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THE BOLTZMANN EQUATION WITH FORCE TERM NEAR THE VACUUM (*)

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Abstract. We prove a theorem of existence, uniqueness and positivity of the solution for the Boltzmann equation with force term and initial data near the Vacuum.

1. INTRODUCTION

The aim in this article it is prove global existence of the problem for small data (near to the vacuum) in the case of a solid sphere. For the density $f = f(t, x, v)$, $t \geq 0$; x, $v \in \mathbb{R}^3$,we write the equation of Boltzmann:

$$\left.\begin{aligned} f_t + v.\nabla_x f + F.\nabla_v f = Q(f, f) \\ f(0, x, v) = f_0(x, v) \end{aligned}\right\} \tag{1}$$

Let us also consider the following problem:

$$\left.\begin{aligned} f_t + v.\nabla_x f + F.\nabla_v f + t\frac{\partial F}{\partial t}.\nabla_v f = Q(f, f) \\ f(0, x, v) = f_0(x, v) \end{aligned}\right\} \tag{2}$$

where

$$\begin{aligned} Q(f, f) &= \sigma \int_{S^2_+} \int_{\mathbb{R}^3} w.(v-u)[f(t, x, v^{'})f(t, x, u^{'}) - f(t, x, u)f(t, x, v)]dudw \\ &= Q_g(f, f) - Q_l(f, f) \end{aligned}$$

here $S^2_+ = \{w \in S^2 : wv \geq wu\}$, σ is a constant proportional to the area of the sphere and

$$\left.\begin{aligned} v^{'} = v - aw \\ u^{'} = u + aw \end{aligned}\right\}$$

$a = w(v - u)$.
The Conservation momentum is given by

$$u^{'} + v^{'} = u + v$$

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and the Energy of conservation is

$$|u^{'}|^2 + |v^{'}|^2 = |u|^2 + |v|^2.$$

We write $Q_l(f,f) = fR(f)$, where

$$R(f)(t,x,v) = \sigma \int_{S_+^2} \int_{\mathbb{R}^3} w(v-u) f(t,x,u) du dw = \pi\sigma \int_{\mathbb{R}^3} |v-u| f(t,x,u) du$$

Let F be a vectorial field not depending on f such that:

$$\begin{aligned} F : &\mathbb{R}^7 \longrightarrow \mathbb{R}^3 \\ &(t,x,v) \longrightarrow F(t,x,v) = (F_1(t,x,v), F_2(t,x,v), F_3(t,x,v)) \end{aligned}$$

and for a given $\beta > 0$, let

$$M = \left\{ f \in C^0([0,\infty) \times \mathbb{R}^3 \times \mathbb{R}^3) : \text{ exists } c > 0 \text{ such that } |f(t,x,v)| \leq c e^{-\beta(|x|^2+|v|^2)} \right\}$$

with norm

$$||f|| = \sup_{t,x,v} e^{\beta(|x|^2+|v|^2)} |f(t,x,v)|.$$

This makes M into a Banach Space [8].
To solve problem (2) we introduce the following notation:

$$f^{\#}(t,x,v) = f(t, x+vt, v+tF(t,x,v)) \tag{3}$$

with F differentiable with regard to time. Therefore

$$\begin{aligned} &Q(f^{\#}, f^{\#})(t,x,v) = Q_g(f^{\#}, f^{\#})(t,x,v) - Q_l(f^{\#}, f^{\#})(t,x,v) \\ &= \sigma \int_{S_+^2} \int_{\mathbb{R}^3} w(v-u) f(t, x+tv, v^{'} + tF(t,x,v)) f(t, x+tv, u^{'} + tF(t,x,v)) du dw \\ &- \pi\sigma f(t, x+tv, v+tF(t,x,v)) \int_{\mathbb{R}^3} |(v-u)| f(t, x+tv, u+tF(t,x,v)) du \end{aligned}$$

we write the equation (1) as

$$\frac{d}{dt} f^{\#}(t,x,v) = Q(f^{\#}, f^{\#})(t,x,v) \tag{4}$$

Integrating (3) with respect to time, we obtain

$$f^{\#}(t,x,v) = f_0(x,v) + \int_0^t Q(f^{\#}, f^{\#}) d\tau \tag{5}$$

We observe that if F is constant in time, (2) reduces to (1). We will prove the following theorems:

Theorem 1. *There exists a constant R_0, such that if $||f_0||$ and $\sigma\beta^{-2}R_0$ are sufficiently small and F is differentiable with regard to the time t, then the equation (4) has a unique solution $f^{\#} \in M_{R_0} = \{f \in M : ||f|| \leq R_0\}$.*

Theorem 2. *Let us consider*

$$f^{\#}(t,x,v) = f_0(x,v) + \int_0^t Q(f^{\#}, f^{\#}) d\tau. \tag{6}$$

There are constants c_0 and R_0 such that if $||f_0|| \leq c_0 R_0$ and $\alpha\beta^{-2}R_0$ are sufficiently small, then the equation (5) has a unique non-negative solution $f^{\#} \in M_{R_0}$.

