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ON THE CHARACTERIZATION OF CONVEX FUNCTIONS

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Abstract. A simple characterization of convex functions as indefinite integrals of non-decreasing ones is obtained, using only Riemann integrals.

1.- INTRODUCTION

Characterization of convex functions ([3], App. III, theor. 2) is usually performed within Lebesgue's integration theory, despite the fact that the involved integrands are non-decreasing (therefore Riemann integrable) functions. We transcribe its statement as it appears in the cited book:

Theorem 1. The class of functions which are convex downward on the interval (a, b) coincides with the class of indefinite integrals of functions which are increasing on (a, b) and bounded on every $[p, q] \subset (a, b)$.

The same result can be achieved with a much less expensive treatment, by using only Riemann integrals. In order to remember the usual proof and show the simpler one, we prefer to adopt the following point of view.

Theorem 1 is an immediate corollary of the next result:

Theorem 2. Let (a, b) be an interval and $x_0 \in (a, b)$. Let X be the space of (a. e. classes of) non-decreasing functions on (a, b) and let Y be the space of convex functions on the same interval vanishing at x_0. Then the operators "indefinite integration from x_0" and "differentiation" are inverse to each other. So,

$$\frac{d}{dx}\int_{x_0}^{x} f(t)dt = f(x) \qquad \text{a. e.,} \quad f \in X \tag{1}$$

and

$$\int_{x_0}^{x} F'(t)dt = F(x), \qquad F \in Y. \tag{2}$$

An elementary general result furnishes the suitable framework for this point of view.

Theorem 3. Let $\Phi : X \to Y$ and $\Psi : Y \to X$ be two mappings such that

$$\Psi \circ \Phi = id_X, \tag{3}$$

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Then, they are inverse to each other, i. e.

$$\Phi \circ \Psi = id_Y, \tag{4}$$

if and only if one of the following conditions is satisfied

$$\Psi \text{ is one to one} \tag{5}$$

$$\Phi \text{ is onto} \tag{6}$$

In our case, hypothesis (3) is identity (1), which is assured by Lebesgue's differentiation theorem (see [5] (7.2)). Then identity (2), which implies the characterization of convex functions, is given by conclusion (4) of theorem 3 if condition (5) is fulfilled: differentiation is a one to one mapping from convex functions to non-decreasing ones. This is another well-known result. Since convex functions satisfy a Lipschitz condition (see [5] (7.4)), they are absolutely continuous. Hence the difference of two convex functions which have the same derivative a.e. is an absolutely continuous and singular function and it must reduce to a constant. This fact is proved by using sophisticated tools as Vitali's covering lemma ([5] (7.28)) or F. Riesz's lemma ([2], Ch. 9). Of course, the conditions $\Phi : X \to Y$ (the indefinite integral of a non-decreasing function is convex) and $\Psi : Y \to X$ (the a. e. derivative of a convex function is non-decreasing) must be verified as well.

However, theorem 1 may be derived alternatively from the following (Riemannian) version of theorem 2.

Theorem 4. Let (a, b) be an interval and $x_0 \in (a, b)$. Let X be the space of right continuous non-decreasing functions on (a, b) and let Y be the space of convex functions on the same interval vanishing at x_0. Then the operators

$$\Phi(f) = \int_{x_0}^{x} f(t)dt, \qquad f \in X$$

and

$$\Psi(F) = D^+F, \qquad F \in Y$$

are inverse to each other.

The proof of theorem 4 is obtained by applying theorem 3. In order to verify that the hypotheses are fulfilled, we must prove the following propositions:

(1) If f is (Riemann) integrable, the limit
$$f(x+) = \lim_{t \to x+} f(t) \tag{7}$$
exists and $F(x) = \int_{x_0}^{x} f(t)dt$, then $D^+F(x) = f(x+)$.
This proposition is hypothesis (3).

