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SMALL OSCILLATIONS ON $\mathbb{R}^2$ AND LIE THEORY

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Abstract. Making use of Lie theory we propose a model for the simple harmonic oscillator and for the linear inverse pendulum of $\mathbb{R}^2$. In both cases the phase space are orbits of the coadjoint representation of the Heisenberg Lie group. These orbits and the Heisenberg Lie algebra are included in a solvable Lie algebra admitting an ad-invariant metric. The corresponding quadratic form induces the Hamiltonian and the associated Hamiltonian system is a Lax equation.

1. Introduction

In classical mechanics a simple harmonic oscillator with one degree of freedom has a Hamiltonian H of the form

$$H(p,q) = \frac{1}{2}(p^2 + q^2) \tag{1}$$

where q is the position and $p = \dot{q}$ is the canonical momentum. This yields the following equation of motion

$$\begin{aligned} \tfrac{dq}{dt} &= -p \\ \tfrac{dp}{dt} &= q \end{aligned} \tag{2}$$

Another possible motion to consider in $\mathbb{R}^2$ is the linear inverse pendulum which has a Hamiltonian of the form

$$H(p,q) = \frac{1}{2}(p^2 - q^2) \tag{3}$$

which induces the equation of motion

$$\begin{aligned} \tfrac{dq}{dt} &= p \\ \tfrac{dp}{dt} &= q \end{aligned} \tag{4}$$

These equations predict the position and the velocity at any time if initial conditions $q(t_0)$, $p(t_0) = \dot{q}(t_0)$ are known. Both systems correspond to quadratic Hamiltonians on $\mathbb{R}^2$ given by

$$H(x) = \frac{1}{2}(Ax, x) \quad \text{with } x = (q,p) \in \mathbb{R}^2$$

where $(,)$ denotes the canonical inner product and A is a symmetric transformation for $(,)$. For the harmonic oscillator we are taking A as the identity and for the linear inverse pendulum we have

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In quantum mechanics a good approach to the harmonic oscillator is through the Heisenberg Lie algebra. In dimension three this is the Lie algebra generated by the position operator $Q = \textit{multiplication by } x$, the momentum operator $P = -i\frac{d}{dx}$ and 1 with the only non trivial commutation relation

$$[Q, P] = 1$$

These operators evolve according to the Heisenberg equations

$$\frac{dP}{dt} = -Q \qquad\qquad \frac{dQ}{dt} = P$$

In this work we shall show that the Heisenberg Lie algebra also allows an approach to the classical harmonic oscillator and to the linear inverse pendulum. To this end we shall make use of Lie theory, which was proved to be successful when studying some mechanical systems [A] [Ko2] [Sy]. This gives a general setting so that the Hamiltonian system is a Lax equation and functions in involution arise from the ad-invariant ones or with other weaker conditions as in [R1]. This algebraic framework was used with semisimple Lie algebras and the power of representation theory to describe for instance generalised Toda lattices. What we need is a Lie algebra with an ad-invariant metric, a splitting of this Lie algebra into a direct sum as vector subspaces of two subalgebras and a given function. In the case of semisimple Lie algebra the Killing form is the natural candidate for the ad-invariant metric.

However there are more Lie algebras admitting an ad-invariant metric. We shall endow a four dimensional solvable Lie algebra with such metric and we use the algebraic tools to construct a Hamiltonian system whose phase space are orbits of the coadjoint action of the Heisenberg Lie group, which is a proper normal subgroup of the corresponding solvable Lie group. By considering the restriction to the orbits of the function induced by the quadratic form of the metric, one constructs a Hamiltonian system which in one of the orbits is equivalent to (2). Since the function is ad-invariant, the Hamiltonian system takes the form of a Lax equation and the solution can be computed with help of the Adjoint map. We show a matrix realization for this system.

A similar procedure can be done for the inverse pendulum. It can be constructed also from a four dimensional solvable Lie algebra admitting an ad-invariant neutral metric. A certain splitting of the Lie algebra gives rise to the phase space as orbits of the coadjoint representation of the Heisenberg Lie algebra. The Hamiltonian system takes also the form of a Lax equation. But in this case the trajectories are not bounded.

