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CLASSIFICATORY PROBLEMS IN AFFINE GEOMETRY APPROACHED BY DIFFERENTIAL EQUATIONS METHODS

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Abstract. We present in this survey article an account of some examples on the use of nonlinear differential equations, both partial and ordinary, that have been applied to the treatment of classificatory problems in affine geometry of hypersurfaces. Locally strongly convex, complete affine hyperspheres is the first topic explained, then hypersurfaces of decomposable type, and, finally, those with parallel second fundamental (cubic) form.

INTRODUCTION

While the origins of Affine Differential Geometry may be traced back to the beginning of the 20th. Century, with the articles by Tzitzeica [31], the full development of the theory is due to W. Blaschke and his school (Berwald, Franck, Gross, Liebmann, Pick, Radon, Reidemeister, etc.), since 1916. The progress was so fast that Blaschke, with Reidemeister as coauthor, published the first monography on the topic in 1923, [1].

Later, and through the development of topics like the classificatory problem of locally strongly convex affine hyperspheres, initiated by Calabi [2, 3], there appeared a renewed interest in the area, highlighted by the dedication, too, of renowned geometers. Here there appear the methods of partial differential equations of Monge-Ampère type for the treatment of the problem, that received the contributions of several authors before the complete and final solution was achieved: Schneider [29], Pogorelov [27], Cheng and Yau [5], Sasaki [28], Li [23], and the present author [7, 8, 9, 13].

As for the use of ordinary differential equations in classificatory problems in the area, the first, historic treatment in the mathematical literature appears with the article by Gläsner [22]. However, it was shown later that the results published in that work were incomplete [11, 13, 14]. The further development of this particular topic was constituted by the analysis of characterization and classification of the so-called class of *affine hypersurfaces of decomposable type*, also known as *translation hypersurfaces with plane translation curves* or, even, *Scherk's type of hypersurfaces.*

More recently, there appear again the methods of partial differential equations of Monge-Ampère type for the treatment of problems. In fact, in [17] we developed a new method for approaching the Classificatory Problem of Affine Hypersurfaces with parallel Second Fundamental (Cubic) Form. This problem had been considered by other authors, who obtained results only for dimensions $n = 2, 3$ [25, 26, 30]. The above mentioned method allowed to approach the problem for every case of dimension $n \geq 2$ and is based, primarily, in the reduction of the, eminently geometric, problem in terms of the classification of certain types of solutions to equations of Monge-Ampère type and, from there, to the implementation of the so-called *algorithmic sequence of coordinate changes* which facilitate integration of the respective Hessian matrix.

This article is organized as follows: in Section 1 we present, in a condensed form, the main necessary tools, that shall be used in the rest of the paper, concerning the theory of affine hypersurfaces, by using the method of moving frames. In Section 2 we explain the historical development on the classification problem of locally strongly convex, complete affine hyperspheres, to its final solution. The topic of affine hypersurfaces of decomposable type occupies section 3; while in Section 4, affine hypersurfaces with affine normal parallel second fundamental (cubic) form is also explained and developed to its final solution.

1. AFFINE HYPERSURFACE GEOMETRY

Let $X : M^n \to E^{n+1}$ be a differentiable, codimension-one immersion of the real, oriented, n-dimensional, abstract differentiable manifold M, into the $(n+1)$-dimensional real vector space E (we could take, for example $E := \mathrm{R}^{n+1}$, by considering only the real vector space structure of R^{n+1}). From this we can construct several geometrical objects, as induced in suitable manifolds. Thus, the tangent map of X, dX, can be thought of as a section in the bundle $Hom(TM, E)$, which is canonically isomorphic to $E \otimes T^*M$, i.e., $dX = DX \in \Gamma\left(Hom(TM, E) = \Gamma\left(E \otimes T^*M\right)\right.$, where TM represents the tangent bundle of M, T^*M its cotangent bundle, and D is the ambient space covariant derivative.

