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CHARACTERISATIONS OF NELSON ALGEBRAS

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Abstract. Nelson algebras arise naturally in algebraic logic as the algebraic models of Nelson's constructive logic with strong negation. This note gives two characterisations of the variety of Nelson algebras up to term equivalence, together with a characterisation of the finite Nelson algebras up to polynomial equivalence. The results answer a question of Blok and Pigozzi and clarify some earlier work of Brignole and Monteiro.

1. Introduction

Recall from the theory of distributive lattices [2, Chapter XI] that a *De Morgan algebra* is an algebra $\langle A; \wedge, \vee, \sim, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ where $\langle A; \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice and for all $a, b \in A$, $\sim\sim a = a$, $\sim(a \wedge b) = \sim a \vee \sim b$ and $\sim(a \vee b) = \sim a \wedge \sim b$.

A *Nelson algebra* (also *$\mathcal{N}$-lattice* or *quasi-pseudo-Boolean algebra* in the literature) is an algebra $\langle A; \wedge, \vee, \to, \sim, 0, 1\rangle$ of type $\langle 2, 2, 2, 1, 0, 0\rangle$ such that the following conditions are satisfied for all $a, b, c \in A$ [26, Section 0]:

(N1) $\langle A; \wedge, \vee, \sim, 0, 1\rangle$ is a De Morgan algebra with lattice ordering $\leq$;
(N2) The relation $\preceq$ defined for all $a, b \in A$ by $a \preceq b$ if and only if $a \to b = 1$ is a quasiordering (reflexive and transitive relation) on A;
(N3) $a \wedge b \preceq c$ if and only if $a \preceq b \to c$;
(N4) $a \leq b$ if and only if $a \preceq b$ and $\sim b \preceq \sim a$;
(N5) $a \preceq c$ and $b \preceq c$ implies $a \vee b \preceq c$;
(N6) $a \preceq b$ and $a \preceq c$ implies $a \preceq b \wedge c$;
(N7) $a \wedge \sim b \preceq \sim(a \to b)$ and $\sim(a \to b) \preceq a \wedge \sim b$;
(N8) $\sim(a \to 0) \preceq a$ and $a \preceq \sim(a \to 0)$;
(N9) $a \wedge \sim a \preceq b$.

The class N of all Nelson algebras is a variety [11], which arises naturally in algebraic logic as the equivalent quasivariety semantics (in the sense of [4]) of Nelson's constructive logic with strong negation [25, Chapter XII]. For studies of

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Nelson algebras see in particular Sendlewski [26], Vakarelov [31], and Rasiowa [25, Chapter V].

A *commutative, integral residuated lattice* is an algebra $\langle A; \wedge, \vee, *, \Rightarrow, 1\rangle$ of type $\langle 2,2,2,2,0\rangle$, where: (i) $\langle A; \wedge, \vee\rangle$ is a lattice with lattice ordering $\leq$ such that $d \leq 1$ for all $d \in A$; (ii) $\langle A; *, 1\rangle$ is a commutative monoid; and (iii) for all $a, b, c \in A$, $a * b \leq c$ if and only if $a \leq b \Rightarrow c$. By Blount and Tsinakis [9, Proposition 4.1] the class CIRL of all commutative, integral residuated lattices is a variety. An FL_{ew}*-algebra* $\langle A; \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ is a commutative, integral residuated lattice with distinguished least element $0 \in A$. The variety FL_{ew} of all FL_{ew}-algebras arises naturally in algebraic logic in connection with the study of substructural logics; see [19, 21, 22, 23] for details.

An FL_{ew}-algebra $\mathbf{A}$ is said to be *3-potent* when $a * a * a = a * a$ for all $a \in A$, *distributive* when its lattice reduct is distributive, and *classical* when $(a \Rightarrow 0) \Rightarrow 0 = a$ for all $a \in A$. Rewriting $b \Rightarrow 0$ as $\sim b$ for all $b \in A$, classicality expresses the law of double negation in algebraic form. A *Nelson* FL_{ew}*-algebra* is a 3-potent, distributive classical FL_{ew}-algebra such that $\big(a \Rightarrow (a \Rightarrow b)\big) \wedge \big(\sim b \Rightarrow (\sim b \Rightarrow \sim a)\big) = a \Rightarrow b$ for all $a, b \in A$.

The following description (to within term equivalence) of the variety of Nelson algebras was obtained by the authors in [29, 30].

Theorem 1.1. [29, Theorem 1.1]

(1) *Let* $\mathbf{A}$ *be a Nelson algebra. Define the derived binary terms* $*$ *and* $\Rightarrow$ *by:*

$$x * y := \sim(x \to \sim y) \vee \sim(y \to \sim x) \qquad (*_{\text{def}})$$

$$x \Rightarrow y := (x \to y) \wedge (\sim y \to \sim x). \qquad (\Rightarrow_{\text{def}})$$

Then the term reduct $\mathbf{A}^F := \langle A; \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ *is a Nelson* FL_{ew}*-algebra.*

(2) *Let* $\mathbf{A}$ *be a Nelson* FL_{ew}*-algebra. Define the derived binary term* $\to$ *and the derived unary term* $\sim$ *by:*

$$x \to y := x \Rightarrow (x \Rightarrow y) \qquad (\to_{\text{def}})$$

$$\sim x := x \Rightarrow 0. \qquad (\sim_{\text{def}})$$

Then the term reduct $\mathbf{A}^N := \langle A; \wedge, \vee, \to, \sim, 0, 1\rangle$ *is a Nelson algebra.*

(3) *Let* $\mathbf{A}$ *be a Nelson algebra. Then* $(\mathbf{A}^F)^N = \mathbf{A}$.