(2) If f is a non-decreasing function then $F(x) = \int_{x_0}^{x} f(t)dt$ is convex on (a, b) (i.e. $\Phi : X \to Y$).

(3) The right derivative of a convex function is right continuous (i.e. $\Psi : Y \to X$).

(4) Two convex functions with coincident right derivative at **every** point of an interval which have the same value at the point x_0 are equal. This proposition says that Ψ is one to one, which is hypothesis (5) of theorem 3.

The proof of the first proposition is very simple in Riemann integration theory. The second one is a well-known fact from convex functions theory and will be discussed in section 2. The third statement will be proved in section 3. The fourth one will be proved in section 4. It constitutes a simplified version of a theorem of Scheeffer (see [4*a*] or [1], Ch 5).

2.- WELL-KNOWN FACTS ABOUT CONVEX FUNCTIONS

Convexity of the function F on the interval (a,b), characterized by

$$(v-u)\,F(x) \le (v-x)\,F(u) + (x-u)\,F(v), \qquad a<u<x<v<b, \tag{8}$$

is equivalent to the increase of the following function

$$\Gamma\,(u,v) = \frac{F(v)-F(u)}{v-u},$$

in one of its variables.

Since Γ is symmetric, if it is non-decreasing with respect to one variable, then it is also non-decreasing with respect to the other. Hence, if $u<x<v$, then $\Gamma\,(u,x) \le \Gamma\,(u,v) \le \Gamma\,(x,v)$. As a consequence,

$$\frac{F(x)-F(u)}{x-u} \le \frac{F(v)-F(x)}{v-x}, \qquad u<x<v, \tag{9}$$

This condition is also sufficient for convexity, since it implies (8)

From (9), the existence of lateral derivatives at **every** point of the interval (and hence continuity of F) becomes clear:

$$D^-F(x) = \sup_{u<x} \Gamma\,(u,x) \le \inf_{v>x} \Gamma\,(x,v) = D^+F(x).$$

Moreover, if $u<v$, then

$$D^+F(u) \le \Gamma\,(u,v) \le D^-F(v).$$

Hence we have the following series of inequalities:

$$D^-F(u) \le D^+F(u) \le D^-F(v) \le D^+F(v), \tag{10}$$

which implies that D^-F and D^+F are non-decreasing functions. Hence, lateral derivatives are continuous functions with an exceptional denumerable set of jump discontinuities.

At any point where one of the lateral derivatives is continuous (i.e. at every point but in a denumerable set), the other one exists and takes the same value (i.e. there exists the derivative). If, for instance, D^-F is continuous at x, it follows from (10) that

$$D^-F(x) = \lim_{v\to x+} D^-F(v) \ge D^+F(x).$$

One can ask if continuity points of lateral derivatives are the only ones for which the derivative exists. The answer is yes and it follows from theorem 6 below.

Another immediate consequence of the sufficiency of (9) is that indefinite integrals of non-decreasing functions are convex. In fact, if $F(x) = \int_{x_0}^{x} f(t)dt$ and $u < x < v$,

$$\Gamma(u,x) = \frac{1}{x-u}\int_u^x f(t)dt \le f(x) \le \frac{1}{v-x}\int_x^v f(t)dt = \Gamma(x,v). \tag{11}$$

Then statement 2 is proved.

3.- LATERAL CONTINUITY OF LATERAL DERIVATIVES

Firstly we will demonstrate a convex version of the mean value theorem.