We shall use local representations such as $dX = D_iX \otimes dt^i = X_i \otimes dt^i$, where we agree to use, now and in most parts of what follows, the summation convention for repeated indices, in this case with reference to a system of local coordinates $\left(t^1, t^2, ..., t^n\right)$ of M, $D_iX := X_i$ denoting partial derivatives. In a similar fashion we can represent locally the Hessian of X by $D^2X := D_i(D_jX) \otimes \left(dt^i dt^j\right) = X_{ij} \otimes \left(dt^i dt^j\right)$

Next, we choose a non-zero exterior $(n+1)$-form in E, or determinant function, and denote its action, as usual, by square brackets $\omega^{n+1} := [\ ,.\ .\ .\ .\ .\ ,\ ] = \det$. This is the essential object that is used in the construction of a geometrical theory of invariants, for the hypersurface $X(M)$, under the action of the unimodular affine group $ASL(n+1, \mathrm{R})$. Thus, by considering the form $[D^2X, (DX)^n] := [X_{ij}, X_1, ..., X_n](dt^i dt^j) \otimes (dt^1 \wedge ... \wedge dt^n)$, the first and natural restriction, that is customarily made in affine geometry in order to develop a meaningful theory, is that the latter form be nondegenerate. This condition allows to normalize the scalar components of the given form and introduce the first fundamental form of the geometry.

In what follows we proceed to use the "method of moving frames" with the following notation for indices: small Latin letters shall range from 1 to $n = \dim(M)$, i.e., $1 \leq i, j, k, p, q.... \leq n$. Small Greek letters shall range from 1 to $n+1 = \dim(E)$: $1 \leq \alpha, \beta, \gamma, ... \leq n+1$. Thus, if $(f_1, f_2, ..., f_n)$ denotes a positively oriented frame field, locally defined on an open subset U of M, and

$(\sigma^1, \sigma^1, ..., \sigma^n)$ is the corresponding dual coframe, we can introduce a general affine frame field $(X, (e_1, e_2, ..., e_{n+1}))$ on the image hypersurface $X(U)$, by writing $e_i = dX(f_i)$ and prescribing that e_{n+1} be a non-zero differentiable vector field, transversal to $X(U)$ at each point. For this purpose it is enough to require that $[e_1, e_2, ..., e_{n+1}] \neq 0$. Hence, if $(\omega^1, \omega^2, ..., \omega^{n+1})$ denotes the coframe of one forms, dual to $(e_1, e_2, ..., e_{n+1})$ one obtains

$$\sigma^i = X^*(\omega^i), \ \omega^{n+1}_{|X(U)} = 0. \tag{1.1}$$

Then, the tangent map dX can be expressed by

$$dX = \sigma^i \otimes e_i. \tag{1.2}$$

Next, by writing the exterior differentials of the vector components of the frame in terms of the frame itself, one may introduce the attitude matrix of one forms (σ^β_α) by means of equations

$$de_\alpha = \sigma^\beta_\alpha e_\beta. \tag{1.3}$$

By exterior differentiating the last two numbered equations, and also considering (1.1), one further obtains

$$d\sigma^i = \sigma^j \wedge \sigma^i_j, \ \sigma^i \wedge \sigma^{n+1}_i = 0 \quad . \tag{1.4}$$

$$d\sigma^\beta_\alpha = \sum_\gamma \sigma^\gamma_\alpha \wedge \sigma^\beta_\gamma. \tag{1.5}$$

Moreover, since we are to work in unimodular affine geometry, the frame could, if necessary, be chosen to be equiaffine, i.e., $[e_1, e_2, ..., e_{n+1}] = 1$, and this relation implies

$$\sum_\alpha \sigma^\alpha_\alpha = \sigma^1_1 + \sigma^2_2 + \cdots + \sigma^{n+1}_{n+1} = 0. \tag{1.6}$$

Together, (1.4), (1.5) and (1.6) are the *Maurer-Cartan structural equations* of the immersion X, under the action of the unimodular affine group $ASL(n+1, \mathrm{R})$. They represent the *integrability conditions* in the language of moving frames.

Cartan's lemma, applied to the second equation in (1.4), implies the existence of differentiable functions h_{ij} such that

$$\sigma^{n+1}_i = h_{ij}\sigma^j; \ h_{ij} = h_{ji}, \tag{1.7}$$

where besides $H := \det(h_{ij}) \neq 0$, because X was assumed to be non-degenerate.

From the latter relation one obtains the *first fundamental form* of unimodular affine geometry I_{ua}, namely,

$$I_{ua} := \sum g_{ij}\sigma^i\sigma^j; \ g_{ij} := |H|^{-1/(n+2)}\, h_{ij}. \tag{1.8}$$

This is a real-valued, globally defined symmetric two-tensor representing the so-called Blaschke-Berwald, unimodular affine invariant, pseudo-Riemannian structure of the immersed manifold $X(M)$.