(4) *Let* $\mathbf{A}$ *be a Nelson* FL_{ew}*-algebra. Then* $(\mathbf{A}^N)^F = \mathbf{A}$.

Hence the varieties of Nelson algebras and Nelson FL_{ew}*-algebras are term equivalent.* □

In this note we give several further characterisations of Nelson algebras, all of which may be understood as corollaries of Theorem 1.1. We shall make implicit use of Theorem 1.1 without further reference throughout the paper.

A *BCK-algebra* is a $\langle \Rightarrow, 1\rangle$-subreduct of a commutative, integral residuated lattice [32, Theorem 5.6]; for an equivalent quasi-equational definition, see Section 2. We show in Section 2 that every finite Nelson algebra $\mathbf{A}$ is polynomially equivalent to its own BCK-algebra term reduct $\langle A; \Rightarrow, 1\rangle$.

A *pseudo-interior algebra* is a hybrid of a (topological) interior algebra and a residuated partially ordered monoid; for a precise definition, see Section 3 below. We prove in Section 3 that the variety of Nelson algebras is term equivalent to a congruence permutable variety of pseudo-interior algebras with compatible operations. We obtain this result as a byproduct of the solution to a problem of Blok and Pigozzi [5].

A *lower BCK-semilattice* is the conjunction of a meet semilattice with a BCK-algebra such that the natural partial orderings on both algebras coincide; for a formal definition see Section 4 below. We verify in Section 4 that the variety of Nelson algebras is term equivalent to a variety of bounded BCK-semilattices. The result clarifies earlier work on the axiomatics of Nelson algebras due to Brignole [10].

2. Finite Nelson algebras as BCK-algebras

A *BCK-algebra* is an algebra $\langle A; \Rightarrow, 1\rangle$ of type $\langle 2, 0\rangle$ such that the following identities and quasi-identity are satisfied:

$$\begin{aligned} (x \Rightarrow y) \Rightarrow \big((y \Rightarrow z) \Rightarrow (x \Rightarrow z)\big) &\approx \mathbf{1} \\ \mathbf{1} \Rightarrow x &\approx x \\ x \Rightarrow \mathbf{1} &\approx \mathbf{1} \end{aligned} \tag{1}$$

$$x \Rightarrow y \approx \mathbf{1} \;\&\; y \Rightarrow x \approx \mathbf{1} \supset x \approx y.$$

By Wroński [34] the quasivariety BCK of all BCK-algebras is not a variety. BCK-algebras have been considered extensively in the literature; for surveys, see Iséki and Tanaka [17] or Cornish [13]. Here we simply recall that for any BCK-algebra $\mathbf{A}$, the relation $\sqsubseteq$ on A defined for all $a, b \in A$ by $a \sqsubseteq b$ if and only if $a \Rightarrow b = 1$ is a partial ordering, which has the property that for any $f, g, h \in A$,

$$f \sqsubseteq g \quad \text{implies} \quad g \Rightarrow h \sqsubseteq f \Rightarrow h \tag{2}$$

$$f \sqsubseteq g \quad \text{implies} \quad h \Rightarrow f \sqsubseteq h \Rightarrow g. \tag{3}$$

A non-empty subset F of a BCK-algebra $\mathbf{A}$ is said to be a *BCK-filter* if $1 \in F$ and $a, a \Rightarrow b \in F$ implies $b \in F$ for all $a, b \in A$. Also, a non-empty subset of an FL_{ew}-algebra $\mathbf{A}$ is said to be an FL_{ew}*-filter* if: (i) $a \leq b$ and $a \in F$ implies $b \in F$; and (ii) $a, b \in F$ implies $a * b \in F$ for all $a, b \in A$. It is easy to see that a non-empty subset of an FL_{ew}-algebra $\mathbf{A}$ is an FL_{ew}-filter if and only if it is a BCK-filter of the BCK-algebra reduct of $\mathbf{A}$ [19, p. 12].

Let $\mathbf{A}$ be an FL_{ew}-algebra [resp. BCK-algebra]. It is well known and easy to see that every congruence ϕ on $\mathbf{A}$ [resp. congruence ϕ on $\mathbf{A}$ such that $\mathbf{A}/\phi$ is a BCK-algebra] is of the form $\theta(F)$ for some FL_{ew}-filter [resp. BCK-filter] F, where for all $a, b \in A$, $a \equiv b \pmod{\theta(F)}$ if and only if $a \Rightarrow b, b \Rightarrow a \in F$ (put $F := 1/\phi$). See [19, Proposition 1.3] for the case of FL_{ew}-algebras and [7, Proposition 1] for the case of BCK-algebras.

For any $\mathbf{A} \in \mathsf{BCK}$, let $\mathrm{Con}_{\mathsf{BCK}}\, \mathbf{A} := \{\theta \in \mathrm{Con}\, \mathbf{A} : \mathbf{A}/\theta \in \mathsf{BCK}\}$. In view of the preceding discussion, we have

Lemma 2.1. *For any* FL_{ew}*-algebra* $\mathbf{A}$*,* $\mathrm{Con}\,\mathbf{A} = \mathrm{Con}_{\mathsf{BCK}}\langle A; \Rightarrow, 1\rangle$*. In particular, if* $\mathbf{H}(\langle A; \Rightarrow, 1\rangle) \subseteq \mathsf{BCK}$*, then* $\mathrm{Con}\,\mathbf{A} = \mathrm{Con}\langle A; \Rightarrow, 1\rangle$*.* □

Recall from [3] that a class K of similar algebras has *definable principal congruences* (DPC) if and only if there exists a formula $\phi(x, y, z, w)$ in the first-order language of K (whose only free variables are x, y, z, w) such that for any $\mathbf{A} \in \mathsf{K}$ and $a, b, c, d \in A$, $c \equiv d \pmod{\Theta^{\mathbf{A}}(a, b)}$ if and only if $\mathbf{A} \models \phi[a, b, c, d]$. When $\phi(x, y, z, w)$ can be taken as a conjunction (*viz.*, finite set) of equations, then K is said to have *equationally definable principal congruences* (EDPC) [15].

