REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Volumen 48, Número 1, 2007, Páginas 17–25

GEODESICS AND NORMAL SECTIONS ON REAL FLAG MANIFOLDS

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ABSTRACT. In the present paper we study Riemannian and canonical geodesics in a real flag manifold M, considered as curves in the ambient Euclidean space of the natural embedding of M.

1. INTRODUCTION.

In this paper we present some results concerning an interesting problem in the geometry of submanifolds of Euclidean spaces. Our note originated in a paper by Ferus and Schirrmacher [8] where those authors considered the problem of determining all the submanifolds of Euclidean space whose geodesics (considered as curves in the Euclidean space) are W-curves i.e. Frenet curves with constant curvatures along the curve, see next section for definitions. Ferus and Schirrmacher obtained an important result which we will describe.

W. Strübing [13] had shown earlier that symmetric R-spaces (a particular case of real flag manifolds) have the property that all their geodesics, considered as curves in the Euclidean space, are W-curves. Ferus and Schirrmacher obtained the following characterization of symmetric R-spaces based on the behavior of their geodesics as curves in the Euclidean space.

Theorem 1.1. [8]*Let M be a closed connected Riemannian manifold and $f : M \to R^N$ an isometric immersion. Then the following properties are equivalent:*

(i) For almost every unit-speed geodesic $\gamma : R \to M$, the image curve $c = f \circ \gamma$ is a generic *W-curve (definition in the next section).*

(ii) M is an extrinsic symmetric submanifold in the sense of [7].□

Here the phrase *almost every unit-speed geodesic* means that the tangent vectors of these geodesics fill the unit-tangent bundle up to a closed set of measure zero. Recall that by [7], extrinsic symmetric submanifold is equivalent to naturally embedded symmetric real flag manifold.

This result has an immediate consequence which suggests some interesting questions namely:

2000 *Mathematics Subject Classification.* Primary 53C30, 53C22; Secondary 53B15, 53A04.

This research was partially supported by CONICET, SECYT-UNCba and Universidad Nacional de San Luis, Argentina.

- For those real flag manifolds which are not symmetric, the geodesics which are generic W-curves must have their tangent vectors contained in a subset of the unit-tangent bundle whose complement does not have measure zero.

This suggests studying Riemannian geodesics on general real flag manifolds (or other submanifolds of Euclidean spaces) trying to consider the following problems:
(i) *Determine some subset of the unit-tangent bundle which contains the tangent vectors to those geodesics which are* generic *W-curves and*
(ii) *Determine, if possible, some subset of the unit tangent bundle which contains the tangent vectors to* all *those geodesics which are W-curves.*

The present paper is devoted to these problems and contains some information concerning problem (i) for every compact submanifold of an Euclidean space, in particular, for real flag manifolds (Theorem 6.1). It contains also a result about generic canonical geodesics (Theorem 6.2) and an answer to problem (ii) in the case of isoparametric submanifolds of rank 2 in an Euclidean space, (Corollary 7.2).

In every real flag manifold there exists an affine connection naturally associated to its homogeneous structure which is the so called "canonical connection" (see Subsection 3.1). It has proven to be useful to characterize these submanifolds of Euclidean spaces [10]. It is known, [11], that all the geodesics of this canonical connection, considered as curves in the Euclidean space, are W-curves. This suggests to start by studying the set of "coincidence" of the two types of geodesics. This is an easy task in terms of the difference tensor D of the two connections (see Proposition 4.2). This may lead one to think that this set contains all the unit vectors generating Riemannian geodesics which are W-curves but this may not be the case.

In the present paper we introduce the subset of the unit tangent bundle containing all the tangent vectors to Riemannian geodesics that are *generic* W-curves. This subset, which we shall denote by $\Xi[M]$, is that whose fibre at each point $p \in M$ is the real algebraic variety $\widehat{X}_p[M]$ (definition in Section 5). This variety has been introduced in [4] and many of its properties studied also in [12], [6] and [5].