To solve problem (1) we consider:

$$f^{\#}(t,x,v) = f(t, x+vt, v + \int_0^t F(\tau,x,v)d\tau) \tag{7}$$

Assuming F integrable with regard to the time and therefore

$$\frac{df^{\#}}{dt}(t,x,v) = Q(f^{\#}, f^{\#}) \tag{8}$$

and integrating we obtain again (5) and the theorems are completed, but with F integrable with regard to the time.

This topic is developed in [4], as well as in the articles [11], [9], [5] and [15].

Article [11] refers to a gas with a strong sphere and to a initial condition which tends exponentially to zero at infinity in phase space. Article [9] generalizes results of [11] to the Boltzmann equation with interaction potential between two particles. In both articles, the mass of the gas is infinite. In another direction the result of [5] refers to a gas with infinite mass where the initial condition is assumed to decay in a physical space with behavior of power inverse. In this case for sufficiently small decay the mass of the gas can be infinite. Another generalization is given in the article [15] where global existence is proven for initial conditions decaying with behavior of power inverse in the whole phase space. The articles [5] and [13] refer both to Boltzmann equation with interaction of general potential for the couples of internal forces of particles. Besides these previously mentioned articles, we also have to mention two results given by Polewczak. In the first one [13], the author generalizes the theory of previously limited existence to the case of mild solution, also to the case of classic solution. In the second [14] the mathematical results are generalized to the case of initial conditions with more general decay at infinity in phase space.

On the Problem of Cauchy with field of External force

First, let us consider the spatially inhomogeneous equation for neuter particles in a force field $F = F(t,x,v)$. In relationship with this topic we should mention the article of Asano [1] in which local existence is proven for general initial conditions. The formulation of Asano is the starting point of all the studies developed in this article.

Some studies of the problem can be seen in the articles [2], [7], [10] and [6]. In particular, the articles of Asano [2] and Grunfeld [7] provide a global existence proof for the solution with initial condition close to equilibrium and a conservative force field. Hamdache [10] gives us a result with initial conditions decaying exponentially to zero at infinity in phase space and trajectories prescribed by a oscillating field. In [6] it is given a result of global existence for decaying conditions and for a force field acting in an interval of time $[0, T]$ with T big, but finite. Recently in [12] Lions developed the problem of Cauchy in L^1 in the presence of an external field. The important aspect of the result in [12] is that refers to force fields depending on the distribution function, this case is entirely different from the cases previously prescribed.

This article divides into two parts, the first one has to do with a theorem which, as opposed to the articles previously mentioned, solves the problem for fields differentiable with regard to time, and subsequently we extend it to a theorem of positivity of the solutions.

Demonstration of the theorem 1. Indeed, let us consider :

$$Q(f^{\#}, f^{\#})(t,x,v) = Q_g(f^{\#}, f^{\#})(t,x,v) - Q_l(f^{\#}, f^{\#})(t,x,v),$$

where

(a)

$$\begin{aligned}
&Q_g(f^{\#}, f^{\#})(t,x,v) = \\
&\sigma \int_{S_+^2} \int_{\mathbb{R}^3} w \cdot (v-u) f(t, x+tv, v' + tF(t,x,v)) f(t, x+tv, u' + tF(t,x,v)) dwdu = \\
&\sigma \int_{S_+^2} \int_{\mathbb{R}^3} w \cdot (v-u) f^{\#}(t, x+t(v-v'), v') f^{\#}(t, x+t(v-u'), u') dwdu
\end{aligned}$$

