = k > 0$ for each $\alpha \in \Sigma \cup C(J)$, and $0 < \lambda^0_\alpha \leq k$ otherwise. According to Lemma 4.4, $(\mathbb{F}, J, ds^2_{\Lambda^0}) \in W_1 \oplus W_3$. We can prove the following theorem.

Theorem 4.5. *Let $\psi = (\psi_\alpha) : M^2 \to \mathbb{F}$ be an arbitrary holomorphic–horizontal frame. Then, $\psi : (M^2, g) \to (\mathbb{F}, ds^2_{\Lambda^0})$ is unstable.*

5. Stability results on $\mathbb{F}$

According to the results obtained in [25], among all the invariant Hermitian structures, the main cases are $W_1 \oplus W_2$ and $W_1 \oplus W_3$. We will now discuss the stability phenomenon of holomorphic–horizontal frames in these two main classes.

Based on a crucial result derived in [25] we present the following definition.

Definition 5.1. *Let $\epsilon' = (\epsilon'_\alpha)_{\alpha \in \Pi}$, $\epsilon'_\alpha = \pm 1$ and $\epsilon'_{-\alpha} = -\epsilon'_\alpha$. We fix a Kähler structure $(\mathbb{F}, (\epsilon_\alpha)_{\alpha \in \Pi}, (\lambda_\alpha)_{\alpha \in \Pi})$. The metric $\Lambda' = (\lambda'_\alpha)_{\alpha \in \Pi}$ is said a perturbation of type (1,2)-symplectic of $\Lambda = (\lambda_\alpha)_{\alpha \in \Pi}$ if*

1. for each $\alpha \in \Pi^+$ with $\epsilon'_\alpha = +1$ and, $\alpha = \alpha_1 + \cdots + \alpha_s$ we have $\lambda'_\alpha = \lambda_{\alpha_1} + \cdots + \lambda_{\alpha_s}$.

2. for each $\alpha \in \Pi^+$ with $\epsilon_\alpha^{'} = -1$ and if $\alpha = \mu - \alpha_1 - \cdots - \alpha_k$, where μ is the highest root, and each $\alpha_i \in \Sigma$, then $\lambda_\alpha^{'} = \lambda_\mu + \lambda_{\alpha_1} + \cdots + \lambda_s$.

We now are ready to discuss the $W_1 \oplus W_2$ case. We consider $(\mathbb{F}, (\epsilon_\alpha^{'}), ds^2_{\Lambda'})$ equipped with an invariant Hermitian structure that comes from a perturbation of type (1,2)-symplectic of a Kähler structure $(\mathbb{F}, (\epsilon_\alpha), ds^2_\Lambda)$. Thus, in [23] we prove

Theorem 5.1. *Let $\psi : (M^2, g) \to (\mathbb{F}, ds^2_{\Lambda'})$ be a holomorphic–horizontal frame. Then ψ is stable.*

We will now concentrate our attention on the family of invariant Hermitian structures on $\mathbb{F}$ that are in $W_1 \oplus W_3$. We begin our discussion mentioning the classification of Einstein metrics on $\mathbb{F}(3)$ and $\mathbb{F}(4)$ derived in [17] and [22], exploiting these results and obtaining (see [23])

Theorem 5.2. *Let $\psi : (M^2, g) \to (\mathbb{F}(n), ds^2_{\Lambda=(\lambda_{ij})})$ a holomorphic and horizontal frame with $n = 3$ or 4 and $ds^2_{\Lambda=(\lambda_{ij})}$ a Einstein and non-Kähler metric. Then, the map ψ is unstable.*

A basic result due to Arvanitoyeorgos [1] and Kimura [17] is the following one.

Theorem 5.3. *The space $\mathbb{F}(n)$ for $n = 3$ admits as Einstein metrics only the normal and the Kähler-Einstein metrics. If $n \geq 4$ it admits at least $\frac{n!}{2} + n + 1$ Einstein metrics. The $\frac{n!}{2}$ metrics are the already mentioned Kähler-Einstein metrics described by Borel, one is the usual normal metric and the remaining n are given explicitly as follows:*

1. $\lambda_{si} = \lambda_{sj} = n - 1$, $i \neq s,\ j \neq s$
2. $\lambda_{kl} = n + 1$, $k, l \neq s \quad (1 \leq s \leq n)$.

More generally, in his Ph.D. thesis ([12]), dos Santos has found new families of Einstein non-Kähler metrics on arbitrary $\mathbb{F}(n)$. See also [9] and [13] for additional details.

We notice that any known invariant Einstein metric on $\mathbb{F}(n)$ has a common feature: either it is Kähler or is in $W_1 \oplus W_3$. In fact, we believe that this fact is true for any Einstein metric on $\mathbb{F}(n)$.

Using an appropriate Cartan-Killing perturbation (as in Theorem 4.5) we can prove.

Theorem 5.4. *Let $(\mathbb{F}(n), ds^2_{\Lambda=(\lambda_{ij})})$ equipped with any of the known Einstein non-Kähler metrics above described, and $\psi : (M^2, g) \to (\mathbb{F}(n), ds^2_{\Lambda=(\lambda_{ij})})$ be any arbitrary holomorphic–horizontal frame. Then, ψ is unstable.*

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