It seems hard to determine which is the subset of the unit tangent bundle of a general real flag manifold that contains the tangent vectors to all Riemannian geodesics which are W-curves. So it seems surprising that if our real flag manifold is a homogeneous isoparametric submanifold of rank 2 in an Euclidean space $M^m \to R^{m+2}$, then the unit tangent vectors to all Riemannian geodesics that are W-curves are contained in $\Xi[M]$ (see Corollary 7.2).

If our real flag manifold M is *symmetric* then $\Xi[M]$ coincides with the unit tangent bundle. But if M *is not symmetric* then $\Xi[M]$ is a set of *measure zero* in the unit tangent bundle of M and this seems to be a nice identification of the set mentioned in the above consequence of Theorem 1.1. The complement of this set is, in this case, open in the unit tangent bundle of M.

The paper is organized as follows. In the next section we have the definition of W-curves. In Section 3 we recall the definition of real flag manifold and include a subsection recalling also the definition of the canonical connection. In Section 4 we study the coincidence of Riemannian and canonical geodesics. Section 5 contains the definition of the varieties $\widehat{X}[M]$ and $D[M]$ and in Section 6 we include the results concerning generic Riemannian and canonical geodesics. Finally Section 7 contains Proposition 7.1 and Corollary 7.2 which concern the question (ii) indicated above.

The authors express their appreciation to the anonymous referee whose comments lead to a considerable improvement of the present paper.

2. W-curves.

Following Ferus and Schirrmacher [8] we shall say that a regular curve $c : J \subset R \to R^N$ is a *W-curve of rank r* if, for all $s \in J$, the derivatives $c'(s), \ldots, c^{(r)}(s)$ are linearly independent, the derivatives $c'(s), \ldots, c^{(r+1)}(s)$ are linearly dependent and if the (therefore well defined) Frenet curvatures $k_1, k_2, \ldots, k_{r-1} : J \to R^+ = \{u \in R : u > 0\}$ are constant (independent of the parameter but depending on the geodesic).

We reproduce now a result from [8] which we need.

Lemma 2.1. *Let $c : J \subset R \to R^N$ be a W-curve of infinite length and parameterized by arc length. If the image $c(J)$ is bounded, then the rank of c is even, $r = 2m$. Furthermore, there are m pairs of positive constants $(a_1, r_1), (a_2, r_2), \ldots, (a_m, r_m)$ (unique up to order) and a set of $2m$ orthonormal vectors $\{e_j : 1 \leq j \leq 2m\}$ in R^{n+q} such that*

$$c(s) = c_o + \sum_{i=1}^{m} r_i \left(e_{2i-1} \sin(a_i s) + e_{2i} \cos(a_i s)\right).$$

Also following [8], we shall say that a W-curve in R^{n+q} is *generic* if the real numbers $\{a_i : 1 \leq i \leq m\}$ are independent over the rationals. This is equivalent to say that the closure $\overline{c(R)}$ of the image of c in R^{n+q} is a torus $S^1(r_1) \times S^1(r_2) \times \ldots \times S^1(r_m) \subset R^{n+q}$.

3. Real Flag Manifolds.

Real flag manifolds can be informally defined as follows: Let N be an irreducible symmetric space (compact or noncompact) p a point in N and $T_p(N)$ its tangent space at that point. The corresponding isotropy group K at p acts on $T_p(N)$ by the derivatives (at p) of its elements and its orbits on $T_p(N)$ are the so called R-spaces or real flag manifolds. By their definition, these spaces are compact homogeneous and have a natural embedding in the Euclidean space $T_p(N)$. The Riemannian metric that is usually considered on them is the induced one. We shall denote by $\langle X, Y \rangle$ the inner product in $T_p(N)$ and by ∇^E the Levi-Civita connection associated to the Euclidean metric in $T_p(N)$.