The solvable Lie algebras we are making use of are the two solvable non abelian low-dimensional Lie algebras having an ad-invariant metric, which is not definite (see for instance [B-K] [O1]). In particular the ad-invariant metric in the first case is of Lorentzian type.

2. Preliminaries

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and exponential map $\exp : \mathfrak{g} \to G$. Let M be a smooth manifold and $\phi : G \times M \to M$ be a smooth action of G on M. The vector fields on M

$$\tilde{X}(m) = \frac{d}{dt}_{|t=0} \phi(\exp tX, m) \qquad m \in M, \quad X \in \mathfrak{g}, \quad t \in \mathbb{R}$$

will denote the infinitesimal generators of this action. If $G \cdot m = \{\phi(g, m),\, g \in G\}$ denotes the G-orbit through $m \in M$ its tangent space is the set

$$T_m(G \cdot m) = \{\tilde{X}(m) \,/\, X \in \mathfrak{g}\}.$$

Here we also make use of the notation $g \cdot m = \phi(g, m)$. The following actions are important in our setting:

- the adjoint action $\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$ whose infinitesimal generators are $\tilde{X} = \mathrm{ad}_X$, where $\mathrm{ad}_X Y = [X, Y]$ denotes the Lie bracket of $X, Y \in \mathfrak{g}$;

- the coadjoint action of G on $\mathfrak{g}^*$ is the dual of the adjoint action and it is given by $g \to \mathrm{Ad}^*(g^{-1})$, for $g \in G$, whose infinitesimal generator is $\tilde{X} = -\,\mathrm{ad}^*_X$.

The coadjoint orbits are examples of symplectic manifolds. Recall that they are endowed with the Kirillov-Kostant-Souriau symplectic structure given by:

$$\omega_\beta(\tilde{X}, \tilde{Y}) = -\beta([X, Y]), \qquad \beta \in G \cdot \mu.$$

Assume now that $\mathfrak{g}$ has a bi-invariant non-degenerate symmetric bilinear form $\langle , \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$; bi-invariant means that the maps $\mathrm{Ad}(g)$ are isometries for all $g \in G$. Then ad_X is skew symmetric with respect to $\langle , \rangle$ for any X and $\langle , \rangle$ induces a diffeomorphism between the adjoint orbit $G \cdot X$ and the coadjoint orbit $G \cdot \ell_X$ where $\ell_X(Y) = \langle X, Y \rangle$.

Suppose that the Lie algebra $\mathfrak{g}$ admits a splitting

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$$

as a direct sum of linear subspaces, where $\mathfrak{g}_+$, $\mathfrak{g}_-$ are subalgebras of $\mathfrak{g}$. Then the Lie algebra $\mathfrak{g}$ also splits as $\mathfrak{g} = \mathfrak{g}_+^\perp \oplus \mathfrak{g}_-^\perp$, where $\mathfrak{g}_\pm^\perp$ is isomorphic as vector spaces (via $\langle , \rangle$) to $\mathfrak{g}_\mp^*$. Let G_- denotes a subgroup of G with Lie algebra $\mathfrak{g}_-$. Then the coadjoint action of G_- on $\mathfrak{g}_-^*$ induces an action of G_- on $\mathfrak{g}_+^\perp$:

$$g_- \cdot X = \pi_{\mathfrak{g}_+^\perp}(\mathrm{Ad}(g_-)X) \quad g_- \in G_-, \quad X \in \mathfrak{g}_+^\perp,$$

where $\pi_{\mathfrak{g}_+^\perp}$ denotes the projection of $\mathfrak{g}$ on $\mathfrak{g}_+^\perp$. Thus the infinitesimal generator corresponding to $X_- \in \mathfrak{g}_-$ is

$$\tilde{X}(Y) = \pi_{\mathfrak{g}_+^\perp}([X_-, Y]) \qquad Y \in \mathfrak{g}_+^\perp.$$

The orbit $G_- \cdot Y$ becomes a symplectic manifold with the symplectic structure given by

$$\omega_X(\tilde{U_-}, \tilde{V_-}) = \langle X, [U_-, V_-] \rangle \qquad \text{for } U_-, V_- \in \mathfrak{g}_-, \tag{5}$$

which is induced by the Kostant-Kirillov-Souriau symplectic form on the coadjoint orbits in $\mathfrak{g}_-^*$.