(b)

$$\begin{aligned}
&\int_0^t |Q_g(f^{\#}, f^{\#})| d\tau \leq \\
&\sigma \int_0^t \left| \int_{S_+^2} \int_{\mathbb{R}^3} |w(v-u)| f^{\#}(\tau, x+\tau(v-v'), v') f^{\#}(\tau, x+\tau(v-u'), u') dwdu \right| d\tau \\
&\leq \int_0^t \int_{S_+^2} \int_{\mathbb{R}^3} |w(v-u)| \; ||f^{\#}||^2 e^{-\beta(|x+\tau(v-v')|^2 + |v'|^2)} e^{-\beta(|x+\tau(v-u')|^2 + |u'|^2)} dwdud\tau \\
&= \sigma ||f^{\#}||^2 \int_{S_+^2} \int_{\mathbb{R}^3} |w \cdot (v-u)| e^{-\beta(|v'|^2 + |u'|^2)} \int_0^t e^{-\beta(|x+\tau(v-v')|^2)} e^{-\beta(|x+\tau(v-u')|^2)} dwdud\tau
\end{aligned}$$

now

$$
\begin{aligned}
&-\beta(|x+t(v-v^{'})|^2+|x+t(v-u^{'})|^2)=\\
&-\beta(|(x+tv)-tv^{'}|^2)+|(x+tv)-tu^{'}|^2)=\\
&-\beta\big(2|x+tv|^2+t^2(|v^{'}|^2+|u^{'}|^2)-2t(x+tv)\cdot(v^{'}+u^{'})\big)=\\
&-\beta\big(2|x+tv|^2+t^2(|v^{'}|^2+|u^{'}|^2)-2t(x+tv)\cdot(v+u)\big)=\\
&-\beta(|(x+tv)-tv|^2+|(x+tv)-tu|^2)=\\
&-\beta(|x|^2+|x+t(v-u)|^2)
\end{aligned}
$$

therefore

$$
\begin{aligned}
&\int_0^t|Q_g(f^{\#},f^{\#})|d\tau\leq\\
&\pi\sigma||f^{\#}||^2\rho(x,v)^{-1}\int_{\mathbb{R}^3}|v-u|\left(\int_0^{\infty}e^{-\beta|x+\tau(v-u)|^2}d\tau\right)e^{-\beta|u|^2}du\leq\\
&\pi^3\beta^{-2}\sigma\rho(x,v)^{-1}||f^{\#}||^2
\end{aligned}
$$

(c)

$$
\begin{aligned}
&Q_l(f^{\#},f^{\#})=\\
&\pi\sigma f(t,x+tv,v+tF(t,x,v))\int_{\mathbb{R}^3}|v-u|f(t,x+tv,u+tF(t,x,v))du=\\
&\pi\sigma f^{\#}(t,x,v)\int_{\mathbb{R}^3}|v-u|f^{\#}(t,x+t|v-u|,u)du
\end{aligned}
$$

(d)

$$
\begin{aligned}
&|Q_l(f^{\#},f^{\#})(t,x,v)|=\\
&\left|\pi\sigma f^{\#}(t,x,v)\int_{\mathbb{R}^3}|v-u|f^{\#}(t,x+t(v-u),u)du\right|=\\
&\left|\pi\sigma f^{\#}(t,x,v)\int_{\mathbb{R}^3}|v-u|f^{\#}(t,x+t(v-u),u)e^{\beta(|x+t(v-u)|+|u|^2)}e^{-\beta(|x+t(v-u)|+|u|^2)}du\right|=\\
&\left|\pi\sigma f^{\#}(t,x,v)e^{\beta(|x|^2+|u|^2)}e^{-\beta(|x|^2+|u|^2)}\int_{\mathbb{R}^3}|v-u|\;||f^{\#}||e^{-\beta(|x+t(v-u)|^2+|u|^2)}du\right|\\
&\leq\pi\sigma||f^{\#}||^2\rho(x,v)^{-1}\int_{\mathbb{R}^3}|v-u|e^{-\beta(|x+t(v-u)|^2+|u|^2)}du
\end{aligned}
$$

here $\rho(x,v)^{-1} = e^{-\beta(|x|^2+|u|^2)}$, therefore

$$
\begin{aligned}
&|Q_l(f^{\#},f^{\#})(t,x,v)| \\
&\leq \pi\sigma||f^{\#}||^2\rho(x,v)^{-1}\int_{\mathbb{R}^3}|v-u|e^{-\beta|u|^2}\int_0^t e^{-\beta|x+\tau(v-u)|^2}d\tau du \\
&\leq \pi\sigma||f^{\#}||^2\rho(x,v)^{-1}\int_{\mathbb{R}^3}|v-u|e^{-\beta|u|^2}\int_0^\infty e^{-\beta|x+\tau(v-u)|^2}d\tau du
\end{aligned}
$$

now $\int_0^\infty e^{-\beta|x+\tau(v-u)|^2}d\tau \leq \sqrt{\frac{\pi}{\beta}}\frac{1}{|v-u|}$; [8, pag. 28]