In the homogeneous space M we also have the "canonical" connection which we describe briefly. To that end we "formalize" the definition of real flag manifold given above. The necessary ingredients to construct the arbitrary real flag manifold M and its canonical embedding are the following. Let $\mathfrak{g}$ be a real semisimple Lie algebra without compact factors, $\mathfrak{k}$ a maximal compactly embedded subalgebra of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition of $\mathfrak{g}$ relative to $\mathfrak{k}$. Let B denote the Killing form of $\mathfrak{g}$; then $\mathfrak{p}$ can be considered an Euclidean space with the inner product defined by the restriction of B to $\mathfrak{p}$. Let $G = Int(G)$ be the group of inner automorphisms of $\mathfrak{g}$. The Lie algebra of G may be identified with $\mathfrak{g}$. Let K be the analytic subgroup of G corresponding to $\mathfrak{k}$; K is compact and acts on $\mathfrak{p}$ as a group of isometries. The real flag manifold M is, by definition, the orbit of a non zero vector $E \in \mathfrak{p}$ i.e. $M = Ad(K)E$.

This defines also the *natural embedding* $\dot{j} : M \to \mathfrak{p}$ of the real flag manifold M in the Euclidean space $(\mathfrak{p}, B)$. We take on M the Riemannian metric induced by the embedding. Furthermore we shall assume that the embedding is *"substantial"* or *"full"* that is, $j(M)$ is not contained in any affine hyperplane of $\mathfrak{p}$. Let us denote by K_E the isotropy subgroup of E. Then, as a homogeneous space, $M = K/K_E$.

In general the group K_E is not connected and we shall denote by $[K_E]_e$ the connected component of the identity. Let $\mathfrak{k}_E$ be the Lie subalgebra corresponding to $[K_E]_e$ in $\mathfrak{k}$. Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{k}_E$ with respect to the restriction of B to $\mathfrak{k}$ (it is negative definite in $\mathfrak{k}$). Then $\mathfrak{k}$=$\mathfrak{k}_E \oplus \mathfrak{m}$ is a reductive decomposition, that is, it satisfies $Ad(K_E)\mathfrak{m} \subset \mathfrak{m}$. We also have

$$T_E(M) = [\mathfrak{m}, E] = [\mathfrak{k}, E] \tag{1}$$

and $ad(E)$ is one to one from $\mathfrak{m}$ onto $T_E(M)$.

3.1. **The two connections.** We recall the following fundamental facts.

Theorem 3.1. [9, p. 43]*Let K/K_E be a reductive homogeneous space with a fixed decomposition of the Lie algebra $\mathfrak{k} = \mathfrak{k}_E \oplus \mathfrak{m}$, $Ad(K_E)\,\mathfrak{m} \subset \mathfrak{m}$. Then there exists a one to one correspondence between the set of all the invariant affine connections ∇ over K/K_E and the set of all the bilinear functions $\omega : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ which satisfy*

$$Ad(h)\,\omega(X, Y) = \omega(Ad(h)\,X, Ad(h)\,Y)$$

for each $X,\ Y \in \mathfrak{m}$ and $h \in K_E$.

Let $\pi : K \longrightarrow K/K_E$ be the natural projection and assume that we have an invariant affine connection ∇ over K/K_E. We require the following properties for the connection ∇:

(A1) Let $\exp(tX)$ be the one parameter subgroup of K generated by $X \in \mathfrak{m}$. Then $\pi(\exp(tX)) = (\exp(tX))\,E$ is a regular curve such that the family of its tangent vectors is parallel along the curve itself.

(A2) Let us consider the curve $\pi(\exp(tX)) = (\exp(tX))\,E$ in K/K_E. Let $Y \in \mathfrak{m}$; then the parallel translation of the vector Y, tangent at E, along this curve coincides with translation of Y by the one parameter subgroup $\exp(tX)$.

If the affine connection ∇ has property (A2) then it also satisfies (A1).