For any integer $n \geq 0$, consider the unary $\{*\}$-terms x^n defined recursively by $x^0 := 1$ and $x^{k+1} := x^k * x$ when $0 \leq k \in \omega$. Given $n \in \omega$, an element a of an FL_{ew}-algebra $\mathbf{A}$ is said to be $n+1$*-potent* if $a^{n+1} = a^n$. $\mathbf{A}$ is said to be $n+1$*-potent* if it satisfies an identity of the form

$$x^{n+1} \approx x^n. \tag{E_n^*}$$

Clearly the class E_n^* of all FL_{ew}-algebras satisfying (E_n^*) is equationally definable.

Theorem 2.2. [18, Theorem 2.1] *For a variety* V *of* FL_{ew}*-algebras the following conditions are equivalent:*

(1) V *has DPC;*
(2) V *has EDPC; and*
(3) $\mathsf{V} \subseteq \mathsf{E}_n^*$ *for some* $n \in \omega$*.* □

A ternary term $e(x, y, z)$ is a *ternary deductive* (*TD*) *term* for an algebra $\mathbf{A}$ if $\mathbf{A} \models e(x, x, z) \approx z$, and, for all $a, b, c, d \in A$, $e^{\mathbf{A}}(a, b, c) = e^{\mathbf{A}}(a, b, d)$ if $c \equiv d \pmod{\Theta^{\mathbf{A}}(a, b)}$ [5, Definition 2.1]. $e(x, y, z)$ is said to be a *ternary deductive* (*TD*) *term* for a class K of similar algebras if it is a TD term for every member of K. By [5, Theorem 2.5], $c \equiv d \pmod{\Theta^{\mathbf{A}}(a, b)}$ for any $\mathbf{A} \in \mathsf{K}$ if and only if $e^{\mathbf{A}}(a, b, c) = e^{\mathbf{A}}(a, b, d)$, whence K has EDPC. A TD term $e(x, y, z)$ for an algebra $\mathbf{A}$ is said to be *commutative* if in addition $\mathbf{A} \models e\big(x, y, e(x', y', z)\big) \approx e\big(x', y', e(x, y, z)\big)$. A TD term $e(x, y, z)$ for a class K of similar algebras is said to be *commutative* if it is commutative for every member of K [5, Definition 3.1].

For any integer $n \geq 0$, consider the binary $\{\Rightarrow\}$-terms $x \Rightarrow^n y$ defined recursively by $x \Rightarrow^0 y := y$ and $x \Rightarrow^{k+1} y := x \Rightarrow (x \Rightarrow^k y)$ when $0 \leq k \in \omega$. Given $n \in \omega$, a BCK-algebra is said to be $n+1$*-potent* if it satisfies an identity of the form

$$x \Rightarrow^{n+1} y \approx x \Rightarrow^n y. \tag{$\mathrm{E}_n^{\Rightarrow}$}$$

By Cornish [12, Theorem 1.4] the class $\mathsf{E}_n^{\Rightarrow}$ of all BCK-algebras satisfying ($\mathrm{E}_n^{\Rightarrow}$) is a variety.

Theorem 2.3. [8, Theorem 4.2] *For* $n \in \omega$*, the following conditions are equivalent for a variety* V *of BCK-algebras:*

(1) V *has DPC;*
(2) V *has EDPC;*
(3) $\mathsf{V} \subseteq \mathsf{E}_n^{\Rightarrow}$*; and*
(4) $(x \Rightarrow y) \Rightarrow^n \big((y \Rightarrow x) \Rightarrow^n z\big)$ *is a commutative TD term for* V*.* □

Let V be a variety of FL_{ew}-algebras. Suppose $\mathsf{V} \subseteq \mathsf{E}_n^*$ for some $n \in \omega$ and let $\mathbf{A} \in \mathsf{V}$. By [7, Proposition 13, Lemma 14], $\mathbf{A} \models (\mathrm{E}_n^*)$ if and only if $\mathbf{A} \models (\mathrm{E}_n^{\Rightarrow})$ if and only if $\langle A; \Rightarrow, 1\rangle \models (\mathrm{E}_n^{\Rightarrow})$, whence $\langle A; \Rightarrow, 1\rangle$ is $n+1$-potent. Since $\langle A; \Rightarrow, 1\rangle \in \mathsf{E}_n^{\Rightarrow}$ and $\mathsf{E}_n^{\Rightarrow}$ is a variety of BCK-algebras, $\mathbf{H}(\langle A; \Rightarrow, 1\rangle) \subseteq \mathsf{BCK}$. By Lemma 2.1, therefore, $\operatorname{Con} \mathbf{A} = \operatorname{Con}\langle A; \Rightarrow, 1\rangle$ and hence

$$\begin{aligned} c \equiv d \pmod{\Theta^{\mathbf{A}}(a,b)} \quad &\text{iff} \quad c \equiv d \pmod{\Theta^{\langle A; \Rightarrow, 1\rangle}(a,b)} \\ &\text{iff} \quad e^{\langle A; \Rightarrow, 1\rangle}(a,b,c) = e^{\langle A; \Rightarrow, 1\rangle}(a,b,d) \\ &\text{iff} \quad e^{\mathbf{A}}(a,b,c) = e^{\mathbf{A}}(a,b,d) \end{aligned}$$

where $e(x,y,z)$ denotes the commutative TD term of Theorem 2.3. Of course, $\mathbf{A} \models e(x,x,z) \approx z$. Thus $e(x,y,z)$ is a commutative TD term for V.