It is also a consequence of the nondegeneracy assumed for the immersion X that there exists, at each point of the image hypersurface $X(M)$, a unique general affine invariant straight line, called the *affine normal line*, which can be characterized by the condition that, the last component of the frame e_{n+1} lies in that direction if, and only if,

$$\sigma_{n+1}^{n+1} + \frac{1}{n+2} d\log|H| - \frac{2}{n+2} d\log[e_1, e_2, ..., e_{n+1}] = 0. \tag{1.9}$$

We shall denote by $X_N(M)$ *affine normal bundle*, i.e., the disjoint union of all affine normal lines. Then, the local expression of the unimodular affine normal N_{ua}, which is a section in the bundle $X_N(M)$, is given by $N_{ua} := |H|^{1/(n+2)} e_{n+1}$.

Next, we introduce the *affine normal connection* ∇ and the *second fundamental (Fubini-Pick cubic) form* II_{ua}: the first one being defined by projecting the ambient space covariant derivative D, along the affine normal direction, onto the corresponding image tangent space, and then pulling back to M. As for the second one we may proceed as follows: compute the exterior differential in equation (1.7) to obtain

$$d\sigma_i^{n+1} = dh_{ij} \wedge \sigma^j + h_{ij} d\sigma^j. \tag{1.10}$$

By applying into the latter the structural identities (1.4) and (1.5) we find that

$$\sum (dh_{ij} - h_{iq}\sigma_j^q - h_{qj}\sigma_i^q + h_{ij}\sigma_{n+1}^{n+1}) \wedge \sigma^j = 0. \tag{1.11}$$

It follows, by Cartan's Lemma, that

$$dh_{ij} - h_{iq}\sigma_j^q - h_{qj}\sigma_i^q + h_{ij}\sigma_{n+1}^{n+1} = h_{ijk}\sigma^k, \tag{1.12}$$

with the differentiable functions h_{ijk} symmetric in all of their indices i, j, k.

By using the latter relationship one obtains the *second fundamental form* of unimodular affine geometry II_{ua}, also known otherwise as the cubic form C, [10, 24, 25, 26], i.e.,

$$C = II_{ua} := \sum g_{ijk}\sigma^i\sigma^j\sigma^k, \;\; g_{ijk} := |H|^{-1/(n+1)} h_{ijk}. \tag{1.13}$$

This is a real-valued, globally defined symmetric three-tensor. This geometrical object, in conjunction with the first fundamental form I_{ua}, plays a roll quite similar to the first and second fundamental forms of euclidean geometry. See Integrability Conditions ahead and also [10, 13]. It is also easy to verify, from the above equations, the following relation involving both fundamental forms and the affine normal connection ∇:

$$II_{ua} = \nabla(I_{ua}) \tag{1.14}$$

A second covariant derivative can be introduced in a natural fashion within the geometry: the *Levi-Civita connection* associated to the first fundamental form I_{ua}, denoted by $\widetilde{\nabla}$. The two mentioned connections and the two first fundamental forms are also related, among them, by means of the following equation

$$\widetilde{\Gamma}_{jk}^i - \Gamma_{jk}^i = \frac{1}{2}\sum g^{ip}g_{pjk} := A_{jk}^i \tag{1.15}$$

where Γ_{jk}^i denote the local components (Christoffel symbols) of the affine normal connection, $\widetilde{\Gamma}_{jk}^i$ are the corresponding components of the Levi-Civita connection. The (1,2) tensor $A := \sum A_{jk}^i f_i \otimes (\sigma^j\sigma^k)$ defined by the latter equation is also known in the theory as the *difference tensor*.

We introduce now the *third fundamental form*:

Exterior differentiation, as applied to (1.9), and the consequent use of equation (1.5), allow us to conclude that

$$d\sigma_{n+1}^{n+1} = \sum \sigma_{n+1}^{\gamma} \wedge \sigma_{\gamma}^{n+1} = \sum \sigma_{n+1}^{k} \wedge \sigma_{k}^{n+1} = 0. \tag{1.16}$$

It follows, by Cartan's lemma, that there exist local differentiable functions L^{ij}, such that

$$\sigma_{n+1}^{i} = \sum L^{ij} \sigma_{j}^{n+1}, \;\; L^{ij} = L^{ji}. \tag{1.17}$$

It also follows, by means of a straightforward verification, that the fourth order object represented locally by the expression

$$III_{ga} := \sum L_{ij} \sigma^{i} \sigma^{j}. \tag{1.18}$$

with $L_{ij} := \sum L^{pq} g_{ip} g_{jq}$ is invariant under the action of the full general affine group. Similarly, the (1,1)-tensor whose scalar components are defined by $L_i^j := L^{jk} g_{ik}$ is obviously a unimodular affine invariant, known as the *affine shape operator*.