Proposition 3.2. [9] *The invariant affine connection defined by the function ω has property (A2) if and only if $\omega(X, Y) = 0$ for each $X,\ Y \in \mathfrak{m}$. Then, over the reductive homogeneous space, there exists one and only one affine connection ∇^c which satisfies (A2). It is the one defined by the connection function which is identically zero on $\mathfrak{m} \times \mathfrak{m}$.*

This invariant affine connection ∇^c is called the *canonical affine connection of second class* over K/K_E with respect to the fixed decomposition of the Lie algebra $\mathfrak{k} = \mathfrak{k}_E \oplus \mathfrak{m}$, We call it the *canonical connection*. We denote by ∇ the Levi-Civita connection associated to the induced metric on M from $\mathfrak{p}$ and by D the difference tensor defined by $D(X,Y) = \nabla_X Y - \nabla^c_X Y$. The *canonical* connection is *metric* and satisfies $\nabla^c D = 0$. A modern view of the canonical connection can be found in [1, p.203-205].

4. Canonical and Riemannian Geodesics.

It is very difficult to determine the Riemannian geodesics of the real flag manifolds which are not symmetric. However, in these spaces the geodesics of the canonical connection ∇^c are well known. Their geometric properties as curves in the Euclidean space of the natural embedding, were studied in [11]. From that paper we recall

Proposition 4.1. *Let $i : M^n \to R^{n+q}$ be a canonical embedding of the real flag manifold M. For each point $p \in M^n$ and each unitary vector $X \in T_p(M)$, if γ is the ∇^c-geodesic defined by X, then $c(s) = i(\gamma(s))$ is a W-curve in R^{n+q}.*

Then we immediately have the following Proposition which is valid for any real flag manifold.

Proposition 4.2. *Let $i : M^n \to R^{n+q}$ be a canonical embedding of the real flag manifold. If $X \in T_E(M)$ is a unitary vector then $D_E(X,X) = 0$ if and only if the Riemannian and canonical geodesics passing through E defined by the vector X coincide for all values of their parameter.*

Then all these Riemannian geodesics are W-curves.

5. The varieties $X[M]$ and $D[M]$.

Let $j : M \to R^N$ be an isometric immersion and p a point in M. We may identify a neighborhood of p with its image by j and consider, in the tangent space $T_p(M)$, a unitary vector X. If $T_p(M)^\perp$ denotes the normal space to M at p, we may define an affine subspace of R^N by

$$S(p,X) = p + Span\left\{X, T_p(M)^\perp\right\}.$$

If U is a small enough neighborhood of p in M, then the intersection $U \cap S(p,X)$ can be considered the image of a C^∞ regular curve $\gamma(s)$, parametrized by arc-length, such that $\gamma(0) = p$, $\gamma'(0) = X$. This curve is called a *normal section of M at the point p in the direction of X*. In a strict sense, we ought to speak of the "germ" of a normal section at p determined by the unit vector X because a change in the neighborhood U will change the curve. However, this new curve will coincide with γ in the proximity of $s = 0$. Since our computations with the curve γ are done at the point p, we may take any one of these curves. We may also assume that j is an embedding. Following B.Y. Chen, we say that the normal section γ of

M at p in the direction of X is *pointwise planar* at p if its first three derivatives $\gamma'(0), \gamma''(0)$ and $\gamma'''(0)$ are linearly dependent.

We say that the submanifold $j : M \to R^N$ is *extrinsically homogeneous* [1, p.35] if for any two points $p, q \in M$ there is an isometry g of $\mathbb{R}^{n+k}$ such that $g(M) = M$ and $g(p) = q$.

Given a point p in the submanifold M we shall denote, as in [4],

$$\widehat{X}_p[M] = \left\{X \in T_p(M) : \|X\| = 1, \left(\overline{\nabla}_X \alpha\right)(X, X) = 0\right\}. \tag{2}$$

Since $Y \in \widehat{X}_p[M]$ clearly implies $-Y \in \widehat{X}_p[M]$, we may take the image $X_p[M]$ of this set in the real projective space $RP(T_p(M))$.

If M is extrinsic homogeneous in the ambient space R^{n+k}, it is clear that $\widehat{X}_p[M]$ does not "depend" on the point p and we may denote it by $\widehat{X}[M]$. In this case $X[M]$ is a real algebraic variety of $RP(T_p(M))$. Its natural complexification $X_c[M]$, is a complex algebraic variety of CP^{n-1}. Clearly real flag manifolds are extrinsically homogeneous in their canonical embeddings.