This is,

$$
\int_0^t |Q_l(f^{\#},f^{\#})|d\tau \leq \pi^{\frac{3}{2}}\sigma\rho(x,v)^{-1}\beta^{-\frac{1}{2}}||f^{\#}||^2\int_{\mathbb{R}^3}e^{-\beta|u|^2}du = \\
\pi^3\sigma\rho(x,v)^{-1}\beta^{-2}||f^{\#}||^2
$$

(e) We define the operator H on M by

$$
Hf^{\#} = f_0(x,v) + \int_0^t Q(f^{\#},f^{\#})|d\tau
$$

and let $M_R = \{f \in M : ||f|| \leq R\}$

The previous estimates show that if $||f_0|| \leq \frac{R}{2}$ and $f^{\#} \in M_R$, then

$$
\begin{aligned}
|Hf^{\#}| &\leq \rho(x,v)^{-1}||f_0|| + 2\pi^3\beta^{-2}\sigma\rho(x,v)^{-1}||f^{\#}||^2 \\
&\leq \rho(x,v)^{-1}\Big[\frac{R}{2} + 2\pi^3\beta^{-2}\sigma R^2\Big].
\end{aligned}
$$

We choose $2\pi^3\beta^{-2}\sigma R \leq \frac{1}{2}$. Therefore H applies M_R into itself for R sufficiently small. Now H is a contraction on M_R. Since the elements of M_R are continuous, by the fixed point theorem the statement is proven. □

Demonstration of the theorem 2. If $\sigma\beta^{-2}R$ and $||f_0||$ are sufficiently small, then

$$
0 \leq l_0(t) \leq l_1(t) \leq u_1(t) \leq u_0(t), \qquad 0 \leq t \leq T \tag{9}
$$

and therefore the system

$$
\left.
\begin{aligned}
l^{\#}(t) + \int_0^t l^{\#}R^{\#}(u)(\tau)d\tau = f_0 + \int_0^t Q_g(l^{\#},l^{\#})(\tau)d\tau \\
u^{\#}(t) + \int_0^t u^{\#}R^{\#}(l)(\tau)d\tau = f_0 + \int_0^t Q_g(u^{\#},u^{\#})(\tau)d\tau
\end{aligned}
\right\} \tag{10}
$$

has a unique global solution (l,u) with $(l^{\#},u^{\#}) \in M_R \times M_R$. Indeed, if $l_0 = 0$

$$
u_1^{\#}(\tau) = f_0 + \int_0^t Q_g(u_0^{\#},u_0^{\#})d\tau.
$$

This is , with $q = w \cdot (v-u)$

$u_1(t, x+vt, v+tF(t,x,v)) =$

$$f_0(x,v) + \sigma \int_0^t \int_{S_+^2} \int_{\mathbb{R}^3} q u_0(\tau, x+\tau v, v^{'} + \tau F(t,x,v)) u_0(\tau, x+\tau v, u^{'} + tF(t,x,v)) dwdud\tau =$$

$$f_0(x,v) + \sigma \int_0^t \int_{S_+^2} \int_{\mathbb{R}^3} q u_0^{\#}(\tau, x+\tau(v-v^{'}), v^{'}) u_0^{\#}(\tau, x+\tau(v-u^{'}), u^{'}) dwdud\tau$$

That is, condition (6) is true if

$$\sigma \int_0^t \int_{S_+^2} \int_{\mathbb{R}^3} q u_0^{\#}(\tau, x+\tau(v-v^{'}), v^{'}) u_0^{\#}(\tau, x+\tau(v-u^{'}), u^{'}) dwdud\tau \leq$$
$$u_0(t, x+vt, v+tF(t,x,v)) - f_0(x,v) =$$
$$u_0^{\#}(\tau, x, v) - f_0(x,v)$$

If we define $u_0^{\#}(\tau,x,v) = e^{-\beta|x-tv|^2} w(u)$, we have that

$$\left.\begin{array}{r} u_0^{\#}(\tau, x+\tau(v-v^{'}), v^{'}) = e^{-\beta|x+t(v-v^{'})|^2} w(v^{'}) \\ u_0^{\#}(\tau, x+\tau(v-u^{'}), u^{'}) = e^{-\beta|x+t(v-u^{'})|^2} w(u^{'}) \end{array}\right\}$$

and therefore

$$\sigma \int_0^t \int_{S_+^2} \int_{\mathbb{R}^3} q w(v^{'}) w(u^{'}) e^{-\beta|x+\tau(v-v^{'})|^2} e^{-\beta|x+\tau(v-u^{'})|^2} dwdud\tau \leq$$
$$U_0^{\#}(\tau,x,v) - f_0(x,v).$$