We show, next, that the third fundamental form and, in particular, the scalar components related with it can be computed in terms of the other objects already introduced in the geometry. In fact, if we denote by (Ω_i^j) the *curvature matrix of the normal connection*, then, by the structural equation (1.5), we can write

$$\Omega_i^j = d\sigma_i^j - \sum \sigma_i^k \wedge \sigma_k^j = \sigma_i^{n+1} \wedge \sigma_{n+1}^j. \tag{1.19}$$

Then,

$$\Omega_i^j = \sum L_q^j g_{ik} \sigma^k \wedge \sigma^q. \tag{1.20}$$

This shows that the local components representing the curvature tensor of the normal connection can be written

$$R_{ikq}^j = L_q^j g_{ik} - L_k^j g_{iq}, \tag{1.21}$$

equation which expresses the mentioned curvature tensor in terms of the third fundamental form III_{ga}, the first fundamental form I_{ua}, and which is also known as the *equation of Gauss*. From (1.21) we obtain immediately

$$L_q^j = \frac{1}{n-1} \sum g^{ik} R_{qik}^j, \tag{1.22}$$

proving our former assertion.

We enumerate next a complete set of *integrability conditions* representing the essential requirement that is needed in the formulation, and proof, of the so-called *Fundamental Existence Theorem* for the unimodular affine theory of hypersurfaces, in general dimensions greater or equal than two. The corresponding exposition can be made in a similar fashion to that in [10].

Integrability conditions.

(1) The scalar components g_{ij} and g_{ijk} are symmetric in all of their indices.
(2) The matrix (g_{ij}) is nonsingular, i.e., $\det(g_{ij}) \neq 0$.
(3) II_{ua} is apolar with respect to I_{ua}, i.e., $\sum g^{ij} g_{ijk} \equiv 0, \; k = 1, ..., n$.
(4) A *Mainardi-Codazzi* type condition holds. Namely,

$$g_{ijk,q} - g_{ijq,k} = L_{iq} g_{jk} + L_{jq} g_{ik} - L_{ik} g_{jq} - L_{jk} g_{iq}, \tag{1.23}$$

the comma , indicating covariant derivation with respect to the Levi-Civita connection.

(5) A *Gauss type condition* holds:

$$2\widetilde{R}_{jikq} = L_{jq}g_{ik} - L_{jk}g_{iq} + L_{ik}g_{jq} - L_{iq}g_{jk} - \sum A^s_{iq}g_{sjk} + \sum A^s_{ik}g_{sjq}, \quad (1.24)$$

with $\widetilde{R}_{jikq}$ denoting the components of the curvature tensor corresponding to the Blaschke-Berwald pseudo-Riemannian metric I_{ua}.

(6) The affine normal derivative of the shape operator satisfies

$$L^k_{i;j} = L^k_{j;i}. \quad (1.25)$$

This last condition is only needed for the case of dimension $n = 2$. In fact, for $n \geq 3$ the latter equation is a direct consequence of the Bianchi identity, as applied to the curvature tensor of the affine normal connection.

Other, related geometrical objects may be introduced in the theory. For example, the *unimodular affine principal curvatures* $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of the third fundamental form III_{ga} respect to the first one I_{ua}, i.e., determined as the roots of the equation $\det(L_{ij} + \lambda g_{ij}) = 0$, and their normalized elementary symmetric functions $L_1, L_2, ..., L_n$ are the *unimodular affine curvature symmetric functions*. Thus, the *unimodular affine mean curvature* is obtained by averaging the contraction of the third fundamental form with respect to the first (unimodular affine) fundamental form, i.e.,

$$L := L_1 = -\frac{1}{n}\sum g^{ij}L_{ij} = -\frac{1}{n}\sum L^k_k. \quad (1.26)$$

