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A QUALITATIVE UNCERTAINTY PRINCIPLE FOR COMPLETELY SOLVABLE LIE GROUPS.

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Abstract. In this paper, we study a qualitative uncertainty principle for completely solvable Lie groups.

1. Introduction

Let G be a connected, simply connected, and completely solvable Lie group, with Lie algebra $\mathcal{G}$. Let $\mathcal{G}^*$ be the dual of $\mathcal{G}$. The equivalence classes of irreducible unitary representations $\hat{G}$ of G is parameterized by the coadjoint orbits $\mathcal{G}^*/G$ via the Kirillov bijective map

$$K\colon \hat{G} \to \mathcal{G}^*/G$$

We recall that if $(V_\rho, \rho) \in \hat{G}$ and $l \in K(\rho)$, then there exists an analytic subgroup H of G and a unitary character ξ of H, such that the induced representation ρ is equivalent to $Ind_H^G \xi$. Moreover the push forward of a Plancherel measure in $\hat{G}$ is a measure equivalent to a Lebesguian measure on convenient set of representatives in $\mathcal{G}^*$ for $\hat{G}$.

Let f in $L^1(\mathbb{R}^n)$ and set $\hat{f}$ its Fourier transform, let $A_f = \{x \in \mathbb{R}^n\colon f(x) \neq 0\}$ and $B_f = \{x \in \mathbb{R}^n\colon \hat{f}(x) \neq 0\}$. By Bénédicks theorem [1, Theorem 2], if $\lambda(A_f) < \infty$ and $\lambda(B_f) < \infty$ then $f = 0$ a.e. Here, λ denote Lebesgue measure on $\mathbb{R}^n$. That is, for $\mathbb{R}^n$ the qualitative uncertainty principle holds.

In this note we prove that a completely solvable Lie group has the qualitative uncertainty principle. In [4] we showed the theorem for nilpotent Lie groups, by induction on the dimension of G. To prove the theorem we apply induction, for this, we need an explicit description of the dual space $\hat{G}$ of G as well as an explicit description of Plancherel measure on $\hat{G}$. For our approach we use a result of B.N. Currey [3], which is a generalization of a result of L. Pukanszky. Let G be a locally compact group. Denote a fixed Haar measure on G by m and the corresponding Plancherel measure on $\hat{G}$ by μ.

Let $A_f = \{x \in G\colon f(x) \neq 0\}$ and $B_f = \{\pi \in \hat{G}\colon \hat{f}(\pi) \neq 0\}$,

Definition 1.1. *G has the qualitative uncertainty principle if $m(A_f) < \infty$ and $\mu(B_f) < \infty$, then $f = 0$ m-a.e.*

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Remark 1.1. *The group ($\mathbb{R}^n, +$) has the qualitative uncertainty principle [1, Theorem 2].*

2. Preliminaries

Let G be a connected, simply connected, and completely solvable Lie group, with the Lie algebra $\mathcal{G}$. Let $\mathcal{G}^*$ be its dual. Since G is completely solvable, there exists a chain of ideals of $\mathcal{G}$

$$(0) = \mathcal{G}_0 \subset \mathcal{G}_1 \subset ... \subset \mathcal{G}_n = \mathcal{G}$$

such that the dimension of $\mathcal{G}_j$ is j, for all $j \leq n$. We fix an ordered basis $\mathcal{B} = \{X_1, X_2, .., X_n\}$ of $\mathcal{G}$ such that $\mathcal{G}_j$ is spanned by the vectors $X_1, .., X_j$,$1 \leq j \leq n$. Let $\mathcal{B}^* = \{X_1^*, X_2^*, .., X_n^*\}$ be the dual basis of $\mathcal{B}$. We fix a Lebesgue measure dX on $\mathcal{G}$ and a right invariant Haar measure m on G such that $m(expX) = J_G(X)dX$ where

$$J_G(X) = |det(\frac{1 - e^{-adX}}{adX})|$$

Let δ be the modular function such that for all $g \in G$, $m(gg') = \delta(g)m(g')$. Let O be a co-adjoint orbit in $\mathcal{G}^*$ and $l \in O$. The bilinear form $B_l \colon (X, Y) \to l([X, Y])$ defines a skew-symmetric and nondegenerate bilinear form on $\mathcal{G}/\mathcal{G}^l$. Since the map $X \to X.l$ induces an isomorphism between $\mathcal{G}/\mathcal{G}^l$ and the tangent space of O at l, the bilinear form B_l defines a nondegenerate 2-form w_l on this tangent space. If $2k$ is the dimension of O we note that