Recall that the gradient of a function $f:\mathfrak{g}\to\mathbb{R}$ at the vector $X\in\mathfrak{g}$ is defined by

$$\langle\nabla f(X),Y\rangle = df_X(Y) \qquad\qquad Y\in\mathfrak{g}.$$

Consider the restriction of the function $f:\mathfrak{g}\to\mathbb{R}$ to an orbit $G_-\cdot X =: \mathcal{M}\subset\mathfrak{g}_+^{\perp}$. Then the Hamiltonian vector field of $H=f_{|\mathcal{M}}$ is given by

$$X_H(Y) = -\pi_{\mathfrak{g}_+^{\perp}}([\nabla f_-(Y),Y]) \tag{6}$$

where $Z_{\pm}$ denotes the projection of $Z\in\mathfrak{g}$ with respect to the decomposition $\mathfrak{g}=\mathfrak{g}_+\oplus\mathfrak{g}_-$. In fact for $Y\in\mathfrak{g}_+^{\perp}$, $V_-\in\mathfrak{g}_-$ we have

$$\begin{aligned}\omega_Y(\tilde{V_-},X_H) &= dH_Y(\tilde{V}) = \langle\nabla f(Y),\pi_{\mathfrak{g}_+^{\perp}}([V_-,Y])\rangle = \langle\nabla f_-(Y),[V_-,Y]\rangle\\ &= \langle Y,[\nabla f_-(Y),V_-]\rangle = \omega_Y(\nabla \tilde{f_-}(Y),\tilde{V}).\end{aligned}$$

Since ω is non degenerate, one gets (6). Therefore the Hamiltonian equation for $x:\mathbb{R}\to\mathfrak{g}$ follows

$$x'(t) = -\pi_{\mathfrak{g}_+^{\perp}}([\nabla f_-(x),x]). \tag{7}$$

In particular if f is ad-invariant then $0=[\nabla f(Y),Y]=[\nabla f_-(Y),Y]+[\nabla f_+(Y),Y]$. Since the metric is ad-invariant it holds $[\mathfrak{g}_+,\mathfrak{g}_+^{\perp}]\subset\mathfrak{g}_+^{\perp}$ and thus equation (7) takes the form

$$x'(t) = [\nabla f_+(x),x] = [x,\nabla f_-(x)], \tag{8}$$

that is, the equation (7) becomes a Lax equation.

If we assume now that the multiplication map $G_+\times G_-\to G$, $(g_+,g_-)\to g_+g_-$, is a diffeomorphism, then the initial value problem

$$\begin{cases}\frac{dx}{dt} &= -[x,\nabla f_+(x)]\\ x(0) &= x_0\end{cases} \tag{9}$$

can be solved by factorization. In fact if $\exp t\nabla f(x_0) = g_+(t)g_-(t)$, then $x(t) = \mathrm{Ad}(g_+(t))x_0$ is the solution of (9).

Remark. If the multiplication map $G_+\times G_-\to G$ is a bijection onto an open subset of G, then equation (7) has a local solution in an interval $(-\varepsilon,\varepsilon)$ for some $\varepsilon>0$.

3. The motion of the Harmonic oscillator

Let us now go back to the harmonic oscillator. The phase space in this case is $\mathbb{R}^2$, which is a symplectic manifold with the canonical structure given by

$$\omega = dq\wedge dp.$$

This has an associated Poisson structure, which for smooth functions f,g on $\mathbb{R}^2$ is defined by

$$\{f,g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}.$$

Consider the vector space over $\mathbb{R}$ generated by the functions $H=\frac{1}{2}(p^2+q^2)$, q, p, and 1. Since it is a closed subspace for the bracket $\{,\}$ then it becomes a solvable Lie algebra of dimension four. In fact we have the following rules

$$\{q,p\}=1 \qquad \{H,q\}=-p \qquad \{H,p\}=q.$$

In order to simplify notations let us rename these elements identifying X_3 with H, X_1 with q, X_2 with p and X_0 with the constant function 1 and set $\mathfrak{g}$ for the Lie algebra generated by these vectors with the Lie bracket $[,]$ derived from the Poisson structure.