Conversely, suppose $e(x,y,z) := (x \Rightarrow y) \Rightarrow^n \big((y \Rightarrow x) \Rightarrow^n z\big)$ is a commutative TD term for V. Let $\mathbf{A} \in \mathsf{V}$. By [5, Theorem 2.3, Corollary 2.4], $\mathbf{A} \models e(x,y,x) \approx e(x,y,y)$, which is to say $\mathbf{A}$ satisfies

$$(x \Rightarrow y) \Rightarrow^n \big((y \Rightarrow x) \Rightarrow^n x\big) \approx (x \Rightarrow y) \Rightarrow^n \big((y \Rightarrow x) \Rightarrow^n y\big). \tag{4}$$

Therefore $\langle A; \Rightarrow, 1\rangle \models (4)$. But by [7, Proposition 13], a BCK-algebra satisfies (4) if and only if it satisfies $(\mathrm{E}_n^{\Rightarrow})$. Hence $\langle A; \Rightarrow, 1\rangle$ is $n+1$-potent. By the remarks following Theorem 2.3 we infer that $\mathbf{A}$ is $n+1$-potent, whence $\mathbf{A} \in \mathsf{E}_n^*$. We have established

Proposition 2.4. *For $n \in \omega$, $(x \Rightarrow y) \Rightarrow^n \big((y \Rightarrow x) \Rightarrow^n z\big)$ is a commutative TD term for a variety V of FL_{ew}-algebras if and only if $\mathsf{V} \subseteq \mathsf{E}_n^*$.* □

In [29, Proposition 3.2] the authors showed that the variety of Nelson algebras satisfies the identity $x \Rightarrow (x \Rightarrow y) \approx x \to y$, where $\Rightarrow$ denotes the derived binary term defined as in $(\Rightarrow_{\mathrm{def}})$. See also Viglizzo [33, Chapter 1]. Since the variety of all Nelson FL_{ew}-algebras is 3-potent, we have

Corollary 2.5. [28, Theorem 3.3, Remark 3.5] *$(x \Rightarrow y) \to \big((y \Rightarrow x) \to z\big)$ is a commutative TD term for the variety of Nelson algebras, where $\Rightarrow$ denotes the derived binary term defined as in $(\Rightarrow_{\mathrm{def}})$.*

Proof. By Blok and Pigozzi [5, Theorem 2.3(iii)] the property of being a commutative TD term for a variety can be characterised solely by equations, so the result follows from the remarks preceding the corollary and Proposition 2.4. □

Next, recall the following classic result from the theory of FL_{ew}-algebras.

Proposition 2.6. [16, Theorem 2] *The variety of FL_{ew}-algebras is arithmetical. A Mal'cev term for FL_{ew} is $\big((x \Rightarrow y) \Rightarrow z\big) \wedge \big((z \Rightarrow y) \Rightarrow x\big)$.* □

From Proposition 2.6 we infer

Theorem 2.7. [28, Theorem 4.4] *The variety of Nelson algebras is arithmetical. A Mal'cev term for N is $\big((x \Rightarrow y) \Rightarrow z\big) \wedge \big((z \Rightarrow y) \Rightarrow x\big)$, where $\Rightarrow$ denotes the derived binary term defined as in $(\Rightarrow_{\mathrm{def}})$.* □

Let $\mathbf{A}$ be a finite FL_{ew}-algebra. Because the monoid reduct of $\mathbf{A}$ is finite, $\mathbf{A}$ must be $n+1$-potent for some $n \in \omega$. See also Cornish [13, p. 419]. By the remarks following Theorem 2.3, we infer that $\langle A; \Rightarrow, 1\rangle$ is $n+1$-potent and hence that $\mathbf{H}(\langle A; \Rightarrow, 1\rangle) \subseteq \mathsf{BCK}$. We therefore have

Theorem 2.8. *Every finite* FL_{ew}*-algebra* $\mathbf{A}$ *is polynomially equivalent to its BCK-algebra reduct* $\langle A; \Rightarrow, 1\rangle$.

Proof. The result follows Lemma 2.1, Proposition 2.6 and a result due to Pixley [24, Theorem 1], which asserts that if $\mathbf{B}$ is a finite algebra in an arithmetical variety and $f : B^m \to B$ is a function preserving congruences on $\mathbf{B}$ then f is a polynomial of $\mathbf{B}$. □

Corollary 2.9. *Every finite Nelson algebra* $\mathbf{A}$ *is polynomially equivalent to its BCK-algebra term reduct* $\langle A; \Rightarrow, 1\rangle$. □

The class $\mathsf{N}^{\{\Rightarrow,1\}}$ of all $\langle \Rightarrow, 1\rangle$-term reducts of Nelson algebras is strictly contained within the variety of all 3-potent BCK-algebras. In particular, it can be shown that $\mathsf{N}^{\{\Rightarrow,1\}}$ satisfies the identity

$$\big((x \Rightarrow y) \Rightarrow (y \Rightarrow x)\big) \Rightarrow (y \Rightarrow x) \approx \mathbf{1}. \tag{L}$$

(Commutative) BCK-algebras satisfying the identity (L) have been studied extensively by Dvurečenskij and his collaborators in a series of papers beginning with [14].