In general for any submanifold of R^N we may define the subset $\Xi[M]$ of the unit tangent bundle mentioned above as:

$$\Xi[M] = \left\{X \in T(M) : \|X\| = 1, \left(\overline{\nabla}_X \alpha\right)(X, X) = 0\right\}$$

In [4] the variety $X[M]$ *of directions of pointwise planar normal sections*, of a natural embedding of a real flag manifold M was introduced. We refer the reader to [4], [12] and [5] for the description of diverse properties of this variety. Here we need essentially the definition (2).

If the submanifold M at hand is a naturally embedded real flag manifold then we may also define

$$D[M] = \{X \in T(M) : \|X\| = 1, D(X, X) = 0\}.$$

which is the set of coincidence of the canonical and Riemannian geodesics. The set

$$D_p[M] = \{X \in T_p(M) : D_p(X, X) = 0\}$$

is in fact a real algebraic set intersection of the unit sphere in $T_p(M)$ and the affine algebraic variety in $T_p(M)$ defined by the finite set of quadratic polynomials implicit in the condition $D_p(X, X) = 0$. Clearly $D_p[M] \subset \widehat{X}_p[M]$ because, for real flag manifolds, one has the identity

$$\left(\overline{\nabla}_X \alpha\right)(X, X) = -2\alpha_p(X, D_p(X, X)) \tag{3}$$

proved in [4].

6. Generic geodesics in M.

Let $i : M^n \to R^{n+q}$ be an isometric embedding of a compact connected manifold M^n. For each point $p \in M^n$ and each unitary vector $X \in T_p(M)$, let us consider the ∇-geodesic $\gamma(s)$ defined by $\gamma(0) = p$, $\gamma'(0) = X$. We are going to study the curve $c(s) = i(\gamma(s))$ in R^{n+q} considering i as the inclusion. We easily compute

$$\begin{aligned} c'(0) &= \gamma'(0) = X \\ c''(0) &= \nabla^E_X \gamma' = \nabla^R_X \gamma' + \alpha_p(X,X) = \alpha_p(X,X) \\ c'''(0) &= -A_{\alpha_p(X,X)} X + \left(\overline{\nabla}_X \alpha\right)(X,X) \end{aligned} \tag{4}$$

We have now the following general result.

Theorem 6.1. *Let $i : M^n \to R^{n+q}$ be an isometric embedding of a compact connected manifold. If $\gamma(s)$ is a Riemannian geodesic and $c(s) = i(\gamma(s))$ is a generic W-curve in R^{n+q} then, for each $s \in R$, $c'(s) = \gamma'(s) \in \widehat{X}_{\gamma(s)}[M]$.*

Proof. Since $c(s)$ is a generic W-curve we have that $c(R)$ is a dense set in the torus given by Lemma 2.1. Then the torus is $\overline{c(R)}$ and since $\overline{c(R)} \subset \overline{M} = M$ we see that the whole torus is contained in M. Now it follows from Lemma 2.1 that $c'''(0)$ is tangent to the torus $\overline{c(R)}$ and hence tangent to M at p. Then (4) yields that $\left(\overline{\nabla}_X \alpha\right)(X,X) = 0$. Since the same thing can be proved for any point in the geodesic $\gamma(s)$ we have that $\gamma'(s) \in \widehat{X}_{\gamma(s)}[M]$ as claimed. □

A similar result can be proved for the canonical geodesics namely.