The quadratic form on $\mathfrak{g}$ which for $X = x_0X_0 + x_1X_1 + x_2X_2 + x_3X_3$ is given by

$$q(X) = \frac{1}{2}(x_1^2 + x_2^2) + x_0x_3$$

induces an ad-invariant metric on $\mathfrak{g}$ denoted by $\langle , \rangle$.

The restriction of the quadratic form to $span\{X_1, X_2\}$ coincides with the canonical one on $\mathbb{R}^2$. In other words, the Lie algebra $\mathfrak{g}$ is the double extension of $\mathbb{R}^2$ with the canonical metric by the skew symmetric linear map which acts on $span\{X_1, X_2\}$ as the restriction of $\mathrm{ad}(X_3)$ to this space (see for instance [M-R] for the double extension procedure).

Consider the splitting of $\mathfrak{g}$ into a vector space direct sum $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, where $\mathfrak{g}_\pm$ denote the Lie subalgebras

$$\mathfrak{g}_- = span\{X_0, X_1, X_2\}, \qquad \mathfrak{g}_+ = span\{X_3\}. \tag{10}$$

Notice that the ideal $\mathfrak{g}_-$ is isomorphic to the 3-dimensional Heisenberg Lie algebra we denote by $\mathfrak{h}$. The metric induces a decomposition of the Lie algebra $\mathfrak{g}$ into a vector subspace direct sum of $\mathfrak{g}_+^\perp$ and $\mathfrak{g}_-^\perp$ where

$$\mathfrak{g}_-^\perp = span\{X_0\} \qquad \mathfrak{g}_+^\perp = span\{X_1, X_2, X_3\},$$

and it also induces linear isomorphisms $\mathfrak{g}_\pm^* \simeq \mathfrak{g}_\mp^\perp$. Let G denote a Lie group with Lie algebra $\mathfrak{g}$ and let $G_\pm \subset G$ be a Lie subgroup whose Lie algebra is $\mathfrak{g}_\pm$. The Lie subgroup G_- acts on $\mathfrak{g}_+^\perp$ by the "coadjoint" representation; indeed in terms of $U \in \mathfrak{g}_-$ and $V \in \mathfrak{g}_+^\perp$ we have

$$exp\ U \cdot V = [x_3(V)x_2(U) + x_1(V)]X_1 + [-x_3(V)x_1(U) + x_2(V)]X_2 + x_3(V)X_3. \tag{11}$$

Therefore the infinitesimal action of $\mathfrak{g}_-$ on $\mathfrak{g}_+^\perp$ is

$$\mathrm{ad}^*_U V \;\; = \;\; x_3(V)(x_2(U)X_1 - x_1(U)X_2) \tag{12}$$

It is not difficult to see that the orbits are trivial or 2-dimensional if $x_3(V) \neq 0$ and furthermore U and V belong to the same orbit if and only if $x_3(U) = x_3(V)$, hence the orbits are parametrized by the x_3-coordinate; so we denote them by $\mathcal{M}_{x_3}$. They are topologically like $\mathbb{R}^2$. In fact $\mathcal{M}_{x_3} = G_- \cdot V \simeq \mathbb{H}/Z(\mathbb{H})$, where $\mathbb{H}$ denotes the Heisenberg Lie group and $Z(\mathbb{H})$ its center.

Let us endow the orbits with the symplectic structure defined as in (5). Computing this explicitly for the orbit $\mathcal{M}_1$ we have

$$\omega_Y(\tilde{U_-}, \tilde{V_-}) = x_1(U_-)x_2(V_-) - x_1(V_-)x_2(U_-) \quad U_-, V_- \in \mathfrak{g}_-,$$

that is, the coordinates $x_i,\ i = 1, 2$, are the canonical symplectic coordinates and one can identify this orbit with $\mathbb{R}^2$, fact which will be reforced in the following.