$$\mathcal{B}_O := (2k)^{-k}(k!)^{-1} w_l \wedge w_l \wedge .. \wedge w_l \ (k\ times)$$

is a canonical measure on O. Lemma 3.2.2 in [2] says that there exists a nonzero rational function ψ on $\mathcal{G}^*$ such that

$$\psi(g.l) = \delta(g)^{-1}\psi(l), g \in G, l \in \mathcal{G}^*$$

and there exists a unique measure m_ψ on $\mathcal{G}^*/G$ such that

$$\int_{\mathcal{G}^*} \phi(l)|\psi(l)|dl = \int_{\mathcal{G}^*/G} (\int_O \phi(l) d\mathcal{B}_O(l)) dm_\psi(O)$$

for all Borel function ϕ on $\mathcal{G}^*$. B.N. Currey [3,] gave an explicit description of the measure m_ψ with the help of the coadjoint orbits. We recall the theorem proved by B.N. Currey which is a essential tool to prove our main theorem.

Theorem 2.1. *Let G be a connected, simply connected and completely solvable Lie group. There exists a Zariski open subset U in $\mathcal{G}^*$, a subset $J = \{j_1 < j_2 < ... < j_{2k}\}$ of $\{1, 2, ..., n\}$, a subset $M = \{j_{r_1} < j_{r_2} < ... < j_{r_a}\}$ of J, for each $j \in M$ a real valued rational function q_j, non vanishing on U, and real analytic functions P_j in the variables $w_1, w_2, .., w_{2k}, l_1, l_2, .., l_n$ such that the following hold.*

(1) *If a denotes the number of elements of M, for each $\epsilon \in \{1, -1\}^a$, the set*

$$U_\epsilon = \{l \in U \mid \ sign\ of\ q_{j_{r_m}}(l) = \epsilon_m, 1 \leq m \leq a\}$$

is a non empty open subset in $\mathcal{G}^*$.

(2) *Define* $V \subset \mathbb{R}^{2k}$ *by* $V = \prod R_r$, *where* $R_r =]0, \infty[$ *if* $j_r \in M$ *and* $R_r = \mathbb{R}$ *otherwise. Let* $\epsilon \in \{1, -1\}^a$, *for* $v \in V$, *define* $\epsilon v \in \mathbb{R}^{2k}$ *by* $(\epsilon v)_j = \epsilon_m v_j$ *if* $j = j_{r_m} \in M$ *and* $(\epsilon v)_j = v_j$ *otherwise. Then for each* $l \in U_\epsilon$, *the mapping* $v \longrightarrow \sum_{j \in J} P_j(\epsilon v, l) X_j^*$ *is a diffeomorphism of* V *with the coadjoint orbit of* l.

(3) *Define* W_D *as the subspace spanned by the vectors* $\{X_i^* \mid i \notin J\}$ *and* W_M *the subspace spanned by the vectors* $\{X_i^* \mid i \in M\}$ *Then the set*

$$W = \{l \in (W_D \oplus W_M) \cap U \mid\mid q_j(l) \mid = 1, j \in M\}$$

is a cross-section for the coadjoint orbits in U. *for each* $j \in M$ *the rational function* q_j *is of the form* $q_j(l) = l_j + p_j(l_1, l_2, .., l_{j-1})$, *where* p_j *is a rational function.*

(4) *For each* $l \in U$, *let* $\epsilon(l) \in \{1, -1\}^a$ *such that* $l \in U_{\epsilon(l)}$. *Then the mapping* $P\colon V \times W \longrightarrow U$, *defined by* $P(v, l) = \sum_j P_j(\epsilon(l)v, l) X_j^*$, *is a diffeomorphism.*

If the subset M is empty, then $W = W_D \cap U$ and the coordinates for W are obtained by identifying W_D with $\mathbb{R}^{n-2k}$, which is the parametrization of $\hat{G}$ in the nilpotent case. If M is not empty and a the number of elements in M. From [3], for each $\epsilon \in \{1, -1\}^a$ U_ϵ is a non empty Zariski open subset and $U = \cup_\epsilon U_\epsilon$ (disjoint union). Set $W_\epsilon = W \cap U_\epsilon$. from [3] we have:

$$W_\epsilon = \{l \in (W_D \oplus W_M) \cap U \mid \text{ for each } j = j_{r_m} \in M, l_j = \epsilon_m - p_j(l_1, ..., l_{j-1})\}$$

p_j is a rational nonsingular function on U.