It is easy to see that 3-potent BCK-algebras need not satisfy (L) in general. Hence, Corollary 2.9 prompts the following

Problem 2.10. Characterise the $\langle \Rightarrow, 1\rangle$-reducts of Nelson algebras. □

3. Nelson algebras as pseudo-interior algebras

A *BCI-monoid* is an algebra $\langle A; \wedge, *, \Rightarrow, 1\rangle$ where: (i) $\langle A; \wedge\rangle$ is a semilattice; (ii) $\langle A; *, 1\rangle$ is a commutative monoid; and for all $a, b, c \in A$, both (iii) $a \leq b$ implies $a * c \leq b * c$ and $c * a \leq c * b$; and (iv) $a \leq b \Rightarrow c$ if and only if $a * b \leq c$ [1, Section 2]. An *integral* BCI-monoid is a BCI-monoid $\mathbf{A}$ satisfying $a \leq 1$ for all $a \in A$. By [1, Proposition 2.8] the class of all (integral) BCI-monoids is equationally definable. For a recent study of BCI-monoids, see Olson [20].

For any integer $n \geq 0$, consider again the unary $\{*\}$-terms x^n defined recursively by $x^0 := 1$ and $x^{k+1} := x^k * x$ when $0 \leq k \in \omega$. A unary operation $f(c)$ on an integral BCI-monoid $\mathbf{A}$ is said to be *compatible* if for any $a, b \in A$ there is an $n \in \omega$ such that $\big((a \Rightarrow b) \wedge (b \Rightarrow a)\big)^n \leq f(a) \Rightarrow f(b)$ [1, Section 2]. An m-ary operation $f(c_1, \ldots, c_m)$ on $\mathbf{A}$ is said to be *compatible* if $f_i(c) := f(a_1, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_m)$ is compatible for any $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m \in A$ and $i = 1, \ldots, m$. An *integral BCI-monoid with compatible operations* is an algebra $\langle A; \wedge, \Rightarrow, *, 1, f_i\rangle_{i \in I}$ such that $\langle A; \wedge, \Rightarrow, *, 1\rangle$ is an integral BCI-monoid and any f_i is compatible. By Aglianó [1, Remarks following Proposition 2.13] $\mathbf{A} := \langle A; \wedge, \Rightarrow, *, 1, f_i\rangle_{i \in I}$ is an integral BCI-monoid with compatible operations if and only if $\mathbf{A}' := \langle A; \wedge, \Rightarrow, *, 1\rangle$ is an integral BCI-monoid and the congruences on $\mathbf{A}$ are determined by $\mathbf{A}'$ in the sense that $\operatorname{Con} \mathbf{A} = \operatorname{Con} \mathbf{A}'$.

Lemma 3.1. *The variety of commutative, integral residuated lattices satisfies the identity:*

$$\big((x \Rightarrow y) \wedge (y \Rightarrow x)\big) \Rightarrow \big((x \vee z) \Rightarrow (y \vee z)\big) \approx \mathbf{1}. \tag{5}$$

Proof. Let $\mathbf{A} \in \mathsf{CIRL}$ and let $a, b, c \in A$. To establish (5), note first that CIRL satisfies the identities

$$(x \Rightarrow y) \Rightarrow \big((x \wedge z) \Rightarrow y\big) \approx \mathbf{1} \tag{6}$$

$$(x \Rightarrow y) \Rightarrow \big(x \Rightarrow (y \vee z)\big) \approx \mathbf{1}. \tag{7}$$

Indeed, from $a \wedge c \leq a$ and (2) we have $a \Rightarrow b \leq (a \wedge c) \Rightarrow b$, which yields (6). Similarly, from $b \leq b \vee c$ and (3) we have $a \Rightarrow b \leq a \Rightarrow (b \vee c)$, which gives (7).

Next, note that CIRL satisfies the identity

$$x \Rightarrow (y \vee z) \approx (x \vee z) \Rightarrow (y \vee z). \tag{8}$$

Indeed, from the theory of residuated lattices [9, Lemma 3.2] we have that CIRL satisfies the identity

$$(x \vee y) \Rightarrow z \approx (x \Rightarrow z) \wedge (y \Rightarrow z). \tag{9}$$

But then

$$\begin{aligned}
(a \vee c) \Rightarrow (b \vee c) &= \big(a \Rightarrow (b \vee c)\big) \wedge \big(c \Rightarrow (b \vee c)\big) && \text{by (9)}\\
&= \big(a \Rightarrow (b \vee c)\big) \wedge 1 && \text{since } c \leq b \vee c\\
&= a \Rightarrow (b \vee c)
\end{aligned}$$

which yields (8) as claimed.

Now it is clear that

$$\begin{aligned}
1 = \big((a \Rightarrow b) &\Rightarrow (a \Rightarrow (b \vee c))\big) \Rightarrow\\
&\big(((a \Rightarrow b) \wedge (b \Rightarrow a)) \Rightarrow (a \Rightarrow (b \vee c))\big) && \text{by (6)}\\
= \big((a \Rightarrow b) &\wedge (b \Rightarrow a)\big) \Rightarrow \big(a \Rightarrow (b \vee c)\big) && \text{by (7),(1)}\\
= \big((a \Rightarrow b) &\wedge (b \Rightarrow a)\big) \Rightarrow \big((a \vee c) \Rightarrow (b \vee c)\big) && \text{by (8)}
\end{aligned}$$

which gives (5) as desired. □

It is well known and easy to see that for any $\mathbf{A} \in \mathsf{CIRL}$, $a \leq b$ implies $a * c \leq b * c$ and $c * a \leq c * b$ for all $a, b, c \in A$. The $\langle \wedge, *, \Rightarrow, 1\rangle$-reduct of any commutative, integral residuated lattice is thus an integral BCI-monoid. Moreover, commutativity of the monoid operation $*$ together with Lemma 3.1 guarantees that the lattice join $\vee$ is compatible with $\langle A; \wedge, *, \Rightarrow, 1\rangle$. Hence we have