To end this section, we calculate the local components of the *Ricci tensor*, $\widetilde{Ric} := \sum \widetilde{R}_{ij}\sigma^i\sigma^j$, and the *scalar curvature*, $\widetilde{R} := \sum \widetilde{R}_{ij}g^{ij}$, by using previous equations (1.24) and (1.26), as being:

$$\widetilde{R}_{ij} = \frac{1}{2}(-nLg_{ij} + (n-2)L_{ij}) - \sum A^s_{iq}A^q_{js} \quad , \quad (1.27)$$

$$\sum g^{jk}g^{iq}\widetilde{R}_{jikq} = (1-n)\sum g^{iq}L_{iq} + \sum g^{iq}A^s_{ij}A^j_{qs}. \quad (1.28)$$

Then, the latter can be written, for brevity,

$$\widetilde{R} = L + J, \quad (1.29)$$

this equality being known as the *Higher Dimensional Affine Theorema Egregium*, where the "normalized" scalar curvature is defined by $\widetilde{R} := \frac{1}{n(n-1)}\sum g^{jk}g^{iq}\widetilde{R}_{jikq}$, and the Pick invariant J can be expressed, in terms of the components of the first two fundamental forms by $J := \frac{1}{n(n-1)}\sum g^{iq}A^s_{ij}A^j_{qs}$.

2. COMPLETE AFFINE HYPERSPHERES

An immersed, oriented hypersurface $X(M)$ is called an *affine hypersphere*, if the affine normal lines through each point of the image either all intersect at one point, called the *center of* $X(M)$, or else are all mutually parallel (*center at infinity*). In the first case it is also called *proper*, while in the second case it is called *improper* (or *parabolic*).

One of the outstanding problems in the unimodular affine geometry of hypersurfaces, which called the attention of quite a few distinguished geometers, was that of classifying all "*complete*" affine hyperspheres, under the additional hypothesis of local strong convexity for the immersed hypersurface $X(M)$. It is easy see that, in this case, $X(M)$ can be oriented in such a way that the first fundamental form of unimodular affine geometry I_{ua} becomes a (proper) Riemannian metric.

With the geometrical objects introduced so far, one may consider two (seemingly) different notions of completeness for the hypersurface $X(M)$:

(1) *Unimodular affine metric completeness*, i.e., completeness with regards to the Levi-Civita connection determined by the first fundamental form I_{ua}.

(2) *Geometrical completeness*, meaning that $X(M)$ is complete if, and only if, it is a closed set with respect to the ambient space topology, induced by the vector space structure of E. Some authors prefer to refer to this as *Euclidian completeness*, i.e., with respect to the (also) Riemannian metric induced on $X(M)$ from a Euclidian metric assumed to be further defined on the ambient vector space.

That the first two notions of completeness are different was shown by Schneider [29], who gave an example of a surface in affine 3-space which is geometrically complete but not unimodular affine metric complete. So far, locally strongly convex affine hyperspheres have been classified with respect to those two notions of completeness. In fact, the classification was first achieved for the improper (parabolic) case: Calabi proved, in [2], that every unimodular affine metric complete, locally strongly convex affine hypersphere of parabolic type and dimension $n \geq 2$ is indeed a paraboloid. As for geometrical completeness, the first result is due to Jörgens, who under this hypothesis reached the same conclusion as before, for the case of dimension $n = 2$, by using complex variable theory techniques. Calabi, also in [2], then extended the result for dimensions $n \leq 5$, by providing *a priori* third-order estimates for a convex solution of the nonlinear, partial differential equation $\det(\partial_{ij} u) = 1$. Finally, Pogorelov [27] proved the classification theorem for improper, geometrically complete, locally strongly convex affine hyperspheres in any dimension, by using convex theory and the previously mentioned result of Calabi. Pogorelov's method provided estimates for the second-order terms of the above differential equation.

As for the proper affine hyperspheres, Schneider proved, in the mentioned article [29], that every unimodular affine metric complete, or geometrically complete, hypersphere of elliptic type is an ellipsoid. The first case of completeness was also treated, and proved, by Calabi in [3].

In the last cited article, Calabi also gave examples of complete affine hyperspheres of hyperbolic type which were not hyperboloids (see also [1]), and made a conjecture as to the classification of these hyperspheres. Calabi's conjecture can be divided in two parts:

(1) "Every complete, n-dimensional affine hypersphere with mean curvature $L < 0$ is asymptotic to the boundary of a convex cone with vertex at the center; ...." .

(2) "....; every pointed, nondegenerate convex cone K determines an affine hypersphere of hyperbolic type, asymptotic to the boundary of K, and uniquely determined by the constant value $L < 0$ of its mean curvature.

Let E be a finite dimensional real vector space, and denote by E^* its dual.