Let $\epsilon \in \{-1, 1\}^a$. From [3], there is a Zariski open subset Λ_ϵ of W_D and a rational function $p_\epsilon\colon \Lambda_\epsilon \longrightarrow W_M$ such that W_ϵ is the graph of p_ϵ. From [3], the projection of U_ϵ into W_D parallel to W_J defines a diffeomorphism of W_D with Λ_ϵ.

Summarizing: let G be connected, simply connected and completely solvable Lie group. Let $\{X_1^*, X_2^*, ..., X_n^*\}$ be a Jordan-Holder basis of $\mathcal{G}^*$. Then, there is a finite family of disjoint open subsets U_ϵ of $\mathcal{G}^*$ and there is a subspace W_D of $\mathcal{G}^*$ such that for each ϵ, the orbits in U_ϵ are parameterized by a Zariski open subset Λ_ϵ of W_D. The union of this open sets determines an open dense subset of $\mathcal{G}^*/G$ whose complement has Plancherel measure zero.

3. The $ax + b$ Group.

Consider the group

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a > 0, b \in \mathbb{R} \right\}$$

We use the notation $(a, b) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. Matrix multiplication is:

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2, a_1 b_2 + b_1)$$

and the inverse is

$$(a, b)^{-1} = (a^{-1}, -ba^{-1}).$$

The Lie algebra $\mathcal{G}$ of G is the set of matrices

$$\mathcal{G} = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} x, y \in \mathbb{R} \right\}$$

We choose as ordered base $\mathcal{B}$,

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

We have $[X, Y] = Y$. Thus the group is not nilpotent.

Let $\{X^*, Y^*\}$ the dual basis of $\mathcal{G}^*$. Let $l = \alpha X^* + \beta Y^* \in \mathcal{G}$. The orbits of G in $\mathcal{G}^*$ are: the upper half plane $\beta > 0$, the lower half plan $\beta < 0$ and the points $(\alpha, 0)$. Here, $J = \{j_1, j_2\} = \{1, 2\}$ and $M = \{j_2\} \subset J$, so that $V = \mathbb{R}^2, V =]0, +\infty[\times\mathbb{R}$. $W_D = (0)$ and W_M is spanned by the vector $X^*_{j_2}$. The Zariski open sets U_+ and U_- are the half planes of $\mathcal{G}^*$ and $U = U_+ \cup U_-$. Since there are two orbits, the set

$$W = \{l \in W_M \cap U\colon \mid q_{j_2}(l) \mid = 1, j_2 \in M\}$$

has exactly two points. We have $W_+ = W \cap U_+$ and $W_- = W \cap U_-$. The Zariski open set Λ_+ or Λ_- of W_D, reduces to a point.

4. Fourier transform.

We must consider two cases(see [5]):

(1) All the orbits in general position are saturated with respect to $\mathcal{G}_{n-1}$. That is, for each $l \in \mathcal{G}^*$, $\mathcal{G}^l \subset \mathcal{G}_{n-1}$. Then, we may and will choose a basis of $\mathcal{G}$

$$B_{W_\epsilon} = \{X_1(l), X_2(l), ..., X_{n-1}(l), X_n\}.$$

where the last vector of the basis does not depend on l. We apply the previous setting to $G_{n-1} := exp(\mathcal{G}_{n-1})$. Let $J_1 = J \setminus \{n, j_1\}$ the index set for G_{n-1}, then M_1 is a subset of J_1, let a_1 denote the number elements of M_1. For each $\epsilon_1 \in \{-1, 1\}^{a_1}$, the set U_{ϵ_1} is nonempty open subset of $\mathcal{G}^*_{n-1}$. Let $W_{D_1} = W_D \oplus \mathbb{R}X^*_{j_1}$ and then W_{M_1} is the subspace spanned by $X^*_j, j \in M_1$. We apply the inductive hypothesis to G_{n-1}, hence, there is a Zariski open subset $\Lambda_{\epsilon_1} \subset W_{D_1}$ and a rational function $p_{\epsilon_1} : \Lambda_{\epsilon_1} \to W_{D_1}$ such that W_{ϵ_1} is the graph of p_{ϵ_1}. Let Λ_{ϵ_1} denote the projection of Λ_ϵ on $\mathcal{G}^*_{n-1}$. From [5,lemma 3.2], the measure $d\mu_1$ on W_{ϵ_1} in terms of the measure $d\mu$ on W_ϵ and $dX^*_{j_1}$ is $d\mu_1 = d\mu \times dX^*_{j_1}$

(2) If some orbit $G \cdot l$ in general position is not saturated with respect to $\mathcal{G}_{n-1}$, we can still obtain a basis of $\mathcal{G}$ such that the last vector of the basis does not depend on l, $X_n \in \mathcal{G}^l$ and $X_i \in \mathcal{G}^{l_j}_j$ for certain j with $l_j = l \mid \mathcal{G}_j$. In this case since $\mathcal{G}^l = \mathcal{G}^{l_{n-1}}$, we have $W_D = W_{D_1} + \mathbb{R}X_n$. Moreover $\Lambda_\epsilon = \Lambda_{\epsilon_1} + \mathbb{R}{X_n}^*$. The Plancherel measure can be written as $d\mu(l) = d\mu_1 \times dX^*_n$.