Lemma 3.2. *The variety of commutative, integral residuated lattices, hence* FL_{ew}*-algebras, is a variety of integral BCI monoids with compatible operations.* □

A TD term $e(x, y, z)$ for an algebra $\mathbf{A}$ with a constant term $\mathbf{1}$ is said to be *regular* (*for* $\mathbf{A}$) *with respect to* $\mathbf{1}$ if $a \equiv b \pmod{\Theta^{\mathbf{A}}(e^{\mathbf{A}}(a, b, 1), 1)}$ for all $a, b \in A$ [5, Definition 4.1]. A TD term $e(x, y, z)$ for a variety V with a constant term $\mathbf{1}$ is said to be *regular* (*for* V) *with respect to* $\mathbf{1}$ if it is regular with respect to $\mathbf{1}$ for every member of V.

Theorem 3.3. [1, Theorem 3.1, Corollary 3.2] *For $n \in \omega$, the following conditions are equivalent for a variety V of integral BCI-monoids with compatible operations:*

(1) *The ternary term $\big((x \Rightarrow y) \wedge (y \Rightarrow x)\big)^n * z$ is a commutative, regular TD term for V with respect to $\mathbf{1}$;*
(2) *V has EDPC: for any $\mathbf{A} \in \mathsf{V}$ and $a, b, c, d \in A$,*

$$c \equiv d \pmod{\Theta^{\mathbf{A}}(a,b)} \quad \textit{iff} \quad \big((a \Rightarrow b) \wedge (b \Rightarrow a)\big)^n \leq (c \Rightarrow d) \wedge (d \Rightarrow c).$$

□

Let V be a variety of FL_{ew}-algebras. Observe that for any $\mathbf{A} \in \mathsf{V}$ and $a, b, c, d \in A$, the statement

$$c \equiv d \pmod{\Theta^{\mathbf{A}}(a,b)} \quad \text{iff} \quad \big((a \Rightarrow b) \wedge (b \Rightarrow a)\big)^n \leq (c \Rightarrow d) \wedge (d \Rightarrow c) \qquad (10)$$

is equivalent to its corresponding statement about FL_{ew}-filters, *viz.*:

$$c \in \mathrm{F}(a) \quad \text{iff} \quad a^n \leq c, \qquad (11)$$

where $\mathrm{F}(c)$ denotes the principal filter generated by $c \in A$. We claim that (11) is equivalent to the assertion $a^{n+1} = a^n$. So assume a is $n+1$-potent. We have $c \in \mathrm{F}(a)$ if and only if $a^k \leq c$ for some k if and only if $a^n \leq c$ (because $a^r \leq a^s$, for $s \leq r$). Conversely, suppose (11) holds. Clearly, $n+1$-potency is equivalent to $a^n \leq a^{n+1}$, which in view of (11) reduces to $a^{n+1} \in \mathrm{F}(a)$, which statement is true.

From the preceding discussion it follows that $\mathsf{V} \models (\mathrm{E}_n^*)$ if and only if (10) holds for any $\mathbf{A} \in \mathsf{V}$ and $a, b, c, d \in A$. Combining Lemma 3.2 with Theorem 3.3 therefore yields

Proposition 3.4. *For $n \in \omega$, $\big((x \Rightarrow y) \wedge (y \Rightarrow x)\big)^n * z$ is a commutative, regular TD term with respect to $\mathbf{1}$ for a variety V of FL_{ew}-algebras if and only if $\mathsf{V} \subseteq \mathsf{E}_n^*$.* □

In [5, Problem 7.4] Blok and Pigozzi asked whether the variety of Nelson algebras has a commutative, regular TD term, or even a TD term; for a discussion and references, see Spinks [28]. The following corollary, in conjunction with Corollary 2.5, completely resolves this question. But first, for a term $t := t(\vec{x})$ in the language of Nelson algebras, let t^2 abbreviate $t * t$, where $*$ denotes the derived binary term defined as in $(*_{\mathrm{def}})$.

Corollary 3.5. *$\big((x \Rightarrow y) \wedge (y \Rightarrow x)\big)^2 * z$ is a commutative, regular TD term with respect to $\mathbf{1}$ for the variety of Nelson algebras, where $\Rightarrow$ and $*$ denote the derived binary terms defined as in $(\Rightarrow_{\mathrm{def}})$ and $(*_{\mathrm{def}})$ respectively.*

Proof. Since the property of being a commutative, regular TD term for a variety can be characterised solely by equations (by [5, Theorem 2.3(iii)] and [5, Corollary 4.2(i)]), the result follows from 3-potency and Proposition 3.4. □

Let $\langle A; \cdot, 1\rangle$ be a semigroup with a constant 1 that acts as a left identity for $\cdot$. A unary operation $^\circ$ on A is said to be a *pseudo-interior operation on* $\langle A; \cdot, 1\rangle$ if the following identities are satisfied [6, Definition 2.1]:

$$x^\circ \cdot y^\circ \approx y^\circ \cdot x^\circ$$
$$x \cdot y \approx x^\circ \cdot y$$
$$x \cdot x \approx x^\circ$$
$$\mathbf{1}^\circ \approx \mathbf{1}.$$

Given a semigroup $\mathbf{A}$ with left identity 1 and pseudo-interior operation $^\circ$, the *inverse right-divisibility ordering* on $\mathbf{A}$ is the partial ordering $\leq_{\mathrm{r}}$ defined for all $a, b \in A$ by $a \leq_{\mathrm{r}} b$ if and only if there exists $c \in A$ such that $a = c \cdot b$ [6, Lemma 2.3].