Lemma 5. Let F be a convex function on the open interval (a,b) and let $a < u < v < b$. Then there exists an intermediate value $\xi \in (u,v)$ such that

$$u < \xi_1 < \xi < \xi_2 < v \Rightarrow D^+F(\xi_1) \le \frac{F(v)-F(u)}{v-u} \le D^-F(\xi_2).$$

Proof. Let

$$m := \frac{F(v)-F(u)}{v-u},$$

and consider the line $L(x) = F(u) + m(x-u)$. The difference $G(x) = F(x) - L(x)$ is a convex (and therefore continuous) function vanishing at the end points of the interval $[u,v]$. G reaches its minimum in the closed interval $[u,v]$, but since, because of its convexity, $G \le 0$ in (u,v), the minimum is attained at some interior point ξ. Hence, for $u < x < \xi$, we have $G(\xi) \le G(x)$. i.e. $F(\xi) - F(x) \le L(\xi) - L(x) = m(\xi - x)$. Or, equivalently,

$$\frac{F(\xi)-F(x)}{\xi - x} \le \frac{F(v)-F(u)}{v-u}.$$

Hence,

$$D^-F(\xi) = \sup_{x<\xi}\frac{F(\xi)-F(x)}{\xi-x} \le \frac{F(v)-F(u)}{v-u}.$$

As a consequence, by virtue of (10),

$$D^+F(\xi_1) \le D^-F(\xi) \le \frac{F(v)-F(u)}{v-u}, \qquad \text{for } \xi_1 < \xi.$$

The other inequality is proved in the same way.

By the way, it is deduced from the lemma that a convex function with a lateral derivative vanishing in the whole interval must reduce to a constant.

Theorem 6. The right lateral derivative of a convex function is right continuous.

Proof. Since D^+F is non-decreasing, there exists

$$D^+F(x+) := \lim_{v \to x+} D^+F(v) \ge D^+F(x).$$

For the opposite inequality, given $v > x$, applying the lemma in the interval (x, v) we obtain:

$$\frac{F(v) - F(x)}{v - x} \geq D^+F(\xi), \qquad \text{for some } \xi = \xi(v) \in (x, v).$$

The right member has limit for $v \to x+$, since D^+F does and $\xi(v) \to x$. Hence

$$D^+F(x) = \lim_{v \to x+} \frac{F(v) - F(x)}{v - x} \geq D^+F(x+).$$

Obviously, the same result is valid in the left side.

4.- LATERAL DIFFERENTIATION IS A ONE TO ONE OPERATOR

Lemma 7. Let F be a continuous function on the interval (a, b) and suppose in addition that for every $x \in (a, b)$, $D^+F(x)$ exists and is equal to 0. Hence F is constant in (a, b).

Proof. It will be proved that, for any x_1, x_2 with $a < x_1 < x_2 < b$, $F(x_1) = F(x_2)$. It suffices to prove that, for any $\varepsilon > 0$, $|F(x_2) - F(x_1)| \leq \varepsilon\,(x_2 - x_1)$. Let us consider the set

$$A = \{x \in [x_1, x_2] : |F(x) - F(x_1)| \leq \varepsilon\,(x - x_1)\}$$

and let $s = \sup A$. Note that by virtue of the continuity of F, $s \in A$. Then we only need to prove that $s = x_2$. Of course, $s > x_2$ is not possible, since x_2 is an upper bound for A; but neither is $s < x_2$, because $D^+F(s) = 0$ would imply the existence of a point y near s, $y \in (s, x_2)$ such that $|F(y) - F(s)| \leq \varepsilon\,(y - s)$ In such a case, using that $s \in A$ and then $|F(s) - F(x_1)| \leq \varepsilon\,(s - x_1)$, it would yield

$$|F(y) - F(x_1)| \leq |F(y) - F(s)| + |F(s) - F(x_1)| \leq \varepsilon\,(y - x_1)\,.$$

This inequality implies that $y \in A$, which is in contradiction with $y > s$ and therefore $s = \sup A$.

With the aid of lemma 7, the following theorem becomes immediate and constitutes a proof of proposition 4

Theorem 8. If F and G are convex functions on (a, b), with $F(x_0) = G(x_0)$ for certain $x_0 \in (a, b)$, and $D^+F(x) = D^+G(x)$ for every $x \in (a, b)$, then $F = G$ on (a, b).

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