A subset $K \subset E$ is called a *convex cone* if

(1) $X \in K,\ r > 0$ imply $rX \in K$.

(2) $X, Y \in K$ imply $X + Y \in K$

The set defined by $K^* := \{X^* : X^* \in E^*, X^* \cdot X \geq 0, \forall X \in K\}$ is a closed convex cone in E^* called the *dual cone* of K. Then, the latter is said to be *pointed* if

(3) $Int\,(K^*)$ is nonempty (equivalently, K contains no affine line), and to be *nondegenerate* if

(4) $Int\,(K)$ is nonempty.

Around the years 1975-76 there were quite a few people working on the conjecture: E. Calabi himself, L. Nirenberg, S.Y. Cheng, S. T. Yau, T. Sasaki and myself, among others. Most of people approached the solution to the problem by using the Legendre transformation method, suggested by Calabi in [3]. At the beginning of 1976 Cheng and Yau announced to have solved both parts of the conjecture and circulated a preprint among the interested people. However everyone could see, first hand from reading the preprint, that only the first part was treated. Thus, I decided to treat the remaining, second part of the conjecture in a paper which was published in 1981, [9]. Our own method of approaching the solution was different from the previously mentioned one, and is based on the Theory of Convex Cones, exposed in [7], together with properties of solutions to partial differential equations of Monge-Ampère type. The preprint by Cheng-Yau was only published in 1986, and they acknowledged that some of their results had also been obtained by E. Calabi and L. Nirenberg, by somehow different methods (unpublished). The problem was treated too, in [28], by T. Sasaki. Later, in [23], A. M. Li showed to have found, and filled, a gap in the proof by Cheng-Yau. Finally, in the research monograph [13], we treated the first part of the conjecture, too, as before by using the Theory of Convex Cones.

3. AFFINE HYPERSURFACES OF DECOMPOSABLE TYPE

In this section we present the so-called class of hypersurfaces of *decomposable type* in affine geometry. A hypersurface is in this class if it satisfies, basically, two conditions. First of all, it is projectable onto (part of) a hyperplane and can be expressed in the form of Monge, i.e. as an enough differentiable graph function, with respect to a suitable affine system of coordinates. Second, the graph function can be decomposed into a sum of terms, each of them depending on only one of the ambient space independent variables. We have preferred to use the term "decomposable" for such hypersurfaces, as opposed to "translation hypersurfaces" ("schiebflächen" in German), because the latter represents a much wider class of hypersurfaces. Some people prefer to refer to what we call "decomposable" class as "translation hypersurfaces with plane translation curves", or even "Scherk's type of hypersurfaces", because of the famous analogue surface in Euclidian Geometry.

Precisely, the only known previous result in Affine Geometry was the article by E. Gläsner [22] (see also the commentary in [6]). However, it should be noted that the results announced in that article were proven to be incomplete [11, 13].

In our first article dedicated to this topic, [11], we used the *method of qualitative analysis* in order to investigate the solutions of a fourth order, non linear, ordinary differential equation. This equation arose because of the splitting that occurs, in the diagonal terms of the third fundamental form (see equation (1.18) above), for a hypersurface of decomposable type. In this way we were able to classify, both locally and globally, the subclass, of the above class, consisting of those hypersurfaces with constant general affine mean curvature for all cases of dimension $n \geq 2$. Also, we stated there that the same methods could be used to deal with the case of affine minimal, alternatively called maximal, hypersurfaces [1, 4, 6], i.e., when the unimodular affine mean curvature vanishes identically. Moreover, due to the so-called "Higher dimensional affine theorema Egregium" (see equation (1.29) above), the latter scalar invariant is always related to two more invariants: the Pick invariant and the scalar curvature. Thus, continuing with that scheme of work we obtained afterwards the classification of:

- Affine hypersurfaces of decomposable type with constant Pick invariant.
- Affine hypersurfaces of decomposable type with constant unimodular affine mean curvature.
- Affine hypersurfaces of decomposable type with constant scalar curvature.