An algebra $\langle A; \cdot, \rightarrow, ^\circ, 1\rangle$ of type $\langle 2, 2, 1, 0\rangle$ is said to be a *pseudo-interior algebra* if [6, Definition 2.6]: (i) $\langle A; \cdot, 1\rangle$ is a semigroup with left identity 1; (ii) $^\circ$ is a pseudo-interior operation on $\langle A; \cdot, 1\rangle$; and (iii) $\rightarrow$ is an *open left residuation* on $\langle A; \cdot, 1\rangle$ in the sense that $(a \rightarrow b)^\circ = a \rightarrow b$ for all $a, b \in A$, and moreover $c \cdot a \leq_{\mathrm{r}} b$ if and only if $c^\circ \leq_{\mathrm{r}} a \Rightarrow b$ for all $c \in A$. An algebra $\mathbf{A} := \langle A; \cdot, \rightarrow, ^\circ, 1, f_i\rangle_{i \in I}$ is said to be a *pseudo-interior algebra with compatible operations* if $\mathbf{A}' := \langle A; \cdot, \rightarrow, ^\circ, 1\rangle$ is a pseudo-interior algebra and the congruences on $\mathbf{A}$ are determined by $\mathbf{A}'$ in the sense that $\operatorname{Con} \mathbf{A} = \operatorname{Con} \mathbf{A}'$ [6, Definition 2.7, Corollary 2.17]. By [6, Theorem 3.1] the class of all pseudo-interior algebras, with or without compatible operations, is equationally definable.

Theorem 3.6. [6, Theorem 4.1, Corollary 4.2] *A variety* V *has a commutative, regular TD term if and only if it is term equivalent to a variety of pseudo-interior algebras with compatible operations. If* $e(x, y, z)$ *is a commutative, regular TD term for* V *with respect to* $\mathbf{1}$*, then*

$$x \cdot y := e(x, \mathbf{1}, y)$$
$$x^\circ := e(x, \mathbf{1}, \mathbf{1})$$
$$x \rightarrow y := e\big(x, e(x, y, x), \mathbf{1}\big)$$

define terms realising the pseudo-interior operations $\cdot$*,* $^\circ$*, and* $\rightarrow$ *on any* $\mathbf{A} \in \mathsf{V}$ *such that all the fundamental operations of* $\mathbf{A}$ *are compatible with* $\langle A; \cdot, \rightarrow, ^\circ, 1\rangle$*.*
□

Let V be a variety of FL_{ew}-algebras. If $\mathsf{V} \subseteq \mathsf{E}_n^*$ for some $n \in \omega$, then V is term equivalent to a congruence permutable variety of pseudo-interior algebras with compatible operations by Proposition 2.6, Proposition 3.4, and Theorem 3.6. Conversely, if V is term equivalent to a congruence permutable variety of pseudo-interior algebras with compatible operations, then $\mathsf{V} \subseteq \mathsf{E}_n^*$ for some $n \in \omega$ by Theorem 3.6, EDPC, and Theorem 2.2. Therefore we have

Theorem 3.7. *A variety* V *of* FL_{ew}*-algebras is term equivalent to a congruence permutable variety of pseudo-interior algebras with compatible operations if and only if* $\mathsf{V} \subseteq \mathsf{E}_n^*$ *for some* $n \in \omega$*. If* $\mathsf{V} \subseteq \mathsf{E}_n^*$*, then for any* $\mathbf{A} \in \mathsf{V}$*, terms realising* *the pseudo-interior operations $\cdot$, $^\circ$, and $\to$ on A are defined by*

$$x \cdot y := x^n * y$$
$$x^\circ := x^n$$
$$x \to y := \big(x \Rightarrow (((x \Rightarrow y) \wedge (y \Rightarrow x))^n * x)\big)^n.$$

Proof. It remains only to establish the second assertion of the theorem. When V is $n+1$-potent, the terms realising the pseudo-interior operations on any member of V may be obtained by instantiating Theorem 3.6 with the TD term of Proposition 3.4 and simplifying the resulting expressions for $x \cdot y$, x° and $x \to y$ using the now well-developed arithmetic of commutative, integral residuated lattices [9, 19]. □

Since the variety of all Nelson FL_{ew}-algebras is 3-potent, from Theorem 3.7 we have

Corollary 3.8. *The variety of Nelson algebras is term equivalent to a congruence permutable variety of pseudo-interior algebras with compatible operations. For any $\mathbf{A} \in \mathsf{N}$, terms realising the pseudo-interior operations $\cdot$, $^\circ$, and $\to$ on A are defined by*

$$x \cdot y := x^2 * y$$
$$x^\circ := x^2$$
$$x \to y := \big(x \Rightarrow (((x \Rightarrow y) \wedge (y \Rightarrow x))^2 * x)\big)^2$$

where the derived binary terms $\Rightarrow$ and $*$ are defined as in ($\Rightarrow_{\mathrm{def}}$) and ($*_{\mathrm{def}}$) respectively. □