Let $f : \mathfrak{g} \to \mathbb{R}$ be the ad-invariant function given by $q(X)$. The gradient of f at a point X is

$$\nabla f(X) = X.$$

The Hamiltonian system of $H = f_{|\mathcal{M}_{x_3}}$, the restriction of the ad-invariant function to the orbit, reduces to

$$\begin{array}{rcl} \frac{dx}{dt} & = & [x_3X_3, x_1X_1 + x_2X_2 + x_3X_3] \\ x(0) & = & x^0 \end{array} \tag{13}$$

where $x^0 = x_1^0X_1 + x_2^0X_2 + x_3^0X_3$.

For $x_3 \equiv x_3^0 \equiv 1$ this system is equivalent to (2). Then the trajectories $(x_1(t) = q(t), x_2(t) = p(t))$ are parametrized circles of angular velocity 1. More generally the trajectories on $\mathcal{M}_{x_3}$ are curves $x(t) = x_1(t)X_1 + x_2(t)X_2 + x_3(t)X_3$ where

$$\begin{array}{rcl} x_1(t) & = & x_1^0\cos(x_3^0t) + x_2^0\sin(x_3^0t) \\ x_2(t) & = & -x_1^0\sin(x_3^0t) + x_2^0\cos(x_3^0t) \\ x_3(t) & = & x_3^0 \end{array}$$

This solution coincides with that computed in the previous section, when we considered systems on coadjoint orbits. In fact it can be written as

$$X(t) = \mathrm{Ad}(exp\ tx_3^0X_3)X^0,$$

and one verifies that the flow at the point $X^0 \in \mathfrak{g}_+^\perp$ is then

$$\Delta^t(X^0) = (x_1^0\cos(x_3^0t) + x_2^0\sin(x_3^0t)X_1 + (-x_1^0\sin(x_3^0t) + x_2^0\cos(x_3^0t)X_2 + x_3^0X_3)) \tag{14}$$

System (13) is a Lax pair equation $L' = [M, L] = ML - LM$, taking as L and M the following matrices:

$$M = \begin{pmatrix} 0 & x_3 & 0 & 0 \\ -x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 0 & x_3 & 0 & x_1 \\ -x_3 & 0 & 0 & x_2 \\ -\frac{1}{2}x_2 & \frac{1}{2}x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

4. The inverse pendulum

Consider a linear system of one degree of freedom with Hamiltonian given by: $H(q,p) = \frac{1}{2}(p^2 - q^2),\ p \in \mathbb{R},\ q \in \mathbb{R}$, which yields the equation of motion

$$\begin{array}{rcccl} \frac{dp}{dt} & = & -\frac{\partial H}{\partial q} & = & q \\ \frac{dq}{dt} & = & \frac{\partial H}{\partial p} & = & p \end{array} \tag{15}$$

The phase space for this system is $\mathbb{R}^2$. Our aim now is to construct a model for this system in a similar setting as that of the previous section.

As in the case of the Harmonic oscillator let us consider the four dimensional real Lie algebra $\mathfrak{g}$ generated by H, q, p and 1 with the Lie bracket induced from the Poisson bracket on $\mathbb{R}^2$. Rename these elements as above identifying H with X_3, q with X_1, p with X_2 and 1 with X_0. Then we have the following non trivial Lie bracket relations:

$$[X_3, X_1] = X_2, \quad [X_3, X_2] = X_1, \quad [X_1, X_2] = X_0.$$

The quadratic form on $\mathfrak{g}$ which for $X = x_0X_0 + x_1X_1 + x_2X_2 + x_3X_3$ is given by

$$q(X) = \frac{1}{2}(x_1^2 - x_2^2) + x_0x_3$$

induces an ad-invariant neutral metric on $\mathfrak{g}$ denoted by $\langle , \rangle$.

The Lie algebra $\mathfrak{g}$ is the double extension of $\mathbb{R}^2$ by the skew symmetric linear map with respect to the neutral metric on $\mathbb{R}^2$ which acts on $span\{X_1, X_2\}$ as the restriction of $\mathrm{ad}(X_3)$ to this space. Notice that this map acts as the map A we mentioned in our introduction.