These results were published in the series of papers referenced by [12, 13, 14, 15, 16]. The method of work consisted, basically, in reducing the problem of classifying all nondegenerate hypersurfaces of each of the mentioned classes, to the analysis of solutions of some nonlinear, fourth order (third order in the case of the Pick invariant), ordinary differential equation. Then, we showed that such a kind of equations can be further reduced to a second-order, and finally to a first order ordinary differential equation. This last is the classifying differential equation for the class of hypersurfaces considered, and it generally depends on two constants: one coming from the hypothesis of belonging to the corresponding class, the other from the first integration that reduces the problem from second to first order. Some of the original types of solutions obtained for the latter are expressible as explicit solutions and can be further integrated in a simple fashion. For the remaining of cases it is more feasible to integrate first the corresponding inverse functions, and then proceed to implement the method of *qualitative analysis* of these solutions, in order to characterize the corresponding analytical and geometrical properties of the corresponding class.

In particular, for the case of unimodular affine mean curvature we compared that class with the class of affine hyperspheres, showing that actually there are very few belonging to both classes simultaneously. Finally, we were also able to show that, even in dimension $n = 2$, it is possible to construct more affine minimal (maximal) surfaces of translation, with plane translation curves, than there were previously reported (in [22]). In fact, the graphs of $f(x, y) = x^2 - (y)^{2/3}$ and $g(x, y) = -x^{2/3} - y^{2/3}$ are both surfaces of Scherk's type in affine geometry.

4. AFFINE HYPERSURFACES WITH PARALLEL CUBIC FORM

The problem of classifying all those affine hypersurfaces with parallel Second Fundamental (Cubic) form $C = II_{ua}$ (see (1.13)), with respect to the normal connection ∇, which are not hyperquadrics, i.e., $C = II_{ua} \neq 0$, was first treated and solved by Nomizu and Pinkall, in [25], proving the following

Theorem 4.1. *Let $X : M^2 \rightarrow E^3$ be a nondegenerate surface with parallel cubic form, $\nabla(C) = \nabla(II_{ua}) = 0$, which is not a quadric, i.e., $C = II_{ua}$ does not vanish identically on M. Then $X(M)$ is affinely congruent to the Cayley Surface, i.e., expressible as the graph function $t_3 = t_1 t_2 + t_1^3$.*

See also the book by Nomizu and Sasaki where a different method of proof is presented [26].

L. Vrancken treated next, in [30], the case of dimension $n = 3$ stated here as:

Theorem 4.2. *Let $X : M^3 \rightarrow E^4$ be a nondegenerate hypersurface with parallel cubic form, $\nabla(C) = \nabla(II_{ua}) = 0$, which is not a hyperquadric, i.e., $C = II_{ua}$ does not vanish identically on M. Then $X(M)$ is affinely congruent to one of the following graph immersions:*

a) $t_4 = t_1 t_2 + t_1^3 + t_3^2$.

b) $t_4 = t_1 t_2 + t_1^2 t_3 + t_3^2$

In our first article dedicated to the problem [17] we introduced a new method of approaching the solution, different to the ones previously used by the other mentioned authors. We called it *algorithmic sequence of coordinate changes* and consists, basically, in referring the hypersurface to a suitable linear coordinate system of the ambient space, and then making algorithmic adjustments into the Hessian matrix so that this can be integrated, fairly easily, to obtain the graph function representing the hypersurface. With this approach the classification depends strongly on two integer constants, that we labeled k and r, with $1 \leq k \leq n/2$, $1 \leq r \leq n-1$, where n equals the dimension of the immersed manifold ($n \geq 2$). These questions are made clear in the next, procedural result:

Lemma 4.3. *Let $X : M^n \rightarrow E^{n+1}$ be a nondegenerate hypersurface with parallel cubic form, $\nabla(C) = \nabla(II_{ua}) = 0$, which is not a hyperquadric, i.e., $C = II_{ua}$ does not vanish identically on M. Then there exists an affine coordinate system in the ambient space such that $X(M)$ is expressible in the form of Monge, i.e., by means of a graph function f and such that the corresponding Hessian matrix is given by $H(f) = (f_{ij}) = J_k + (x_{ij})$ where J_k is a matrix with k (≥ 1) blocks of the form $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in diagonal position, occupying the first $2k$ diagonal entries, the (possible) remaining diagonal elements are equal to 1, and with all of the rest of entries equal to 0; while all of the entries of the matrix (x_{ij}) are linear functions of the (domain) coordinates $t_1, t_2, ..., t_n$, i.e., $x_{ij} = \sum_k a_{ijk} t_k$. Moreover, the matrix of linear functions (x_{ij}) is everywhere singular, whose maximal rank r is attained on an open, dense subset of the domain, and we have $1 \leq r \leq n-1$.*

Moreover, in that article we presented new proofs of the previously stated results and, then, extended the classification to dimension $n = 4$.