4. Nelson algebras as bounded BCK-semilattices

In 1963 D. Brignole resolved a problem, posed by A. Monteiro, that asked whether Nelson algebras could be defined in terms of the connectives $\Rightarrow$, $\wedge$ and the constant 0. See [10] or [33, Chapter 1, p. 21]. Let B (for Brignole) denote the variety of all algebras $\langle A; \wedge, \Rightarrow, 0\rangle$ of type $\langle 2, 2, 0\rangle$ axiomatised by the following collection of identities

$$(x \Rightarrow x) \Rightarrow y \approx y$$
$$(x \Rightarrow y) \wedge y \approx y$$
$$x \wedge \sim(x \wedge \sim y) \approx x \wedge (x \Rightarrow y)$$
$$x \Rightarrow (y \wedge z) \approx (x \Rightarrow y) \wedge (x \Rightarrow z)$$
$$x \Rightarrow y \approx \sim y \Rightarrow \sim x$$
$$x \Rightarrow \big(x \Rightarrow (y \Rightarrow (y \Rightarrow z))\big) \approx (x \wedge y) \Rightarrow \big((x \wedge y) \Rightarrow z\big)$$
$$\sim(\sim x \wedge y) \Rightarrow (x \Rightarrow y) \approx x \Rightarrow y$$
$$x \wedge (x \vee y) \approx x$$
$$x \wedge (y \vee z) \approx (z \wedge x) \vee (z \wedge y)$$
$$(x \wedge \sim x) \vee (y \vee \sim y) \approx x \wedge \sim x$$

where the derived nullary term $\mathbf{1}$ is defined as

$$\mathbf{1} := \mathbf{0} \Rightarrow \mathbf{0}, \tag{$\mathbf{1}_{\mathrm{def}}$}$$

the derived unary term $\sim$ is defined as in ($\sim_{\mathrm{def}}$), and the derived binary term $\vee$ is defined by

$$x \vee y := \sim(\sim x \wedge \sim y). \tag{$\vee_{\mathrm{def}}$}$$

Brignole established the following result:

Theorem 4.1. [10]

(1) *Let* $\mathbf{A}$ *be a Nelson algebra and define the derived binary term* $\Rightarrow$ *as in* ($\Rightarrow_{\mathrm{def}}$). *Then the term reduct* $\mathbf{A}^B := \langle A; \wedge, \Rightarrow, 0\rangle$ *is a member of* B.

(2) *Let* $\mathbf{A}$ *be a member of* B. *Define the derived binary terms* $\vee$ *and* $\rightarrow$ *as in* ($\vee_{\mathrm{def}}$) *and* ($\rightarrow_{\mathrm{def}}$) *respectively, the derived unary term* $\sim$ *as in* ($\sim_{\mathrm{def}}$) *and the derived nullary term* $\mathbf{1}$ *as in* ($\mathbf{1}_{\mathrm{def}}$). *Then the term reduct* $\mathbf{A}^N := \langle A; \wedge, \vee, \rightarrow, \sim, 0, 1\rangle$ *is a Nelson algebra.*

(3) *Let* $\mathbf{A}$ *be a Nelson algebra. Then* $(\mathbf{A}^B)^N = \mathbf{A}$.

(4) *Let* $\mathbf{A}$ *be a member of* B. *Then* $(\mathbf{A}^N)^B = \mathbf{A}$.

Hence the variety of Nelson algebras and the variety B *are term equivalent.* □

A *lower BCK-semilattice* is an algebra $\langle A, \wedge, \Rightarrow, 1\rangle$ where [27, Lemma 1.6.24]: (i) $\langle A; \Rightarrow, 1\rangle$ is a BCK-algebra; (ii) $\langle A; \wedge\rangle$ is a lower semilattice; and (iii) for all $a, b \in A$, $a \leq b$ if and only if $a \sqsubseteq b$, where $\leq$ and $\sqsubseteq$ denote the semilattice and BCK-algebra partial orderings respectively. Lower BCK-semilattices have been studied in particular by Idziak [16]. A *bounded lower BCK-semilattice* $\langle A; \wedge, \Rightarrow, 0, 1\rangle$ is a lower BCK-semilattice with distinguished least element $0 \in A$. A (bounded) lower BCK-semilattice is said to be $n+1$*-potent* if its BCK-algebra reduct is $n+1$-potent.

Let $\mathsf{B}^{\mathbf{1}}$ denote the variety of all algebras $\langle A; \wedge, \Rightarrow, 0, 1\rangle$ having type $\langle 2, 2, 0, 0\rangle$ axiomatised by the identities defining B given above together with the identity $x \Rightarrow x \approx \mathbf{1}$. It is clear that $\mathsf{B}^{\mathbf{1}}$ is term equivalent to B and therefore also to both N and NFL_{ew}. The following result illuminates Brignole's description of Nelson algebras given in Theorem 4.1 above.

Theorem 4.2. *The variety of Nelson algebras is term equivalent to a variety of bounded 3-potent BCK-semilattices.*

Proof. It suffices to show any $\mathbf{A} \in \mathsf{B}^{\mathbf{1}}$ is a bounded 3-potent lower BCK-semilattice. By [29, Theorem 3.7] the $\langle \Rightarrow, 1\rangle$-term reducts of members of NFL_{ew} are 3-potent BCK-algebras. Hence $\langle A; \Rightarrow, 1\rangle$ is a 3-potent BCK-algebra. Of course, $\langle A; \wedge\rangle$ is a lower semilattice. By Rasiowa [25, Theorem V.1.1], $a \leq b$ if and only if $a \sqsubseteq b$ for all $a, b \in B$ for any Nelson algebra $\mathbf{B}$, where $\leq$ and $\sqsubseteq$ denote the lattice and BCK-algebra partial orders respectively. Hence the semilattice partial order and the BCK-algebra partial order coincide on A, and $\mathbf{A}$ is a 3-potent lower BCK-semilattice. Finally, 0 is clearly the least element of $\mathbf{A}$, whence $\mathbf{A}$ is a bounded 3-potent lower BCK-semilattice. □

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