Consider the splitting of $\mathfrak{g}$ into a vector space direct sum $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, where $\mathfrak{g}_\pm$ denote the Lie subalgebras $\mathfrak{g}_+ = \mathbb{R}X_3$, $\mathfrak{g}_- = span\{X_0X_1, X_2\}$. The Lie subalgebra $\mathfrak{g}_-$ is isomorphic to the 3-dimensional Heisenberg Lie algebra $\mathfrak{h}$. The Lie algebra $\mathfrak{g}$ decomposes as a vector space direct sum of $\mathfrak{g}_+^\perp$ and $\mathfrak{g}_-^\perp$ where

$$\mathfrak{g}_-^\perp = span\{X_0\} \qquad\qquad \mathfrak{g}_+^\perp = span\{X_1, X_2, X_3\}.$$

If G denotes a Lie group with Lie algebra $\mathfrak{g}$, set $G_- \subset G$ the Lie subgroup with Lie subalgebra $\mathfrak{g}_-$. Indeed G_- acts on $\mathfrak{g}_+^\perp$ by the coadjoint action which in terms of $U \in \mathfrak{g}_-$ and $V \in \mathfrak{g}_+^\perp$ is given by

$$exp\ U \cdot V \quad = \quad [x_3(V)x_2(U) + x_1(V)]X_1 + [x_3(V)x_1(U) + x_2(V)]X_2 + x_3(V)X_3. \tag{16}$$

Hence the action of $\mathfrak{g}_-$ on $\mathfrak{g}_+^\perp$ is

$$\mathrm{ad}_U^* V \quad = \quad x_3(V)(x_2(U)X_1 - x_1(U)X_2) \tag{17}$$

It is not difficult to verify that the orbits are 2-dimensional if $x_3(V) \neq 0$ and in this case the orbits are topologically like $\mathbb{R}^2$; in fact they are diffeomorphic to $\mathbb{H}/Z(\mathbb{H})$. The orbits can be parametrized by the x_3-coordinate since two vectors $U, V \in \mathfrak{g}_+^\perp$ belong to the same orbit if $x_3(U) = x_3(V)$. So we denote them by $\mathcal{M}_{x_3}$.

Endow the orbits with the symplectic structure induced from the coadjoint orbits (5).

Let $f : \mathfrak{g} \to \mathbb{R}$ be the ad-invariant function given by $q(X)$. The gradient of f at a point X is $\nabla f(X) = X$ and therefore the Hamiltonian system of $H = f_{|\mathcal{M}_{x_3}}$, the restriction of the ad-invariant quadratic function to the orbit, is given by

$$\begin{array}{rcl} \frac{dx}{dt} & = & [x_3X_3, x_1X_1 + x_2X_2 + x_3X_3] \\ x(0) & = & x^0 \end{array} \tag{18}$$

where $x^0 = x_1^0X_1 + x_2^0X_2 + x_3^0X_3$. Notice that the function H on $\mathcal{M}_1$ can be identified with the Hamiltonian for the linear inverse pendulum on $\mathbb{R}^2$ and the Hamiltonian system (18) for $x_3^0 = 1$ is equivalent to (15).

The trajectories $x(t) = x_1(t)X_1 + x_2(t)X_2 + x_3(t)X_3$ are

$$\begin{array}{rcl} x_1(t) & = & x_1^0 \cosh(x_3^0t) + x_2^0 \sinh(x_3^0t) \\ x_2(t) & = & x_1^0 \sinh(x_3^0t) + x_2^0 \cosh(x_3^0t) \\ x_3(t) & = & x_3^0 \end{array}$$

This solution coincides with that computed when we considered systems on coadjoint orbits. In fact it can be written as

$$X(t) = \mathrm{Ad}(exp\ tx_3^0X_3)X^0.$$

One can verify that the flow at the point $X^0 \in \mathfrak{g}_+^\perp$ is then

$$\Delta^t(X^0) = (x_1^0 \cosh(x_3^0 t) - x_2^0 \sinh(x_3^0 t) X_1 + (x_1^0 \sinh(x_3^0 t) + x_2^0 \cosh(x_3^0 t) X_2 + x_3^0 X_3)) \tag{19}$$

System (18) is a Lax pair equation $L' = [M, L] = ML - LM$, taking as L and M the following matrices:

$$M = \begin{pmatrix} 0 & x_3 & 0 & 0 \\ x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 0 & x_3 & 0 & x_1 \\ x_3 & 0 & 0 & x_2 \\ -\frac{1}{2}x_2 & \frac{1}{2}x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

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