The latter, main result reads as follows:

Theorem 4.4. *Let $X : M^4 \to E^5$ be a nondegenerate hypersurface with parallel cubic form, $\nabla(C) = \nabla(II_{ua}) = 0$, which is not a hyperquadric, i.e., $C = II_{ua}$ does not vanish identically on M. Then $X(M)$ is affinely congruent to one of the following graph immersions:*

a) *For $k = r = 1$: $t_5 = t_1t_2 + t_1^3 + t_3^2 + t_4^2$.*

b) *For $k = 1$, $r = 2$: $t_5 = t_1t_2 + t_1^2t_3 + t_3^2 + t_4^2$.*

c) *The case where $k = 1$, $r = 3$ is not possible, i.e., it does not exist any non degenerate hypersurface immersion with the required geometrical properties in the case where $k = 1$, $r = 3$.*

d) *For $k = 2$, $r = 1$: $t_5 = t_1t_2 + t_1^3 + t_3t_4$.*

e) *For $k = r = 2$ we have the following subcases:*

e_{11}) $t_5 = t_1t_2 + \frac{a}{6}t_1^3 + \frac{b}{2}t_1^2t_3 + \frac{c}{2}t_1t_3^2 + t_3t_4 + \frac{d}{6}t_3^3$,

e_{12}) $t_5 = t_1t_2 + \frac{c}{2}t_1t_3^2 + \beta\frac{c}{2}t_2t_3^2 + t_3t_4 + \frac{d}{6}t_3^3$.

f) *For $k = 2$, $r = 3$: $t_5 = t_1t_2 + \frac{a}{2}t_2t_3^2 + t_3t_4 + \frac{b}{2}t_1^2t_3 + \frac{c}{2}t_1t_3^2 + \frac{d}{6}t_1^3 + \frac{e}{6}t_3^3$.*

In each of cases e) *and* f) *the constants must be related among them in order to fulfill the condition that the maximal rank of the complementary matrix equals respectively 2 and 3, i.e., $r = 2, 3$.*

More recently [18] we extended the classification to the case of dimension $n = 5$.

Furthermore, from the results obtained in the last two articles we were led to formulate the possibility of fully classifying the class of hypersurfaces under consideration, coming to the conclusion that two questions had to be treated and developed:

1) Verification, by means of an inductive process, of the Conjecture: There exist no solutions to the problem for those classificatory values where $r > 2k$.

2) Implementation of an inductive scheme on the classificatory values n, k and r which would allow, from the classification process developed so far, to obtain all remaining possible cases of solution.

These considerations were elaborated in full detail in our yet unpublished article [19]. The main results here may be summarized as follows:

Theorem 4.5. *Let $X : M^n \to E^{n+1}$ be a nondegenerate hypersurface with parallel second fundamental (cubic) form, $\nabla(C) = \nabla(II_{ua}) = 0$, which is not a hyperquadric, i.e., $C = II_{ua}$ does not vanish identically on M, and suppose that it has already been expressed in the form indicated in Lemma 4.3, i.e., by means of graph function f with corresponding $n \times n$ Hessian matrix $H(f)$ and values k for the number of blocks of the form $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, where $1 \le k \le n/2$, and r for the maximal rank of the complementary matrix (x_{ij}), with $1 \le r \le n-1$. Then, the classificatory values k and r must satisfy, besides, the condition that r be less or equal than $2k$, i.e., there exist no solution to the classificatory problem in those cases where $r > 2k$.*

Theorem 4.6. *Let us consider the class of all nondegenerate affine hypersurfaces* $\left\{X : M^n \to E^{n+1} : n \geq 2\right\}$ *and the problem of finding and classifying those objects in the class with parallel second fundamental (cubic) form,* $\nabla(C) = \nabla(II_{ua}) = 0$*, which are not hyperquadrics, i.e., with* $C = II_{ua}$ *not vanishing identically on* M*. Then, there always exists a solution to the problem for every one of those classificatory values expressed by using Lemma 4.3, with the further restriction imposed by Theorem 4.5, i.e., for* $1 \leq k \leq n/2$ *and* $1 \leq r \leq \min\{2k, n-1\}$*. Moreover, all of those solutions can be obtained and classified, inductively, from the knowledge of the previously classified ones, exposed in Theorems 4.1, 4.2, 4.4 and, for dimension* $n = 5$*, the one in [18].*

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