Theorem 6.2. *Let $i : M^n \to R^{n+q}$ be a canonical embedding of the real flag manifold. If $\beta(s)$ is a canonical geodesic and $c(s) = i(\beta(s))$ is a generic W-curve in R^{n+q} then, for each $s \in R$, $c'(s) = \gamma'(s) \in \widehat{X}_{\gamma(s)}[M]$.*

Proof. Again $c(R)$ is a dense set in the torus given by Lemma 2.1 and set $p = c(0)$ then

$$\begin{aligned} c'(0) &= X \\ c''(0) &= \nabla^E_X \beta' = D_p(X,X) + \alpha_p(X,X) \\ c'''(0) &= \nabla^E_X (D(\beta',\beta')) + \nabla^E_X \alpha(\beta',\beta') = \\ &= D_p(X, D_p(X,X)) + \alpha_p(X, D_p(X,X)) - \\ &\quad - A_{\alpha_p(X,X)} X + \nabla^\perp_X (\alpha(\beta',\beta')). \end{aligned} \tag{5}$$

Now, since M is a real flag manifold,

$$\nabla^\perp_X (\alpha(\beta',\beta')) = (\nabla^c_X \alpha)(X,X) + 2\alpha_p(\nabla^c_X \beta', X) = 0$$

and because also in this case $c'''(0)$ is tangent to the torus $\overline{c(R)}$ and hence tangent to M at p, we must have (by (5)) $\alpha_p(X, D_p(X,X)) = 0$ and by (3) $X \in \widehat{X}_p[M]$. □

This theorem corrects [11, p.300,14].

Corollary 6.3. *In a non-extrinsically symmetric canonically embedded R-space, the vectors of the unit-tangent bundle that generate canonical geodesics which are* generic *W-curves are contained in $\Xi[M]$.* □.

This Corollary could be rephrased as "in a non-extrinsically symmetric canonically embedded R-space, canonical geodesics are generically non-generic". This is in fact also true for the Riemannian geodesics of any *non extrinsic symmetric* compact embedded submanifold of R^N as Theorem 6.1 shows. On the other hand, for extrinsic symmetric submanifolds (symmetric R-spaces) the Theorem of Chen

[3] (see also [4]) says that $\Xi[M]$ coincides with the unit-tangent bundle and, one may think, this is the reason why on these last spaces generic geodesics are *generic* W-curves.

7. The case of isoparametric submanifolds of rank two.

In this section we assume that M is isoparametric submanifold of rank 2 in an Euclidean space $M^m \to R^{m+2}$. These submanifolds can be considered isoparametric hypersurfaces in the unit sphere S^{n+1}. Many of them are extrinsically homogeneous, but there are examples which are known to be non-homogeneous. A reference for the present section is [1] and references there. Recall that for a regular curve in M parametrized by arc-length considered as curve in the ambient Euclidean space the first Frenet curvature is given by $k_1^R(s) = \|\alpha(\gamma',\gamma')\|$. The objective here is to indicate the following

Proposition 7.1. *Let M^m be a compact isoparametric submanifold of rank 2 in a Euclidean space R^{m+2} and $\gamma(s)$ a regular curve parametrized by arc-length in M^m. Then $k_1^R(s) = \|\alpha(\gamma'(s),\gamma'(s))\|$ is constant, if and only if, $\gamma'(s) \in \widehat{X}_{\gamma(s)}[M]$ for each $s \in dom(\gamma)$.*

Proof. Due to codimension two, $\|\alpha(\gamma'(s),\gamma'(s))\|$ is constant along γ if and only if $\alpha(\gamma'(s),\gamma'(s))$ is parallel in the normal connection and this is equivalent to

$$0 = \left(\overline{\nabla}_{\gamma'}\alpha\right)(\gamma',\gamma').$$

□

We get then, as a corollary in this particular case, an answer to problem (ii) presented in the Introduction.

Corollary 7.2. *Let $\gamma(s)$ be a Riemannian geodesic in M^m which is a W-curve in R^{m+2}. Then for each $s \in dom(\gamma)$ we have $\gamma'(s) \in \widehat{X}_{\gamma(s)}[M]$.* □

Unfortunately we cannot prove that there are on M^m any geodesics $\gamma(s)$ such that $\gamma'(s) \in \widehat{X}_{\gamma(s)}[M]$ (besides those tangent to the eigendistributions), neither that, if there is one, it is a W-curve.

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Recibido: 29 de septiembre de 2005
Aceptado: 28 de marzo de 2007