We write $\psi(\sigma) = \sup_x e^{\beta|x|^2} f_0(x,v)$. This is

$$\psi(v) + \sigma \int_0^t \int_{S_+^2} \int_{\mathbb{R}^3} w\cdot(v-u) w(v^{'}) w(u^{'}) e^{\beta|x|^2} e^{-\beta|x+\tau(v-v^{'})|^2} + e^{-\beta|x+\tau(v-u^{'})|^2} dwdud\tau \leq w(v) \tag{11}$$

We want a non negative solution w for (8), and a sufficient condition is:

$$\psi(v) + \sigma \int_0^t \int_{S_+^2} \int_{\mathbb{R}^3} w \cdot (v-u) w(v^{'}) w(u^{'}) e^{\beta|x+t(v-u)|^2} dwdud\tau = w(v).$$

This is

$$\psi(v) + \sigma \sqrt{\frac{\pi}{\beta}} \int_{S_+^2} \int_{\mathbb{R}^3} w \; w(v^{'}) w(u^{'}) dwdu = w(v) \tag{12}$$

To prove the existence of a solution $w \geq 0$ of (9) let us consider the space

$$G = \left\{ g \in C^0(\mathbb{R}^3) : \text{ It exists } c > 0 \text{ con } |g(v)| \leq ce^{-\beta|v|^2} \right\}$$

with norm $||g||_G = \sup_v e^{\beta|v|^2} |g(v)|$

Let us consider the operator F defined on G by:

$$F(w)(v) = \psi(v) + \epsilon \int_{S_+^2} \int_{\mathbb{R}^3} w(v^{'})w(u^{'})dudw$$

with $\epsilon = \sigma\sqrt{\frac{\pi}{\beta}}$.

F applies a sufficiently small ball in G into itself. Indeed, let $w \geq 0$, since $0 \leq f_0 \in M$, $F(w)(v) \geq 0$ and

$$|F(w)(v)| \leq c_0 e^{-\beta|v|^2} + \epsilon \int_{S_+^2} \int_{\mathbb{R}^3} ||w||_G^2 e^{-\beta(|v|^2+|u|^2)} dudw =$$
$$e^{-\beta|v|^2}\Big(c_0 + 2\sigma\pi^3\beta^{-2}||w||_G^2\Big).$$

This is, for such w, $||Fw||_G \leq c_0 + 2\sigma\pi^3\beta^{-2}||w||_G^2$. Therefore for $w_1, w_2 \in G$

$$||Fw_1 - Fw_2||_G \leq 2\sigma\pi^3\beta^{-2}\Big(||w_1||_G + ||w_2||_G\Big)||w_1 - w_2||_G.$$

Namely, F applies non-negative functions in the ball of radius R_0 into itself, and it is a contraction there if $||f_0||$ and $\alpha\beta^{-2}R_0$ are sufficiently small.

As F is a contraction, $w = \lim_{n\to\infty} F^n(w)$, we can take $F^0(w) = f_0, f_0 \geq 0$ and the solution for (9) is non-negative.

It remains to show that $u = l$. By definition

$$\left.\begin{aligned} l^{\#}(t) + \int_0^t l^{\#}R^{\#}(u)(\tau)d\tau &= f_0 + \int_0^t Q_g(l^{\#}, l^{\#})(\tau)d\tau \\ u^{\#}(t) + \int_0^t u^{\#}R^{\#}(l)(\tau)d\tau &= f_0 + \int_0^t Q_g(u^{\#}, u^{\#})(\tau)d\tau \end{aligned}\right\}$$

then

$$(u^{\#}-l^{\#})(t) = \int_0^t (Q_g(u^{\#}, u^{\#}-l^{\#}) + Q_g(u^{\#}-l^{\#}, l^{\#}) + l^{\#}R(u^{\#}-l^{\#}) - (u^{\#}-l^{\#})R(l^{\#}))d\tau.$$

This is $||u^{\#} - l^{\#}||_M \leq c\sigma\beta^{-2}(||u^{\#}||\,||u^{\#} - l^{\#}|| + ||l^{\#}||\,||u^{\#} - l^{\#}||)$. Now $u^{\#}$, $l^{\#}$ both are in M_R and in this way $||u^{\#}||$ and $||l^{\#}||$ are bounded by cR_0; therefore the conclusion follows if $\sigma\beta^{-2}R$ is sufficiently small. □

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