5. The main theorem.

Theorem 5.1. *Let G be a connected, simply connected, completely solvable Lie group with the unitary dual $\hat{G}$, and let f be integrable function on G ($f \in L^1(G)$). If $m(A_f) < \infty$ and $\mu(B_f) < \infty$ then $f = 0$ almost every where.*

Proof 5.1. *We proceed by induction on the dimension n of G. The result is true if the dimension of G is one, since $G \cong \mathbb{R}$ (see [1,theorem2]). Assume that the result is true for all completely solvable Lie groups of dimension $n-1$. Suppose that $m(A_f), \mu(B_f)$ are finite. From [4, lemma 1.6], $m_1(A_{f^t})$ is finite. To conclude, it remains to show that $\mu_1(B_{f^t})$ is finite. We can assume that B_f is contained in W_ϵ (It suffices to take B_f as the finite union of $B_f \cap W_\epsilon$). We consider tho cases*

(1) *We suppose that $\mathcal{G}^l \subset \mathcal{G}_{n-1}$ for all $l \in W_\epsilon$. That is, all the orbits in general position are saturated with respect to $\mathcal{G}_{n-1}$. For $\phi \in \mathcal{G}^*$, let ϕ_0 be the restriction of ϕ to $\mathcal{G}_{n-1}$, then $\pi_\phi = Ind_{G_{n-1}}^{G} \pi_{\phi_0}$ is irreducible. From [6, proposition 2.5] we have:*

$$\int_{\mathcal{G}^*_{n-1}}^{\oplus} Ind_{G_{n-1}}^{G} \pi_{\phi_0} d\lambda_{\mathcal{G}^*_{n-1}}(\phi_0) \quad \simeq \quad \int_{\mathcal{G}^*}^{\oplus} \pi_\phi d\lambda_{\mathcal{G}^*}(\phi) \tag{1}$$

where $d\lambda_{\mathcal{G}^*}$ is the Lebesgue measure on $\mathcal{G}^*$ and $d\lambda_{\mathcal{G}^*_{n-1}}$ is the Lebesgue measure on $\mathcal{G}^*_{n-1}$. From the formula (1) and the definition of X_n, we conclude that the map $\phi \to \phi_0$ is an isomorphism which respect to the measures $d\lambda_{\mathcal{G}^*_{n-1}}$ and $d\lambda_{\mathcal{G}^*}$, then

$$\mu_1(B_{f^t}) = \mu(B_f) < \infty.$$

By induction hypothesis $f^t = 0$ almost everywhere on G_{n-1} for almost everywhere $t \in \mathbb{R}$, which implies that $f = 0$ almost everywhere on G by using the theorem of Fubini.

(2) *Some orbit $G \cdot l$ is not satured with respect to $\mathcal{G}_{n-1}$. That is, $\mathcal{G}^l \not\subset \mathcal{G}_{n-1}$ for some $l \in W_\epsilon$. For $\phi_0 \in \mathcal{G}^*_{n-1}$, we choose an extension ϕ defined by $\phi(X_n) = 0$. From this we have*

$$ind_{G_{n-1}}^{G} \pi_{\phi_0} \sim \int_{\mathbb{R}}^{\oplus} \pi_{\phi_0 + sX_n^*} ds.$$

Hence

$$\mu(B_f) = \int_{\mathbb{R}} \mu_1(B_{f^t}) dt < \infty.$$

Then for almost everywhere $t \in \mathbb{R}$, $\mu_1(B_{f^t})$ is finite. By inductive hypothesis $f^t = 0$ almost everywhere on G_{n-1} for almost everywhere t in $\mathbb{R}$, which implies that $f = 0$ almost everywhere on G by Fubini's theorem.

Remark 5.1. *The $ax + b$ group has the qualitative uncertainty principle.*

Question 5.1. *Do the exponential solvable Lie groups have the qualitative uncertainty principle ?*

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